Journal of Mathematics Research; Vol. 15, No. 2; April 2023
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Analysis of 2D Maxwell's Equations in a Time-Harmonic Regime

Shuo Sha

Correspondence: School of Physical Sciences, University of California, Irvine, California, United States

Received: January 26, 2023 Accepted: March 11, 2023 Online Published: April 4, 2023

doi:10.5539/jmr.v15n2p1 URL: https://doi.org/10.5539/jmr.v15n2p1

Abstract

The variational formulation is an essential tool to analyze the existence and uniqueness of the solution of certain partial differential equations with boundary conditions. We can further approximate this analytical solution by computing a corresponding numerical solution obtained by the finite element method. In this paper, we studied 2D Maxwell's equations in a time-harmonic regime. We established a corresponding variational formulation and proved its well-posedness in certain conditions. We also constructed a corresponding internal approximation and gave an error estimate within some prior assumptions. This theoretical analysis provides a basis to compute the numerical solution of time-harmonic 2D Maxwell's equations and gives physical significance to the transverse magnetic problem.

Keywords: 2D Maxwell's equations, partial differential equation, variational formulation, finite element method

1. Introduction

In mathematics, more precisely in differential calculus, a partial differential equation (sometimes abbreviated as PDE) is a differential equation that has unknown functions as solutions; these functions depend on several variables that satisfy certain conditions concerning their partial derivatives. (Evans, L. C., 2010; Pinchover, Y. & Rubinstein, J., 2005)

It is hard to analyze the entire set of solutions of a PDE problem, but the boundary conditions often reduce the set of solutions down to a few. Unlike the parameters of the solution sets of an ordinary differential equation, which correspond to the additional conditions, the boundary conditions for PDEs are instead in the form of a function; intuitively, this indicates that it is much more difficult to analyze the solution set, which is true in almost all problems.

The variational point of view enables us to approach certain problems of partial differential equations from an unusual perspective that is rich and powerful. In particular, it allows us to introduce the theoretical elements leading to the solution of the problem (proving the existence and uniqueness of the solution in an adequate framework) and build the finite element method, which depends on the theoretical considerations in order to naturally provide a method to approach the solution (which, very often, is not otherwise explicitly computable).

In the preliminaries, we introduce the variational formulation and Sobolev spaces, which are essential to give the well-posedness of PDE problems. In fact, we can transform PDE problems with boundary conditions into variational formulation problems using Greens formula; then, we can study the corresponding variational formulations. We also introduce an important theorem, the Lax-Milgram theorem, which gives the well-posedness of some variational formulations, and we use it to prove our further research in section 3. In addition, we introduce the finite element method, which is a numerical method that computes solutions of certain boundary problems. The principle of the finite element method is to substitute the Hilbert space V, on which the variational formulation is posed, with a finite dimensional subspace V_h. The internal approximation posed on V_h can be reduced to a simple resolution of a linear system. Moreover, one can carefully construct a V_h so that it accurately approximates V and the solution u_h in V_h is "close" to the actual solution u in V.

Though prior studies have analyzed other Maxwell's equations and curl problems (Ciarlet, Wu & Zou, 2014; Ciarlet, 2020), the investigation of the transverse magnetic problem within the transverse mode of electromagnetic radiation is lacking. Our current research studies the 2D Maxwells equations in a time-harmonic regime that models this problem. By using Greens formula and some analysis, we established a corresponding variational formulation. Then, we studied the well-posedness characteristics of this variational formulation by discussing the different cases of the coefficient ω in the variational formulation. We found that the variational formulation is well-posed when ω is not a real number, and also when ω is a real number and the homogeneous variational formulation has a unique solution. But when ω is a real number and the homogeneous variational formulation has a non-zero solution, the variational formulation may not be well-posed, and we will analyze two situations here. For the numerical approach, we established a corresponding internal approximation, which can be well characterized, to easily obtain its well-posedness. Finally, we adapted some conclusions to derive an estimate for the rate of convergence between the solution of the corresponding internal approximation and that of the variational formulation within some prior assumptions.

2. Preliminaries

In this paper, we explore within the space Ω, which is the domain of $\mathbb{R}^N$, and we denote $\partial\Omega$ its boundary. However, we sometimes assume that Ω is a regular bounded domain, one that locates on only one side of its regular hypersurface boundary. We denote $\vec{n}$ the unit vector normal to $\partial\Omega$ oriented to the external of Ω. Moreover, we denote dx the volume measure in Ω and ds the surface measure on $\partial\Omega$.

Hilbert Space

We first introduce the definition for a Hilbert space.

Definition 2.1. *The vector space V equipped with the inner product $\langle\cdot,\cdot\rangle_V$ is a Hilbert space if it is complete for the norm $\|\cdot\|_V$:*

$$\|v\|_V = \sqrt{\langle v, v\rangle_V}, \quad \forall v \in V. \tag{2.1}$$

A Hilbert space has the following characteristics, which will be frequently used in other subsections.

Definition 2.2. *The sequence $\{v_n\}_{n\in\mathbb{N}}$ in Hilbert space V converges weakly to $v_\infty \in V$ if*

$$\lim_{n\to+\infty} \langle v_n, v\rangle_V = \langle v_\infty, v\rangle_V, \quad \forall v \in V. \tag{2.2}$$

We note in this case $v_n \rightharpoonup v_\infty$ when n tends to $+\infty$.

In fact, if $\{v_n\}_{n\in\mathbb{N}}$ in V converges strongly to $v_\infty \in V$, then $\{v_n\}_{n\in\mathbb{N}}$ converges weakly to $v_\infty \in V$, and this can be proved using Cauchy-Schwarz inequality.

We also have the following theorem that indicates the relationship between weak convergence and boundedness.

Theorem 2.3. *If $\{v_n\}_{n\in N}$ is a bounded sequence in V, then we can extract a sub-sequence $\{v_n\}$ that converges weakly. Conversely, if $\{v_n\}_{n\in N}$ is a sequence that converges weakly in V, then $\{v_n\}$ is bounded.*

Another way to prove that a space is a Hilbert space is using the following lemma:

Lemma 2.4. *Any closed subspace of a Hilbert space is also a Hilbert space.*

Lebesgue Space

Another common function space is Lebesgue space.

Definition 2.5. *For $p \in [1,\infty)$, if any function u in $\Omega \subset \mathbb{R}^N \to \mathbb{R}$ satisfies*

$$\int_\Omega |u|^p dx < +\infty,$$

then we have $u \in L^p(\Omega)$, where $L^p(\Omega)$ is function space with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{1/p}.$$

In this paper, we mainly consider functions in the space $L^2(\Omega)$. Moreover, we admit that $L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega uv dx.$$

Sobolev Space

Before introducing Sobolev spaces, we have the following definitions:

Definition 2.6. *A function u in $L^2(\Omega)$ is weakly differentiable if there exist functions $(w_i)_{1\le i\le N} \in L^2(\Omega)$ such that for any function $\varphi \in C_c^\infty(\Omega)$, we have*

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i} dx = -\int_\Omega w_i \varphi dx.$$

We call w_i the i-th weak partial derivative of u, denoted as $\frac{\partial u}{\partial x_i}$.

Definition 2.7. *Let u be a function of Ω in $\mathbb{R}^N$ whose components belong to $L^2(\Omega)^N$. Then u admits a divergence in the weak sense in $L^2(\Omega)$ if there exists a function $\omega \in L^2(\Omega)$ such that, for any function $\varphi \in C_c^\infty(\Omega)$, we have*

$$\int_\Omega u \cdot \nabla\varphi dx = -\int_\Omega \omega\varphi dx. \tag{2.3}$$

We call ω the weak divergence of u and denote it as div u.

Now, we introduce the definition of the Sobolev space $H^p(\Omega)$, for $p \in \mathbb{N}^*$.

Definition 2.8. *Let Ω be a domain of $\mathbb{R}^N$. The Sobolev space $H^1(\Omega)$ is defined by:*

$$H^1(\Omega) = \left\{u \in L^2(\Omega) \text{ such that } \tfrac{\partial u}{\partial x_i} \in L^2(\Omega), \forall i \in \{1, \dots, N\}\right\},$$

where $\frac{\partial u}{\partial x_i}$ is the weak partial derivative of u.
More generally, $H^p(\Omega)$, for $p \geq 2$, is defined by:

$$H^p(\Omega) = \left\{u \in L^2(\Omega) \text{ such that } \partial^\alpha u \in L^2(\Omega), \forall\alpha \text{ with } |\alpha| \leq p\right\}$$

with

$$\partial^\alpha u = \tfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

where $\partial^\alpha u$ is the weak partial derivative of u. Here, $\alpha = (\alpha_1, \dots, \alpha_N)$ is multi-index with $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^N \alpha_i$.

Proposition 2.9. *The Sobolev space $H^p(\Omega)$ with the inner product*

$$\langle u, v\rangle = \int_\Omega \sum_{|\alpha|\leq n} \partial^\alpha u \partial^\alpha v dx$$

is a Hilbert space.

Proof. Proving for $p > 1$ is similar to that for $p = 1$, so it suffices to prove for the case $p = 1$. We admit that $L^2(\Omega)$ is a Hilbert space. As the inner product is defined, we need to prove that $H^1(\Omega)$ is complete. We assume that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. By the equation

$$||u_n||^2_{H^1(\Omega)} = ||u_n||^2_{L^2(\Omega)} + ||\nabla u_n||^2_{L^2(\Omega)},$$

there exists Cauchy sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{\frac{\partial u_n}{\partial x_i}\}_{n\in\mathbb{N}}$ which converge to u and ω in $L^2(\Omega)$ respectively. From Definition 2.6, for any $\varphi \in C_c^\infty(\Omega)$, we have:

$$\int_\Omega u_n \frac{\partial\varphi}{\partial x_i} dx = -\int_\Omega \frac{\partial u_n}{\partial x_i}\varphi dx. \tag{2.4}$$

As the limit $n \to \infty$, we obtain that $\frac{\partial u}{\partial x_i} = \omega$. Therefore, u_n converges in $H^1(\Omega)$. □

Lemma 2.10. *Let Ω be a regular bounded open set of class C^1, then $C_c^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$ and $H^2(\Omega)$.*

To evaluate the boundary problems later in this paper, we will define the "edge value", or the "trace" of v on the $\partial\Omega$, of a function in $H^1(\Omega)$ through the following theorems.

Theorem 2.11. ***(Trace Theorem for** $H^1(\Omega)$**)*** *Let Ω be a regular bounded open set of class C^1. The trace application T_0 is defined by:*

$$\begin{aligned} H^1(\Omega) \cap C^1(\bar{\Omega}) &\to L^2(\partial\Omega) \cap C(\overline{\partial\Omega}) \\ u &\to T_0(u) = u|_{\partial\Omega}. \end{aligned}$$

This application T_0 extends by continuity into a continuous linear application from $H^1(\Omega)$ into $L^2(\partial\Omega)$, also denoted as T_0. Specifically, there exists a constant $C > 0$ such that for any function $u \in H^1(\Omega)$,

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}.$$

Theorem 2.12. ***(Trace Theorem for** $H^2(\Omega)$**)*** *Let Ω be a regular bounded open set of class C^1. The trace application T_1 is defined by:*

$$\begin{aligned} H^2(\Omega) \cap C^1(\bar{\Omega}) &\to L^2(\partial\Omega) \cap C(\overline{\partial\Omega}) \\ u \to T_1(u) &= \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}, \end{aligned}$$

with $\frac{\partial v}{\partial n} = \nabla u \cdot n$. This application T_1 extends by continuity into a continuous linear application from $H^2(\Omega)$ to $L^2(\Omega)$. Specifically, there exists a constant $C > 0$ such that for any function $u \in H^2(\Omega)$,

$$\|\frac{\partial u}{\partial n}\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^2(\Omega)}.$$

By Definition 2.10 and these trace theorems, we deduce Green's formula below.

Theorem 2.13. ***(Green's Formula)*** *Let Ω be a regular and bounded domain. If u and v are functions of $H^1(\Omega)$, then we have:*

$$\int_\Omega u\frac{\partial v}{\partial x_i}dx = -\int_\Omega v\frac{\partial u}{\partial x_i}dx + \int_{\partial\Omega} uvn_i ds. \tag{2.5}$$

Moreover, if $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then we have:

$$\int_\Omega \Delta uvdx = -\int_\Omega \nabla u \cdot \nabla vdx + \int_{\partial\Omega} \frac{\partial u}{\partial n}vds. \tag{2.6}$$

Remark 2.14. *Since $C_c^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$ and $H^2(\Omega)$, we first construct two sequences $\{u_n\}$ and $\{v_n\}$ in $C_c^\infty(\bar{\Omega})$ which converge to $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$ respectively. We notice that u_n and v_n satisfy (2.5), and as $n \to +\infty$, we can deduce (2.5). The derivation of (2.6) is similar.*

To deduce the equivalence of certain variational formulations later in the paper, we introduce the following lemma.

Lemma 2.15. *Let Ω be an open set in $\mathbb{R}^N$ and g be a continuous function in Ω. Then the function $g = 0$ in Ω if for any function φ of $C_c^\infty(\Omega)$ with compact support in Ω, we have*

$$\int_\Omega g\varphi dx = 0. \tag{2.7}$$

Lax-Milgram theorem and Finite Element Method

To evaluate the well-posedness of a partial differential equation with certain boundary conditions, we follow a general approach. We first abandon the space $C^k(\Omega)$ of continuously differentiable functions in favor of its "generalization", a Hilbert space V. Then, we multiply a test function $v \in V$ to both sides of the equation and integrate within Ω. Moreover, we reduce the order of the equation by one using Green's formula and arrive at a corresponding variational formulation, sometimes known as the weak solution of the original partial differential equation. Next, we evaluate the well-posedness of the variational formulation through the Lax-Milgram theorem. Finally, we use the finite element method to approximate for a numerical solution with certain estimate of error.

Before investigating a specific partial differential equation, we need to briefly introduce the Lax-Milgram theorem and the finite element method. In this paper, we consider variational formulations in the form of

$$\text{Find } u \in V \text{ such that } a(u, v) = L(v), \quad \forall v \in V. \tag{2.8}$$

The assumptions on $a(\cdot, \cdot)$ and $L(\cdot)$ are:

(1) $a(\cdot,\cdot)$ is a continuous bilinear form on V, i.e., there exists $M > 0$ such that:

$$|a(u, v)| \leq M\|u\|_V\|v\|_V, \quad \forall u, v \in V. \tag{2.9}$$

(2) $L(\cdot)$ is a continuous linear form on V, i.e., there exists $C > 0$ such that:

$$|L(v)| \leq C\|v\|_V, \quad \forall v \in V. \tag{2.10}$$

(3) $a(\cdot,\cdot)$ is coercive, i.e., there exists $T > 0$ such that:

$$|a(u, u)| \geq \gamma\|u\|_V^2, \quad \forall u \in V. \tag{2.11}$$

Theorem 2.16. ***(Lax-Milgram Theorem)*** *Let V be a real Hilbert space. If $L(\cdot)$ is a continuous linear form on V and $a(\cdot,\cdot)$ is a continuous and coercive bilinear form on V, then the variational formulation in the form of (2.8) admits a unique solution.*

After analyzing the existence and uniqueness of the solution, we wish to further approximate this solution through a numerical approach – the finite element method. The fundamental idea of this approach is derived from the variational approach introduced previously, where we evaluate within a finite dimensional subspace of V, V_h, reduce the internal approximation to a linear system of matrix, and solve for its solution numerically. Furthermore, to determine the estimated error of the numerical solution, we measure the accuracy of u_h as an approximation of u.

We use the same abstract framework of the variational formulation in (2.8). However, to compute a numerical solution, we use the finite dimensional subspace V_h to replace the Hilbert space V.

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = L(v_h), \quad \forall v_h \in V_h. \tag{2.12}$$

To guarantee the existence and uniqueness of a numerical solution to the internal approximation (2.12), we have the following lemma.

Lemma 2.17. *Let V be a Hilbert space and V_h be its finite dimensional subspace. If $L(v)$ is a continuous linear form on V and $a(u, v)$ is a continuous and coercive bilinear form on V, then the internal approximation (2.12) admits a unique solution that can be computed through solving a linear system of equations with positive definite matrix.*

Proof. Let V_h be a finite dimensional space with $(\phi_j)_{1\leq j\leq N_h}$ as its finite basis. We consider $v_h = \phi_i$ and $u_h = \sum_{j=1}^{N_h} u_j\phi_j$. Then (2.12) is equal to:

$$\text{Find } U_h \in \mathbb{R}^{N_h} \text{ such that } a\left(\textstyle\sum_{j=1}^{N_h} u_j\phi_j, \phi_i\right) = L(\phi_i), \quad \forall 1 \leq i \leq N_h.$$

To express as a linear system, we define the following matrixes:

$$\begin{aligned} U_h &= (u_j)_{1\leq j\leq N_h}, \\ b_h &= L(\phi_i)_{1\leq i\leq N_h}, \\ K_h &= a(\phi_j, \phi_i)_{1\leq i,j\leq N_h}. \end{aligned}$$

Now, for $1 \leq i, j \leq N_h$, the internal approximation becomes:

$$(K_h)_{ij}(U_h)_j = (b_h)_i. \tag{2.13}$$

The coercivity of the bilinear form $a(u, v)$ suggests the positive definiteness of (K_h), which means for any vector $U_h \in \mathbb{R}^{N_h}$, we have

$$(K_hU_h) \cdot U_h \geq \beta\|U_h\|^2, \quad \text{for } \beta > 0$$

Since K_h is invertible, the matrix problem (2.13) admits a unique solution. □

Before introducing the finite element method $\mathbb{P}_k$ in a d-dimensional space($d \geq 2$), where $\mathbb{P}_k$ denotes a k-th order polynomial, we give the definition of the triangulation of a polyhedral domain.

Definition 2.18. *Say that* Ω *is a connected and open polyhedral of* $\mathbb{R}^d$. *We call the set* $\mathcal{T}_h$ *of d-simplexes* $(\kappa_i)_{1\leq i\leq n}$ *a triangulation of* Ω *if it verifies:*
(1) $\kappa_i \subset \bar{\Omega}$ *such that* $\bar{\Omega} = \cup_{i=1}^{n} \kappa i$.
(2) For two distinct d-simplexes κ_i *and* κ_j, *their intersection forms a N-simplexes* $(0 \leq N \leq d-1)$, *whose vertices correspond to those of* κ_i *and* κ_j.

The $\mathcal{T}_h$ *mesh has vertices that correspond to those of the d-simplexes* κ_i *that compose it. Moreover, h denotes the maximum diameters of the d-simplexes* κ_i.

We observe that the finite element method $\mathbb{P}_k$ is only applicable to a polyhedral domain. Then, we give the definition of finite element method $\mathbb{P}_k$:

Definition 2.19. *For a mesh* $\mathcal{T}_h$ *of a connected and open polyhedral domain* Ω, *we define the finite element method* $\mathbb{P}_k$ *by the following space:*

$$Y_h = \left\{u \in C(\bar{\Omega}) \text{ such that } u|_{\kappa_i} \in \mathbb{P}_k, \forall \kappa_i \in \mathcal{T}_h\right\}. \tag{2.14}$$

3. 2D Maxwell's Equations

Establishment of the Variational Formulation

Let $\Omega \subset \mathbb{R}^2$ be a connected bounded domain with a regular (class C^1) connected $\partial\Omega$ boundary. We denote $\vec{n} = (n_x, n_y)$ the exterior normal to $\partial\Omega$. We introduce real-valued functions ω the frequency of the time-harmonic field and $\varepsilon, \mu \in \mathbb{C}^1(\bar{\Omega})$ the dielectric permittivity and magnetic permeability of the medium. We will assume that $\varepsilon^{-1}, \mu^{-1} \in C^1(\bar{\Omega})$ so that there exist constants $\alpha > 0, \beta > 0$ such that

$$\alpha \leq \varepsilon, \mu, \varepsilon^{-1}, \mu^{-1} \leq \beta.$$

The Maxwell problem that we want to study is the transverse magnetic problem, which governs the relationship between the electric field strength $\vec{u}$ and the current density J on a propagation plane. Mathematically, we only consider this relationship in a single period, so we algebraically cancelled the exponential function $e^{i\omega t}$ which shows how the variables change periodically and harmonically with time. The resultant partial differential equation is written below:

$$\begin{cases} \text{Find } \vec{u} \in \boldsymbol{H}(\mathrm{curl}; \Omega) \text{ such that} \\ \vec{\mathrm{curl}}\left(\varepsilon^{-1} \,\mathrm{curl}\, \vec{u}\right) - \omega^2 \mu \vec{u} = \vec{\mathrm{curl}}\left(\varepsilon^{-1} J\right) \text{ in } \Omega \\ \varepsilon^{-1} \,\mathrm{curl}\, \vec{u} = \varepsilon^{-1} J \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

where $\boldsymbol{H}(\mathrm{curl}; \Omega) := \left\{\vec{v} = \left(v_x, v_y\right) \in L^2(\Omega)^2 \mid \mathrm{curl}\, \vec{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \in L^2(\Omega)\right\}$ and $\vec{\mathrm{curl}}\varphi = \left(\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x}\right)$. In this topic, unless otherwise stated, the functions will be complex-valued. Despite ω is real-valued, we still consider the case of $\omega \in \mathbb{C}\backslash\mathbb{R}$ for its easily-derived well-posedness but puts a focus on the case when $\omega \in \mathbb{R}$. In the interest of concision, we avoid the obvious case of $\omega = 0$.

We will need the following (complex) version of the Lax-Milgram theorem:

1. Let X be a Hilbert space of complex-valued functions.

2. Let $l(\cdot)$ be an antilinear form

$$l\left(\lambda u + u'\right) = \bar{\lambda} l(u) + l\left(u'\right), \quad \forall \lambda \in \mathbb{C}, \forall u, u' \in \mathrm{X},$$

continuous on X.

3. Let $a(\cdot, \cdot)$ be a sesquilinear form

$$a\left(\lambda u + u', \gamma v + v'\right) = \lambda \bar{\gamma} a(u, v) + \lambda a\left(u, v'\right) + \bar{\gamma} a\left(u', v\right) + a\left(u', v'\right), \quad \forall \lambda, \gamma \in \mathbb{C}, \quad \forall u, u', v, v' \in \mathrm{X},$$

continuous on $\mathrm{X} \times \mathrm{X}$. Assume that $\Re e\left(e^{i\theta} a(\cdot, \cdot)\right)$ is coercive on X for certain $\theta \in]-\pi; \pi]$, in other words, assume that there exists $\theta \in]-\pi; \pi], \eta > 0$ such that

$$\Re e\left(e^{i\theta} a(u, u)\right) \geq \eta \|u\|_{\mathrm{X}}^2, \quad \forall u \in \mathrm{X}.$$

Then there exists a unique $u \in \mathrm{X}$ verifying $a(u, v) = l(v)$ for all $v \in \mathrm{X}$.

We introduce two functional spaces that will serve in the analysis below. For $\varphi \in C^1(\bar{\Omega})$ and $\vec{v} = \left(v_x, v_y\right) \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$, let us define

$$\overrightarrow{\mathrm{curl}}\varphi = \left(\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x}\right) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}) \quad \text{and} \quad \mathrm{curl}\, \vec{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \in C^0(\bar{\Omega}). \tag{3.2}$$

Corollary 3.1. *By Green's formula, we can establish the identity*

$$\int_\Omega (\overrightarrow{\mathrm{curl}}\varphi \cdot \vec{v} - \varphi\, \mathrm{curl}\, \vec{v}) dxdy = \int_{\partial\Omega} \varphi \vec{v} \cdot \vec{\tau} d\sigma, \quad \forall \varphi \in C^1(\bar{\Omega}), \vec{v} \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega}), \tag{3.3}$$

where $\vec{\tau} = \left(n_y, -n_x\right)$.

Proof. From (2.5), we have

$$\begin{aligned}
\int_\Omega (\overrightarrow{\mathrm{curl}}\varphi \cdot \vec{v} - \varphi\, \mathrm{curl}\, \vec{v}) dxdy &= \int_\Omega \left(\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x}\right)(v_x, v_y) dxdy - \int_\Omega \varphi (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}) dxdy \\
&= \int_\Omega \frac{\partial \varphi}{\partial y} v_x dxdy + \int_\Omega \frac{\partial v_x}{\partial y} \varphi dxdy - \int_\Omega \frac{\partial \varphi}{\partial x} v_y dxdy - \int_\Omega \frac{\partial v_y}{\partial x} \varphi dxdy \\
&= \int_{\partial\Omega} \varphi v_x n_y d\sigma - \int_{\partial\Omega} \varphi v_y n_x d\sigma \\
&= \int_{\partial\Omega} \varphi \vec{v} \cdot \vec{\tau} d\sigma.
\end{aligned}$$

□

Definition 3.2. *A function* $\vec{v} \in \boldsymbol{L}^2(\Omega) := L^2(\Omega) \times L^2(\Omega)$ *admits a weak curl in* $\boldsymbol{L}^2(\Omega)$ *if there exists a function* $V \in L^2(\Omega)$ *such that*

$$\int_\Omega \vec{\mathrm{curl}}\varphi \cdot \vec{v} dxdy = \int_\Omega \varphi V dxdy, \quad \forall \varphi \in C_c^\infty(\Omega).$$

We will then note $\mathrm{curl}\, \vec{v} = V$. *With (3.3), we can verify that this definition extends the one given in (3.2) for regular functions.*

Let us introduce the spaces:

$$\begin{aligned}
\boldsymbol{H}(\mathrm{curl}; \Omega) &:= \left\{ \vec{v} = \left(v_x, v_y\right) \in \boldsymbol{L}^2(\Omega) \mid \mathrm{curl}\, \vec{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \in L^2(\Omega) \right\} \\
\boldsymbol{V}_T(\mu; \Omega) &:= \left\{ \vec{v} \in \boldsymbol{H}(\mathrm{curl}; \Omega) \mid \int_\Omega \mu \vec{v} \cdot \overline{\nabla \varphi} dxdy = 0, \quad \forall \varphi \in H^1(\Omega) \right\}
\end{aligned}$$

where the operator curl is understood in the weak sense according to the previous definition. We will admit that (3.3) is still right for all $\varphi \in H^1(\Omega), \vec{v} \in \boldsymbol{H}(\mathrm{curl}; \Omega)$.

Proposition 3.3. *We admit that* $\boldsymbol{L}^2(\Omega)$ *is a Hilbert space. Then* $\boldsymbol{H}(\mathrm{curl}; \Omega)$ *equipped with the inner product,*

$$(\vec{u}, \vec{v})_{\boldsymbol{H}(\mathrm{curl};\Omega)} = \int_\Omega (\mathrm{curl}\, \vec{u} \overline{\mathrm{curl}\, \vec{v}} + \vec{u} \bar{\vec{v}}) dxdy,$$

and $\boldsymbol{V}_T(\mu; \Omega)$ *with the inner product* $(\cdot, \cdot)_{\boldsymbol{H}(\mathrm{curl};\Omega)}$ *are both Hilbert spaces. We note* $\|\vec{u}\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} = (\vec{u}, \vec{u})^{1/2}_{\boldsymbol{H}(\mathrm{curl};\Omega)}$.

Proof. With the inner products defined as above, it remains to prove that $\boldsymbol{H}(\mathrm{curl}; \Omega)$ and $\boldsymbol{V}_T(\mu; \Omega)$ are complete. We first establish the identity:

$$\|\vec{u}\|^2_{\boldsymbol{H}(\mathrm{curl};\Omega)} = \|\, \mathrm{curl}\, \vec{u}\|^2_{L^2(\Omega)} + \|\vec{u}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

We assume that $\{\vec{u}_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\boldsymbol{H}(\mathrm{curl}; \Omega)$. Since $\boldsymbol{L}^2(\Omega)$ is complete, $\{\vec{u}_n\}$ and $\{\mathrm{curl}\, \vec{u}_n\}$ are both Cauchy sequences that converge in $\boldsymbol{L}^2(\Omega)$. We note that there exists limits $\vec{u} \in \boldsymbol{L}^2(\Omega)$ such that $\vec{u}_n \to \vec{u}$ in $\boldsymbol{L}^2(\Omega)$ and $\omega \in L^2(\Omega)$ such that $\mathrm{curl}\, \vec{u}_n \to \omega$ in $L^2(\Omega)$. To further prove that $\mathrm{curl}\, \vec{u}_n \to \mathrm{curl}\, \vec{u}$, we first use (3.3) to obtain

$$\int_\Omega \vec{\mathrm{curl}}\varphi \cdot \vec{u}_n dxdy = \int_\Omega \varphi\, \mathrm{curl}\, \vec{u}_n dxdy, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{3.4}$$

Moreover, we deduce that $\int_\Omega \vec{\text{curl}}\varphi \vec{u}_n dxdy \to \int_\Omega \vec{\text{curl}}\varphi\vec{u}dxdy$ as $n \to \infty$. Specifically, we notice that

$$\begin{aligned}\left|\int_\Omega \vec{\text{curl}}\varphi\vec{u}_n dxdy - \int_\Omega \vec{\text{curl}}\varphi\vec{u}dxdy\right| &= \left|\int_\Omega \vec{\text{curl}}\varphi(\vec{u}_n - \vec{u})dxdy\right| \\ &\leq \|\operatorname{curl}\varphi\|_{\boldsymbol{L}^2(\Omega)}\|\vec{u}_n - \vec{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &\to 0\end{aligned}$$

as $\|\vec{u}_n - \vec{u}\|_{\boldsymbol{L}^2(\Omega)} \to 0$ and $\|\operatorname{curl}\varphi\|_{L^2(\Omega)}$ is bounded. Similarly, we can also obtain that $\int_\Omega \varphi \operatorname{curl}\vec{u}_n dxdy \to \int_\Omega \varphi\omega dxdy$. Substituting the convergences into (3.4), we have

$$\int_\Omega \vec{\text{curl}}\varphi\vec{u}dxdy = \int_\Omega \varphi\omega dxdy.$$

Based on Definition 3.2, we note that $\omega = \operatorname{curl}\vec{u}$. Since $\operatorname{curl}\vec{u}_n \to \operatorname{curl}\vec{u}$ in $L^2(\Omega)$ and $\vec{u}_n \to \vec{u}$ in $\boldsymbol{L}^2(\Omega)$, we obtain that $\vec{u}_n \to \vec{u}$ in $\boldsymbol{H}(\text{curl};\Omega)$ and $\boldsymbol{H}(\text{curl};\Omega)$ is a Hilbert space.

To prove that $\boldsymbol{V}_T(\mu;\Omega)$ is also a Hilbert space, by Lemma 2.4, we only need to prove $\boldsymbol{V}_T(\mu;\Omega)$ is a closed subspace of $\boldsymbol{H}(\text{curl};\Omega)$. Consider a sequence $\{\vec{u}_n\} \in \boldsymbol{V}_T(\mu;\Omega)$ such that $\vec{u}_n \to \vec{u} \in \boldsymbol{H}(\text{curl};\Omega)$ in $\boldsymbol{H}(\text{curl};\Omega)$. As $n \to \infty$, we notice that,

$$\int_\Omega \mu\vec{u}_n \cdot \overline{\nabla\varphi}dxdy = 0 \overset{n\to\infty}{\to} \int_\Omega \mu\vec{u}\cdot\overline{\nabla\varphi}dxdy = 0, \quad \forall\varphi \in \boldsymbol{H}^1(\Omega).$$

So we have $\vec{u} \in \boldsymbol{V}_T(\mu;\Omega)$, and $\boldsymbol{V}_T(\mu;\Omega)$ is a closed subspace of $\boldsymbol{H}(\text{curl};\Omega)$. Therefore, $\boldsymbol{V}_T(\mu;\Omega)$ is also a Hilbert space. □

Hereinafter, we will admit that the injection of $(\boldsymbol{V}_T(\mu;\Omega), (\cdot,\cdot)_{\boldsymbol{H}(\text{curl};\Omega)})$ into $\boldsymbol{L}^2(\Omega)$ is compact; thereby, from any sequence of elements of $\boldsymbol{V}_T(\mu;\Omega)$ bounded for the norm $\|\cdot\|_{\boldsymbol{H}(\text{curl};\Omega)}$, we can extract a sub-sequence that converges strongly in $\boldsymbol{L}^2(\Omega)$. On the other hand, we specify that the injection of $(\boldsymbol{H}(\text{curl};\Omega), (\cdot,\cdot)_{\boldsymbol{H}(\text{curl};\Omega)})$ into $\boldsymbol{L}^2(\Omega)$ is not compact.

The Maxwell problem that we want to study is:

$$\begin{cases} \text{Find } \vec{u} \in \boldsymbol{H}(\text{curl};\Omega) \text{ such that} \\ \vec{\text{curl}}\left(\varepsilon^{-1}\operatorname{curl}\vec{u}\right) - \omega^2\mu\vec{u} = \vec{\text{curl}}\left(\varepsilon^{-1}J\right) \text{ in } \Omega, \\ \varepsilon^{-1}\operatorname{curl}\vec{u} = \varepsilon^{-1}J \text{ on } \partial\Omega, \end{cases} \tag{3.5}$$

where J denotes a source term belonging to $C^1(\bar{\Omega})$.

Before applying the complex Lax-Milgram theorem to this Maxwell problem, we first establish its variational formulation.

Proposition 3.4. *If $\vec{u}$ verifies (3.5), then $\vec{u}$ is a solution of the problem below:*

$$\begin{cases} \textit{Find } \vec{u} \in \boldsymbol{H}(\text{curl};\Omega) \textit{ such that} \\ a(\vec{u},\vec{v}) = \ell(\vec{v}), \quad \forall\vec{v} \in \boldsymbol{H}(\text{curl};\Omega) \end{cases} \tag{3.6}$$

with $a(\vec{u},\vec{v}) = \int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{u}\overline{\operatorname{curl}\vec{v}} - \omega^2\mu\vec{u}\cdot\bar{\vec{v}})dxdy$ *and* $\ell(\vec{v}) = \int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{v}}dxdy$.

Proof. By multiplying $\bar{\vec{v}} \in \boldsymbol{H}(\text{curl};\Omega)$ to both sides of the equation in (3.5) and integrating within Ω, we obtain the following,

$$\int_\Omega(\vec{\text{curl}}\left(\varepsilon^{-1}\operatorname{curl}\vec{u}\right)\bar{\vec{v}} - \omega^2\mu\vec{u}\bar{\vec{v}})dxdy = \int_\Omega \vec{\text{curl}}\left(\varepsilon^{-1}J\right)\bar{\vec{v}}dxdy.$$

From (3.3), we have

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{u}\overline{\operatorname{curl}\vec{v}} - \omega^2\mu\vec{u}\bar{\vec{v}})dxdy + \int_{\partial\Omega}\left(\varepsilon^{-1}\operatorname{curl}\vec{u}\right)\bar{\vec{v}}\vec{\tau}d\sigma = \int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{v}}dxdy + \int_{\partial\Omega}\varepsilon^{-1}J\bar{\vec{v}}\cdot\vec{\tau}d\sigma.$$

Since $\varepsilon^{-1}\operatorname{curl}\vec{u} = \varepsilon^{-1}J$ on $\partial\Omega$, we can deduce that

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{u}\overline{\operatorname{curl}\vec{v}} - \omega^2\mu\vec{u}\bar{\vec{v}})dxdy = \int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{v}}dxdy.$$

□

Conversely, we also want to show that (3.6) implies (3.5).

Proposition 3.5. *If $\vec{u}$ is a solution in $H^2(\Omega)\times H^2(\Omega)$ of (3.6), then $\vec{u}$ verifies the partial differential equation in (3.5), with the assumption that the image of the application $\vec{\varphi}\mapsto\vec{\varphi}\cdot\vec{\tau}$ defined on $H^1(\Omega)\times H^1(\Omega)$ is dense in $\boldsymbol{L}^2(\partial\Omega)$.*

Proof. (1) The solution of (3.6) verifies (3.5) in Ω.

Using the identity (3.3), we have

$$\int_\Omega(\vec{\text{curl}}(\varepsilon^{-1}\,\text{curl}\,\vec{u})\bar{\vec{v}}-\omega^2\mu\vec{u}\bar{\vec{v}})dxdy-\int_{\partial\Omega}(\varepsilon^{-1}\,\text{curl}\,\vec{u})\bar{\vec{v}}\vec{\tau}d\sigma=\int_\Omega\vec{\text{curl}}(\varepsilon^{-1}J)\bar{\vec{v}}dxdy-\int_{\partial\Omega}\varepsilon^{-1}J\bar{\vec{v}}\cdot\vec{\tau}d\sigma.$$

Since $C_c^\infty(\Omega)$ belongs to $H^1(\Omega)$, we let $v\in C_c^\infty(\Omega)$ and we deduce that

$$\int_{\partial\Omega}\varepsilon^{-1}\,\text{curl}\,\vec{u}\bar{\vec{v}}\cdot\vec{\tau}d\sigma=\int_{\partial\Omega}\varepsilon^{-1}J\bar{\vec{v}}\cdot\vec{\tau}d\sigma=0.$$

Now, the expression coincides with the one in Lemma 2.15:

$$\int_\Omega(\text{curl}\left(\varepsilon^{-1}\,\text{curl}\,\vec{u}\right)-\omega^2\mu\vec{u}-\vec{\text{curl}}\left(\varepsilon^{-1}J\right))\cdot\bar{\vec{v}}dxdy=0.$$

And therefore, by Lemma 2.15, we have

$$\text{curl}\left(\varepsilon^{-1}\,\text{curl}\,\vec{u}\right)-\omega^2\mu\vec{u}=\vec{\text{curl}}\left(\varepsilon^{-1}J\right)\text{ in }\Omega.$$

(2) The solution of (3.6) verifies (3.5) on $\partial\Omega$.

We assume that the image of the application $\vec{\varphi}\mapsto\vec{\varphi}\cdot\vec{\tau}$ defined on $H^1(\Omega)\times H^1(\Omega)$ is dense in $L^2(\partial\Omega)$. As the partial differential equations are equivalent in Ω, we only evaluate the equations on $\partial\Omega$, which is

$$\int_{\partial\Omega}(\varepsilon^{-1}\,\text{curl}\,\vec{u}-\varepsilon^{-1}J)\bar{\vec{v}}\cdot\vec{\tau}d\sigma=0,\ \forall\vec{v}\in\mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega).$$

Since the image of the application $\vec{v}\mapsto\vec{v}\cdot\vec{\tau}$ defined on $\mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)$ is dense in $\mathrm{L}^2(\partial\Omega)$, we deduce that $\varepsilon^{-1}(\text{curl}\,\vec{u}-J)=0$ on $\partial\Omega$. □

Remark 3.6. *However, analyzing the well-posedness characteristics of the variational formulation posed in $\boldsymbol{H}(\text{curl};\Omega)$ is less than satisfactory as the injection of $\boldsymbol{H}(\text{curl};\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is not compact, thereby we cannot extract sub-sequences in $\boldsymbol{H}(\text{curl};\Omega)$ that converge strongly and weakly in $\boldsymbol{L}^2(\Omega)$. Therefore, we want to establish an equivalent variational formulation posed in $\boldsymbol{V}_T(\mu;\Omega)$. To do so, we need to introduce the Rellich-Kondrachov theorem for $\boldsymbol{H}(\text{curl};\Omega)$ and Poincar inequality for $H^1_\#(\Omega)$, a new space defined by $H^1_\#(\Omega):=\left\{\varphi\in H^1(\Omega)\mid\int_\Omega\varphi dxdy=0\right\}$.*

Then we recall a conclusion in (Ciarlet, 2020), which gives a compactness embedding for $\boldsymbol{V}_T(\mu;\Omega)$ into $\boldsymbol{L}^2(\Omega)$.

Lemma 3.7. ***(Rellich-Kondrachov Theorem for $\boldsymbol{V}_T(\mu;\Omega)$)*** *If Ω is a regular and bounded domain, then for any bounded sequence in $\boldsymbol{V}_T(\mu;\Omega)$, we can extract a convergent sub-sequence in $\boldsymbol{L}^2(\Omega)$.*

Lemma 3.8. ***(Poincar Inequality for $H^1_\#(\Omega)$)*** *There exists $C>0$ such that*

$$\int_\Omega|v|^2dxdy\le C\int_\Omega|\nabla v|^2dxdy,\forall v\in H^1_\#(\Omega).\tag{3.7}$$

Proof. We first assume that $H^1_\#(\Omega)$ is a closed subspace of $H^1(\Omega)$, which will be proved in Proposition 3.9. Then we prove by contradiction, where we assume that there exists $\{v_n\}\in H^1_\#(\Omega)$ such that,

$$1=\int_\Omega|v_n|^2dxdy\ge n\int_\Omega|\nabla v_n|^2dxdy,\quad\forall v\in H^1_\#(\Omega).$$

We know that $\{v_n\}\in H^1_\#(\Omega)$ is bounded because we have

$$\|v_n\|^2_{\boldsymbol{H}^1(\Omega)}=\|v_n\|^2_{L^2(\Omega)}+\|\nabla v_n\|^2_{L^2(\Omega)}\le1+\tfrac{1}{n}\le2.$$

By Lemma 3.7, we can find a sub-sequence $\{v_{n'}\}$ which converges in $L^2(\Omega)$. Observing that $\|\nabla v_{n'}\|^2_{L^2(\Omega)} \leq \frac{1}{n'}$, we can deduce that

$$\nabla v_{n'} \overset{L^2(\Omega)}{\to} 0 \text{ as } n' \to \infty.$$

Because both $\|v_{n'}\|_{L^2(\Omega)}$ and $\|\nabla v_{n'}\|_{L^2(\Omega)}$ converge in $L^2(\Omega)$, then $\|v_{n'}\|_{\boldsymbol{H}^1(\Omega)}$ converges in $H^1_\#(\Omega)$. We note that $v_{n'} \to v$ in $H^1_\#(\Omega)$. Moreover, we deduce using Cauchy-Schwarz inequality that

$$\int_\Omega |\nabla v|^2 dxdy = \lim_{n' \to +\infty} \int_\Omega |\nabla v_{n'}|^2\, dxdy \leq \lim_{n' \to +\infty} \frac{1}{n'} = 0,$$

meaning that $v \equiv C$. With the condition of $\int_\Omega v dxdy = 0$, we observe that $v \equiv 0$, which contradicts with $\int_\Omega |v|^2 dxdy = 1$. □

Then we have the following proposition.

Proposition 3.9. *For $\vec{v} \in \boldsymbol{H}(\text{curl}; \Omega)$ given, the problem*

$$\begin{cases} \textit{Find } \varphi \in H^1_\#(\Omega) \textit{ such that} \\ \int_\Omega \mu \nabla\varphi \cdot \overline{\nabla\varphi'} dxdy = \int_\Omega \mu \vec{v} \cdot \overline{\nabla\varphi'} dxdy, \quad \forall \varphi' \in H^1_\#(\Omega) \end{cases} \tag{3.8}$$

admits a unique solution.

Proof. To analyze the existence and uniqueness of the solution, we need to first show that $H^1_\#(\Omega)$ is a Hilbert space by Lemma 2.4. Let $\{\varphi_n\} \in H^1_\#(\Omega)$ and $\varphi_n \to \varphi$ in $H^1(\Omega)$, where $\varphi \in H^1(\Omega)$. By Cauchy-Schwarz inequality, we can deduce that $\int_\Omega \varphi_n dxdy = 0 \to \int_\Omega \varphi dxdy = 0$ in $H^1(\Omega)$, which means that $\varphi \in H^1_\#(\Omega)$. Therefore, $H^1_\#(\Omega)$ is a Hilbert space since it is a closed subspace of $H^1(\Omega)$.

In order to apply the Lax-Milgram theorem, we need to verify the continuity and coercivity of $a(\varphi, \varphi') = \int_\Omega \mu \nabla\varphi \cdot \overline{\nabla\varphi'} dxdy$ and the continuity of $\ell(\varphi') = \int_\Omega \mu \vec{v} \cdot \overline{\nabla\varphi'} dxdy$. From Cauchy-Schwarz inequality and Lemma 3.8, we have:

(1) For $\varphi, \varphi' \in H^1_\#(\Omega)$, there exists $M > 0$ such that:

$$|a(\varphi, \varphi')| \leq \beta \|\nabla\varphi\|_{L^2(\Omega)} \|\nabla\varphi'\|_{L^2(\Omega)} \leq M \|\varphi\|_{H^1(\Omega)} \|\varphi'\|_{H^1(\Omega)}.$$

(2) For $\varphi' \in H^1_\#(\Omega)$, there exists $C > 0$ such that:

$$|\ell(\varphi')| \leq \beta \|\vec{v}\|_{\boldsymbol{L}^2(\Omega)} \|\overline{\nabla\varphi'}\|_{L^2(\Omega)} \leq C \|\varphi'\|_{H^1(\Omega)}.$$

(3) For $\varphi \in H^1_\#(\Omega)$, there exists $\eta > 0$ such that:

$$\Re e(a(\varphi, \varphi)) \geq \eta \|\varphi\|^2_{H^1(\Omega)}.$$

To prove the coercivity, we observe that $\mu \geq \alpha$, obtaining that $\Re e(a(\varphi, \varphi)) = \int_\Omega \mu |\nabla\varphi|^2 dxdy \geq \alpha \|\nabla\varphi\|^2_{L^2(\Omega)}$. Since $\|\varphi\|^2_{H^1(\Omega)} = \|\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|^2_{L^2(\Omega)}$, from Lemma 3.8, we deduce that $\|\nabla\varphi\|^2_{L^2(\Omega)} \geq \frac{1}{C+1} \|\varphi\|^2_{H^1(\Omega)}$, thus obtaining that $\Re e(a(\varphi, \varphi)) \geq \frac{\alpha}{C+1} \|\varphi\|^2_{H^1(\Omega)}$. The coercivity of the sesquilinear form is approved when $\eta = \frac{\alpha}{C+1}$, which is strictly positive. Then by Lax-Milgram theorem, (3.8) admits a unique solution. □

Using Proposition 3.9, we can establish an equivalent variational formulation posed in $\boldsymbol{V}_T(\mu; \Omega)$.

Proposition 3.10. *$\vec{u}$ verifies (3.6) if and only if $\vec{u}$ is a solution of*

$$\begin{cases} \textit{Find } \vec{u} \in \boldsymbol{V}_T(\mu; \Omega) \textit{ such that} \\ a(\vec{u}, \vec{v}) = \ell(\vec{v}), \quad \forall \vec{v} \in \boldsymbol{V}_T(\mu; \Omega). \end{cases} \tag{3.9}$$

Proof. To prove that (3.6) implies (3.9), we notice that $\text{curl}(\nabla\varphi) = \partial x(\partial y \cdot v) - \partial y(\partial x \cdot v) = 0$ for any $\varphi \in C^2(\bar{\Omega})$. Let $\vec{v} = \nabla\varphi$, so that

$$\begin{aligned} a(\vec{u}, \nabla\varphi) &= \int_\Omega (\varepsilon^{-1} \,\text{curl}\, \vec{u} \overline{\text{curl}\, \nabla\varphi} - \omega^2 \mu \vec{u} \cdot \overline{\nabla\varphi}) dxdy \\ &= -\int_\Omega \omega^2 \mu \vec{u} \overline{\nabla\varphi} dxdy \\ &= \ell(\nabla\varphi) \\ &= 0. \end{aligned}$$

Since $C^2(\bar{\Omega})$ is dense in $\boldsymbol{H}^1(\Omega)$, we have $\int_\Omega \omega^2 \mu \vec{u} \overline{\nabla\varphi} dxdy = 0$ for any $\varphi \in H^1(\Omega)$, thus $\vec{u} \in \boldsymbol{V}_T(\mu; \Omega)$.

To prove that (3.9) implies (3.6), we use the fact that (3.8) admits a unique solution. We let φ be that unique solution of (3.8), which corresponds to $\vec{v} \in \boldsymbol{H}(\text{curl}; \Omega)$. From (3.9), we know that $a(\vec{u}, \vec{v}') = \ell(\vec{v}')$ for any $\vec{v}' \in \boldsymbol{V}_T(\mu; \Omega)$. We consider the case $\vec{v}' = \vec{v} - \nabla\varphi$ with $\int_\Omega \varphi dxdy = 0$. We can prove that $\vec{v}' \in \boldsymbol{V}_T(\mu; \Omega)$ by substituting $\vec{v}'$ into (3.8) and derive the condition of $\boldsymbol{V}_T(\mu; \Omega)$ as $\nabla\varphi$ is equivalent in $H^1_\#(\Omega)$ and $H^1(\Omega)$. Then we obtain that

$$a(\vec{u}, \vec{v} - \nabla\varphi) = \ell(\vec{v} - \nabla\varphi), \quad \forall \vec{v} \in \boldsymbol{H}(\text{curl}; \Omega).$$

We know that $a(\vec{u}, \vec{v}) = \ell(\vec{v})$ and $a(\vec{u}, \nabla\varphi) = \ell(\nabla\varphi) = 0$, so we can conclude that (3.9) holds for any $\vec{v} \in \boldsymbol{H}(\text{curl}; \Omega)$. □

Well-Posedness Characteristics

After arriving to the variational formulation (3.9), the weak solution of the Maxwell problem (3.5), we can analyze the existence and uniqueness of the solution $\vec{u}$. We do so under the following cases:

1. Find $\vec{u} \in \boldsymbol{V}_T(\mu; \Omega)$ for $\omega \in \mathbb{C}\backslash\mathbb{R}$.
2. Find $\vec{u} \in \boldsymbol{V}_T(\mu; \Omega)$ for $\omega \in \mathbb{R}$,
   - *i* when P_0 has a unique solution.
   - *ii* when P_0 has a non-zero solution.

In the above, P_0 denotes the problem (3.9) for $\omega_0 \in \mathbb{R}$ and $\ell(\cdot) = 0$:

$$\begin{cases} \text{Find } \vec{u} \in \boldsymbol{V}_T(\mu; \Omega) \text{ such that} \\ a(\vec{u}, \vec{v}) = \ell(\vec{v}) = 0, \quad \forall \vec{v} \in \boldsymbol{V}_T(\mu; \Omega). \end{cases} \tag{3.10}$$

We begin by considering the first case and introducing its corresponding result.

Theorem 3.11. *Problem (3.9) admits a unique solution for $\omega \in \mathbb{C}\backslash\mathbb{R}$.*

Proof. We observe that ω^2 can be written in the form of $\omega^2 = -\rho e^{i\theta}$ where $\theta \in (-\pi, \pi)$ and $\rho > 0$. From the complex Lax-Milgram theorem, (3.9) must be continuous and coercive. Using Cauchy-Schwarz inequality:

(1) For $\vec{u}, \vec{v} \in \boldsymbol{V}_T(\mu; \Omega)$, there exists $M > 0$ such that:

$$\begin{aligned} |a(\vec{u}, \vec{v})| &\leq \beta(\|\,\text{curl}\, \vec{u}\|_{L^2(\Omega)} \|\,\text{curl}\, \vec{v}\|_{L^2(\Omega)} - \|\omega^2\|_{L^\infty(\Omega)} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} \|\vec{v}\|_{\boldsymbol{L}^2(\Omega)}) \\ &\leq M(\|\,\text{curl}\, \vec{u}\|_{H^1(\Omega)} \|\,\text{curl}\, \vec{v}\|_{H^1(\Omega)} - \|\vec{u}\|_{\boldsymbol{H}^1(\Omega)} \|\vec{v}\|_{\boldsymbol{H}^1(\Omega)}). \end{aligned}$$

(2) For $\vec{v} \in \boldsymbol{V}_T(\mu; \Omega)$, there exists $C > 0$ such that:

$$|\ell(\vec{v})| \leq \beta \|J\|_{L^2(\Omega)} \|\,\text{curl}\, \vec{v}\|_{L^2(\Omega)} \leq C \|\,\text{curl}\, \vec{v}\|_{H^1(\Omega)}.$$

(3) For $\vec{u} \in \boldsymbol{V}_T(\mu; \Omega)$, there exists $\eta > 0$ such that:

$$\Re e(e^{i\theta} a(\vec{u}, \vec{u})) \geq \eta \|\vec{u}\|^2_{\boldsymbol{H}(\text{curl};\Omega)}.$$

To prove the coercivity, we can deduce that:

$$\Re e(e^{-i\frac{\theta}{2}}a(\vec{u},\vec{u})) \geq \Re e(e^{-i\frac{\theta}{2}}\int_\Omega \varepsilon^{-1}|\operatorname{curl}\vec{u}|^2 dxdy) + \Re e(e^{i\frac{\theta}{2}}\rho\int_\Omega \mu|\vec{u}|^2 dxdy)$$
$$\geq \alpha\cos(\frac{\theta}{2})\min(1,\rho)||\vec{u}||^2_{\boldsymbol{H}(\mathrm{curl};\Omega)}.$$

Therefore, η exists when $\eta = \alpha\cos(\frac{\theta}{2})\min(1,\rho)$. Then, by the complex Lax-Milgram theorem, we can deduce that (3.9) admits a unique solution. □

Subsequently, we consider the two sub-cases of the second case and introduce their results.

Theorem 3.12. *Suppose that P_0 admits a unique solution of $\vec{u} = 0$, then (3.9) has a solution for $\omega = \omega_0 \in \mathbb{R}$.*

Proof. From Theorem 3.11, we denote $\vec{u}_\delta$ the unique solution of (3.9) for $\omega_\delta = \omega_0 + i\delta$ and $\delta > 0$. We wish to prove the existence of a constant $C > 0$ independent of δ such that

$$\left\|\vec{u}_\delta\right\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} \leq C, \quad \forall\delta \in \left]0,1\right]. \tag{3.11}$$

To do this, we will reason by the absurd by assuming that there is a sequence $\{\delta_n\}$ such that $\lim_{n\to+\infty}\delta_n = 0$ and $\left\|\vec{u}_{\delta_n}\right\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} > n$. We define $\vec{w}_n = \vec{u}_{\delta_n}/||\vec{u}_{\delta_n}||_{\boldsymbol{H}(\mathrm{curl};\Omega)}$.

Here $\{\vec{u}_{\delta_n}\}$ is a sequence of the solution to (3.9) with $\omega_{\delta_n} = \omega_0 + i\delta$. Substituting $\{\vec{u}_{\delta_n}\}$ and $\{\omega_{\delta_n}\}$ into (3.9), we have

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{u}_{\delta_n}\overline{\operatorname{curl}\vec{v}} - \omega^2_{\delta_n}\mu\vec{u}_{\delta_n}\bar{\vec{v}})dxdy = \int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{v}}dxdy. \tag{3.12}$$

Observing that $\vec{u}_{\delta_n} = \vec{w}_n \cdot ||\vec{u}_{\delta_n}||_{\boldsymbol{H}(\mathrm{curl};\Omega)}$, we deduce that

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{w}_n\overline{\operatorname{curl}\vec{v}} - \omega^2_{\delta_n}\mu\vec{w}_n\bar{\vec{v}})dxdy = ||\vec{u}_{\delta_n}||^{-1}_{\boldsymbol{H}(\mathrm{curl};\Omega)}\int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{v}}dxdy. \tag{3.13}$$

As $n \to \infty$, we have $\omega_{\delta_n} \to \omega_0$ and $||\vec{u}_{\delta_n}||^{-1}_{\boldsymbol{H}(\mathrm{curl};\Omega)} \to 0$, we obtain the following equation:

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{w}_n\overline{\operatorname{curl}\vec{v}} - \omega^2_0\mu\vec{w}_n\bar{\vec{v}})dxdy = 0. \tag{3.14}$$

In addition, we notice that $\{\vec{w}_n\}$ is bounded as $||\vec{w}_n||_{\boldsymbol{H}(\mathrm{curl};\Omega)} = \frac{||\vec{u}_{\delta_n}||_{\boldsymbol{H}(\mathrm{curl};\Omega)}}{||\vec{u}_{\delta_n}||_{\boldsymbol{H}(\mathrm{curl};\Omega)}} = 1$. From Lemma 3.7, we can find a sub-sequence of $\{\vec{w}_n\}$ that is convergent in $\boldsymbol{L}^2(\Omega)$. Similarly, from Lemma 2.3, we can find another sub-sequence which is weakly convergent in $\boldsymbol{V}_T(\mu;\Omega)$. Because they are both sub-sequences of $\{\vec{w}_n\}$, we denote them as $\{\vec{w}_n\}$ for simplicity, where we have

$$\vec{w}_n \overset{n\to\infty}{\to} \vec{w} \text{ in } \boldsymbol{L}^2(\Omega) \text{ and}$$

$$\int_\Omega(\vec{w}_n\bar{\vec{v}} + \operatorname{curl}\vec{w}_n\overline{\operatorname{curl}\vec{v}})dxdy \overset{n\to\infty}{\to} \int_\Omega(\vec{\omega}\bar{\vec{v}} + \operatorname{curl}\vec{\omega}\overline{\operatorname{curl}\vec{v}})dxdy.$$

We use Cauchy-Schwarz inequality to prove that $\int_\Omega \vec{w}_n\bar{\vec{v}}dxdy \overset{n\to\infty}{\to} \int_\Omega \vec{w}\bar{\vec{v}}dxdy$, and therefore,

$$\int_\Omega \operatorname{curl}\vec{w}_n\overline{\operatorname{curl}\vec{v}}dxdy \overset{n\to\infty}{\to} \int_\Omega \operatorname{curl}\vec{w}\overline{\operatorname{curl}\vec{v}}dxdy. \tag{3.15}$$

By substituting back to (3.14), we deduce that

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{w}\overline{\operatorname{curl}\vec{v}} - \omega^2_0\mu\vec{w}\bar{\vec{v}})dxdy = 0. \tag{3.16}$$

Since $\vec{w}$ satisfies $\vec{u}$ in P_0 and owing to the uniqueness of a limit, then $\vec{\omega}_n \to 0$ in $\boldsymbol{L}^2(\Omega)$, a convergence that will later be used to show the contradiction. Moreover, we substitute $\vec{v} = \vec{w}_n$ in (3.13) to deduce that

$$\int_\Omega(\varepsilon^{-1}|\operatorname{curl}\vec{w}_n|^2 - \vec{\omega}^2_{\delta_n}\mu|\vec{w}_n|^2)dxdy = ||\vec{u}_{\delta_n}||^{-1}_{\boldsymbol{H}(\mathrm{curl};\Omega)}\int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{w}_n}dxdy.$$

Adding $(1+\vec{\omega}^2_{\delta_n})\int_\Omega \mu|\vec{w}_n|^2 dxdy$ to both sides of the equation, we have

$$\int_\Omega (\varepsilon^{-1}|\operatorname{curl}\vec{w}_n|^2+\mu|\vec{w}_n|^2)dxdy = (1+\vec{\omega}^2_{\delta_n})\int_\Omega \mu|\vec{w}_n|^2 dxdy + \|\vec{u}_{\delta_n}\|^{-1}_{\boldsymbol{H}(\mathrm{curl};\Omega)}\int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{w}_n}dxdy.$$

Then we establish the following inequalities

$$\begin{aligned}\|w_n\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} &\le \alpha^{-1}\int_\Omega (\varepsilon^{-1}|\operatorname{curl}\vec{w}_n|^2+\mu|\vec{w}_n|^2)dxdy\\ &= \alpha^{-1}(1+\vec{\omega_{\delta_n}}^2)\int_\Omega \mu|\vec{w}_n|^2dxdy + \alpha^{-1}\|\vec{u}_{\delta_n}\|^{-1}_{\boldsymbol{H}(\mathrm{curl};\Omega)}\int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{w}_n}dxdy\\ &\le C\|\vec{w}_n\|^2_{\boldsymbol{L}^2(\Omega)} + \alpha^{-1}\|\vec{u}_{\delta_n}\|^{-1}_{\boldsymbol{H}(\mathrm{curl};\Omega)}\int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{w}_n}dxdy. \end{aligned} \tag{3.17}$$

Because $\|\vec{w}_n\|^2_{\boldsymbol{L}^2(\Omega)}\to 0$ and $\frac{1}{\|\vec{u}_{\delta_n}\|_{\boldsymbol{H}(\mathrm{curl};\Omega)}}\to 0$, then we can deduce that $\|\vec{w}_n\|_{\boldsymbol{H}(\mathrm{curl};\Omega)}\to 0$, which contradicts with $\|\vec{w}_n\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} = 1$.

In conclusion, as (3.11) shows that $\|\vec{u}_\delta\|_{\boldsymbol{H}(\mathrm{curl};\Omega)}$ is bounded, we can extract a sub-sequence $\vec{u}_{\delta_n} \overset{n\to\infty}{\rightharpoonup} \vec{u}$ in $\boldsymbol{V}_T(\mu;\Omega)$ with $\delta_n \overset{n\to\infty}{\to} 0$. Since $\vec{u}_{\delta_n} \overset{n\to\infty}{\to} \vec{u}$ in $\boldsymbol{L}^2(\Omega)$ and $\vec{u}_{\delta_n} \overset{n\to\infty}{\rightharpoonup} \vec{u}$ in $\boldsymbol{V}_T(\mu;\Omega)$, we use the same deduction as (3.15) and we have

$$\int_\Omega \operatorname{curl}\vec{u}_{\delta_n}\overline{\operatorname{curl}\vec{v}}dxdy \overset{n\to\infty}{\to} \int_\Omega \operatorname{curl}\vec{u}\overline{\operatorname{curl}\vec{v}}dxdy.$$

Therefore, we can simply show that $\vec{u}$ is a solution of (3.9) with $\omega=\omega_0\in\mathbb{R}$ since

$$\int_\Omega(\operatorname{curl}\vec{u}_{\delta_n}\overline{\operatorname{curl}\vec{v}} - \omega^2_{\delta_n}\mu\vec{u}_{\delta_n}\bar{\vec{v}})dxdy \overset{n\to\infty}{\to} \int_\Omega(\operatorname{curl}\vec{u}\overline{\operatorname{curl}\vec{v}} - \omega_0^2\mu\vec{u}\bar{\vec{v}})dxdy = \ell(\vec{v}). \tag{3.18}$$

□

Remark 3.13. *To discuss the supplement of Theorem 3.12, which is the existence of a solution for (3.9) when $\omega\in\mathbb{R}$ and P_0 has a non-zero solution of $\vec{u}_0$, we consider it in two ways: when $\ell(\vec{u}_0)\neq 0$ and when $\ell(\vec{u}_0)=0$.*

Proposition 3.14. *Suppose that there exists a non-zero solution $\vec{u}_0$ for P_0, then (3.9) does not have a solution when $\omega\in\mathbb{R}$ and $\ell(\vec{u}_0)\neq 0$.*

Proof. For $\delta>0$, we again let $\vec{u}_\delta$ denote the solution of (3.9) for $\omega_\delta=\omega_0+i\delta$. We hope to prove that $\lim_{\delta\to 0}\left\|\vec{u}_\delta\right\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} = +\infty$. To do so, we prove by contradiction by assuming that $\lim_{\delta\to 0}\left\|\vec{u}_{\delta_n}\right\|_{\boldsymbol{H}(\mathrm{curl};\Omega)}\le C$. Then from Lemma 2.3 and 3.7, there exists a limit $\vec{u}\in\boldsymbol{L}^2(\Omega)$ such that $\vec{u}_{\delta_n}\rightharpoonup\vec{u}$ in $\boldsymbol{V}_T(\mu;\Omega)$ and $\vec{u}_{\delta_n}\to\vec{u}$ in $\boldsymbol{L}^2(\Omega)$. Now, substituting $\{\vec{u}_{\delta_n}\}$ and $\{\omega_{\delta_n}\}$ into (3.9) and observing when limit $n\to\infty$, we have

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{u}\overline{\operatorname{curl}\vec{v}} - \omega_0^2\mu\vec{u}\bar{\vec{v}})dxdy = \int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{v}}dxdy, \quad \forall\vec{v}\in\boldsymbol{V}_T(\mu;\Omega).$$

Since $\vec{u}_0$ is a solution of P_0 and $\vec{u}_0\in\boldsymbol{V}_T(\mu;\Omega)$, we substitute $\vec{u}_0$ in place of $\vec{v}$. Now $a(\vec{u},\vec{u}_0)=\ell(u_0)=0$, which contradicts with the condition of $\ell(u_0)\neq 0$. Therefore, $\vec{u}_\delta\to+\infty$ and cannot correspond to certain solution $\vec{u}$ with $\omega=\omega_0$. □

Theorem 3.15. *Suppose that the set of solutions of P_0 coincides with* $\mathrm{vect}(\vec{u}_0)$*, where $\vec{u}_0\not\equiv 0$, then (3.9) has a solution when $\omega=\omega_0\in\mathbb{R}$ and $\ell(\vec{u}_0)=0$.*

Proof. For $\delta>0$, we still denote $\vec{u}_\delta$ the unique solution of (3.9) for $\omega_\delta=\omega_0+i\delta$, which satisfies

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{u}_\delta\overline{\operatorname{curl}\vec{v}} - \omega_\delta^2\mu\vec{u}_\delta\bar{\vec{v}})dxdy = \int_\Omega \varepsilon^{-1}J\overline{\operatorname{curl}\vec{v}}dxdy, \quad \forall\vec{v}\in\boldsymbol{V}_T(\mu;\Omega).$$

Observing that $\ell(\vec{u}_0)=0$ and $\vec{u}_0\in\boldsymbol{V}_T(\mu;\Omega)$, we substitute $\vec{v}=\vec{u}_0$ (equivalent to $\mathrm{vect}(\vec{u}_0)$ for linear equations) into the above

$$\int_\Omega(\varepsilon^{-1}\operatorname{curl}\vec{u}_\delta\overline{\operatorname{curl}(\vec{u}_0)} - \omega_\delta^2\mu\vec{u}_\delta\vec{u}_0)dxdy = 0. \tag{3.19}$$

As a solution of P_0, $\vec{u}_0$ also satisfies

$$\int_\Omega (\varepsilon^{-1}\operatorname{curl}\vec{u}_0\overline{\operatorname{curl}\vec{v}} - \omega_0^2\mu\vec{u}_0\bar{\vec{v}})dxdy = 0, \quad \forall\vec{v}\in \boldsymbol{V}_T(\mu;\Omega).$$

Now, as $\vec{u}_\delta \in \boldsymbol{V}_T(\mu;\Omega)$, we substitute $\vec{v} = \vec{u}_\delta$ into the equation

$$\int_\Omega (\varepsilon^{-1}\operatorname{curl}\vec{u}_0\overline{\operatorname{curl}\vec{u}_\delta} - \omega_0^2\mu\vec{u}_0\vec{u}_\delta)dxdy = 0. \tag{3.20}$$

Subtracting (3.19) from (3.20), we have

$$\int_\Omega (\omega_\delta^2 - \omega_0^2)\mu\vec{u}_\delta\vec{u}_0 dxdy = 0.$$

Since $\omega_\delta^2 - \omega_0^2 \neq 0$, we can deduce that $\int_\Omega \mu\vec{u}_\delta\vec{u}_0 dxdy = 0$. Moreover, we use the same methodology (proof by contradiction) in Theorem 3.12 to show that

$$\left\|\vec{u}_\delta\right\|_{\boldsymbol{H}(\operatorname{curl};\Omega)} \le C, \quad \forall\delta\in\left]0,1\right]$$

with the same definition of $\{\vec{w}_n\}$. As $\{\vec{w}_n\}$ is bounded, we can extract a sub-sequence which strongly converges in $\boldsymbol{L}^2(\Omega)$ and a sub-sequence which weakly converges in $\boldsymbol{V}_T(\mu;\Omega)$. We follow the same deduction process as (3.16) to show that $\vec{w}$ satisfies P_0; thereby $\vec{w} = \operatorname{vect}(\vec{u}_0) = ku_0$ where $k\in\mathbb{C}$. Previously, we established that $\int_\Omega \mu\vec{u}_\delta\vec{u}_0 dxdy = 0$, so we let $\vec{w}_n = \vec{u}_\delta$. As limit $n\to\infty$ we have

$$\int_\Omega \mu\vec{w}\vec{u}_0 dxdy = \int_\Omega \mu k\cdot|\vec{u}_0|^2 dxdy = 0.$$

Since $\vec{u}_0 \not\equiv 0$, we deduce that $k = 0$. Thus, $\vec{w} = 0$ and $\vec{w}_n \to 0$ in $\boldsymbol{L}^2(\Omega)$. Same as Theorem 3.12, we can deduce that

$$\|w_n\|_{\boldsymbol{H}(\operatorname{curl};\Omega)} \to 0,$$

which contradicts with $\|w_n\|_{\boldsymbol{H}(\operatorname{curl};\Omega)} = 1$. So we have proved u_δ is bounded in $\mathbf{V}_T(\mu;\Omega)$ when $\delta\to 0$.

Since $\|\vec{u}_\delta\|_{\boldsymbol{H}(\operatorname{curl};\Omega)} \le C$, we have $\vec{u}_{\delta_n} \to \vec{u}$ in $\boldsymbol{L}^2(\Omega)$ and $\vec{u}_{\delta_n} \rightharpoonup \vec{u}$ in $\boldsymbol{V}_T(\mu;\Omega)$. Using the same deduction as (3.18), we can show that $\vec{u}$ is a solution of (3.9) with $\omega = \omega_0 \in \mathbb{R}$. □

Numerical Approximation

Now, we assume that Ω is a polygon and accept that the previous results remain valid in such a geometry. To study the case $\omega\in\mathbb{R}$, we would like to work on the formulation (3.9) posed in $\boldsymbol{V}_T(\mu;\Omega)$ so that we can use the fact that the injection of $\boldsymbol{V}_T(\mu;\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is compact. However, it is hard to discretize (3.9) because $\boldsymbol{V}_T(\mu;\Omega)$ contains the integral $\int_\Omega \mu\vec{v}\nabla\varphi dxdy = 0$, which cannot be described by an easily-chosen finite basis $\phi_j(x)$. Therefore, we need to apply the finite element method in a new space that is possible to interpret geometrically.

Let us define the Hilbert space Y

$$\mathrm{Y} := \left\{\vec{v}\in \boldsymbol{H}(\operatorname{curl};\Omega) \mid \operatorname{div}(\mu\vec{v}) \in \boldsymbol{L}^2(\Omega), \mu\vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega\right\}$$

with the inner product $(\vec{u},\vec{v})_\mathrm{Y} = (\vec{u},\vec{v})_{\boldsymbol{H}(\operatorname{curl};\Omega)} + \int_\Omega \operatorname{div}(\mu\vec{u})\overline{\operatorname{div}(\mu\vec{v})}dxdy$ and the norm $\|\vec{v}\|_\mathrm{Y} = (\vec{v},\vec{v})_\mathrm{Y}^{1/2}$. To deduce an equivalent variational formulation posed in Y, we can easily prove that $\vec{v}\in\boldsymbol{H}(\operatorname{curl};\Omega)$ belongs to Y if and only if there exists $\omega\in L^2(\Omega)$ such that

$$\int_\Omega \mu\vec{v}\cdot\nabla\varphi dxdy = \int_\Omega w\varphi dxdy, \quad \forall\varphi\in C_c^\infty(\bar{\Omega}). \tag{3.21}$$

We consider, for $\lambda > 0$, the variational formulation

$$\begin{aligned} &\text{Find } \vec{u}\in \mathrm{Y} \text{ such that} \\ &\int_\Omega (\varepsilon^{-1}\operatorname{curl}\vec{u}\overline{\operatorname{curl}\vec{v}} + \lambda\operatorname{div}(\mu\vec{u})\overline{\operatorname{div}(\mu\vec{v})} - \omega^2\mu\vec{u}\cdot\bar{\vec{v}})dxdy = \ell(\vec{v}), \quad \forall\vec{v}\in\mathrm{Y}. \end{aligned} \tag{3.22}$$

Proposition 3.16. *If $\vec{u}$ verifies (3.9), then $\vec{u}$ is a solution of (3.22).*

Proof. If $\vec{u}$ verifies (3.9), we have $\vec{u} \in \boldsymbol{V}_T(\mu;\Omega)$ and thus $\int_\Omega \mu\vec{u}\cdot\nabla\varphi dxdy = 0$ for all $\varphi \in C_c^\infty(\bar{\Omega})$ (because $C_c^\infty(\bar{\Omega})$ belongs to $H^1(\Omega)$). This satisfies (3.21) when we take $\omega = 0$, so $\vec{u} \in \mathrm{Y}$. From Definition 2.7, by taking $\varphi \in C_c^\infty(\Omega)$, we have $\mathrm{div}(\mu\vec{v}) = 0$, indicating that $\vec{u}$ is the solution of (3.22). □

Conversely, to establish that (3.22) implies (3.9), we use the conclusion in (Ciarlet, 2020) that the problem

$$\left|\left\{\begin{array}{l}\text{Find } \varphi \in H^1(\Omega) \text{ such that}\\ \mathrm{div}(\mu\nabla\varphi) + k\varphi = f \quad \text{in } \Omega\\ \mu\nabla\varphi\cdot\vec{n} = 0 \quad \text{on } \partial\Omega.\end{array}\right.\right| \tag{3.23}$$

admits a unique solution for any $f \in L^2(\Omega)$ if and only if $k \notin \{k_0, k_1, \ldots\}$, where (k_n) is a sequence consist of increasing positive real numbers such that $\lim_{n\to+\infty} k_n = +\infty$.

Proposition 3.17. *If $\vec{u}$ verifies (3.22), then $\vec{u}$ is a solution of (3.9) when $\frac{\omega^2}{\lambda} \neq k_n$ for all $n \in \mathbb{N}$.*

Proof. Observing that when $f = \frac{\mathrm{div}(\mu\vec{u})}{\lambda}$ and $k = \frac{\omega^2}{\lambda}$, we have

$$\int_\Omega (\lambda\overline{\mathrm{div}(\mu\nabla\varphi)} + \omega^2\varphi)dxdy = \int_\Omega \mathrm{div}(\mu\vec{u})dxdy,$$

where φ is the unique solution of (3.23). By substituting $\vec{v} = \nabla\varphi$ in (3.22), we have

$$\int_\Omega (\varepsilon^{-1}\,\mathrm{curl}\,\vec{u}\overline{\mathrm{curl}\,\nabla\varphi} + \lambda\,\mathrm{div}(\mu\vec{u})\overline{\mathrm{div}(\mu\nabla\varphi)} - \omega^2\mu\vec{u}\cdot\overline{\nabla\varphi})dxdy = \int_\Omega \varepsilon^{-1}J\overline{\mathrm{curl}\,\nabla\varphi}dxdy.$$

Observing that $\mathrm{curl}(\nabla\varphi) = 0$, the equation becomes

$$\int_\Omega (\lambda\,\mathrm{div}(\mu\vec{u})\overline{\mathrm{div}(\mu\nabla\varphi)} - \omega^2\mu\vec{u}\overline{\nabla\varphi})dxdy = 0.$$

From these equations, we can deduce that the norm of $\mathrm{div}(\mu\vec{u})$ in $\boldsymbol{L}^2(\Omega)$ becomes

$$\begin{aligned}\|\,\mathrm{div}(\mu\vec{u})\|^2_{L^2(\Omega)} &= \int_\Omega \left|\mathrm{div}\,\mu\vec{u}\right|^2 dxdy\\ &= \int_\Omega \mathrm{div}(\mu\vec{u})\cdot(\lambda\overline{\mathrm{div}(\mu\nabla\varphi)} + \omega^2\overline{\varphi})dxdy\\ &= \int_\Omega (\lambda\,\mathrm{div}(\mu\vec{u})\overline{\mathrm{div}(\mu\nabla\varphi)} + \mathrm{div}(\mu\vec{u})\omega^2\overline{\varphi})dxdy\\ &= \int_\Omega (\omega^2\mu\vec{u}\overline{\nabla\varphi} + \omega^2\,\mathrm{div}(\mu\vec{u})\overline{\varphi})dxdy.\end{aligned}$$

Based on the weak divergence in (3.21), we deduce that $\|\,\mathrm{div}(\mu\vec{u})\|_{L^2(\Omega)} = 0$ when $k_n \neq \frac{\omega^2}{\lambda}$. Furthermore, to show that $\vec{u}$ in (3.22) satisfies $\vec{u} \in \boldsymbol{V}_T(\mu;\Omega)$, we use Green's formula (2.5), and we have

$$\int_\Omega \mu\vec{u}\nabla\varphi dxdy = \int_{\partial\Omega} \varphi\mu\vec{u}\cdot\vec{n}ds - \int_\Omega \varphi\,\mathrm{div}(\mu\vec{u})dxdy.$$

Observing that $\int_{\partial\Omega}\varphi\mu\vec{u}\cdot\vec{n}ds = \int_\Omega \varphi\,\mathrm{div}(\mu\vec{u})dxdy = 0$, we obtain that $\int_\Omega \mu\vec{u}\nabla\varphi dxdy = 0$, so we proved $\vec{u}$ is a solution of (3.9). □

Let Y_h be a finite dimensional space such that $\mathrm{Y}_h \subset \mathrm{Y}$. We are interested in approximating the solution of (3.22) in Y_h, but the conditions on $\partial\Omega$ are abstractedly imposed without trying to make explicit of the functions that satisfy it.

For example, we take

$$\mathcal{Y}_h = \left\{\varphi \in C(\bar{\Omega}) | \varphi|_{K_i} \in \mathbb{P}_k(K_i), \forall K_i \in \mathcal{T}_h\right\},$$

where $\mathcal{T}_h$ is a trangulation of Ω and $\mathbb{P}_k(K_i)$ denotes the set of polynomials of degree at most k on the triangle K_i.

We define $\mathrm{Y}_h := \{\vec{v} \in \mathcal{V}_h \times \mathcal{V}_h \mid \mu\vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega\}$ so we can easily see that $\mathrm{Y}_h \subset \mathrm{Y}$. Then we consider the following discretized problem which will be proven as well-posed.

Proposition 3.18. *The discretized problem*

$$\begin{cases} \text{Find } \vec{u_h} \in \mathrm{Y}_h \text{ such that for any } \vec{v}_h \in \mathrm{Y}_h, \\ \int_\Omega (\varepsilon^{-1} \operatorname{curl} \vec{u}_h \overline{\operatorname{curl} \vec{v}_h} + \lambda \operatorname{div}(\mu \vec{u}_h) \overline{\operatorname{div}(\mu \vec{v}_h)} - \omega^2 \mu \vec{u}_h \cdot \overline{\vec{v}_h}) dxdy = \ell(\vec{v}_h), \end{cases} \tag{3.24}$$

where $\omega = i\kappa$ with $\kappa \in \mathbb{R}\backslash\{0\}$ for the simplicity of the following, admits a unique solution.

Proof. We verify the condition of the complex Lax-Milgram theorem using Cauchy-Schwarz inequality.

(1) For $\vec{u}_h, \vec{v}_h \in \mathrm{Y}_h$, there exists $M > 0$ such that:

$$\begin{aligned} |a(\vec{u}_h, \vec{v}_h)| &\le \beta(\|\operatorname{curl} \vec{u}_h\|_{L^2(\Omega)} \|\operatorname{curl} \vec{v}_h\|_{L^2(\Omega)} + \|\omega^2\|_{L^\infty(\Omega)} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} \|\vec{v}\|_{\boldsymbol{L}^2(\Omega)}) + \lambda \|\operatorname{div}(\mu \vec{u}_h)\|_{L^2(\Omega)} \|\operatorname{div}(\mu \vec{v}_h)\|_{L^2(\Omega)} \\ &\le M(\|\operatorname{curl} \vec{u}_h\|_{\mathrm{Y}_h} \|\operatorname{curl} \vec{v}_h\|_{\mathrm{Y}_h} + \|\operatorname{div}(\mu \vec{u}_h)\|_{\mathrm{Y}_h} \|\operatorname{div}(\mu \vec{v}_h)\|_{\mathrm{Y}_h} + \|\vec{u}_h\|_{\mathrm{Y}_h} \|\vec{v}_h\|_{\mathrm{Y}_h}). \end{aligned}$$

(2) For $\vec{v}_h \in \mathrm{Y}_h$, there exists $C > 0$ such that:

$$|\ell(\vec{v}_h)| \le \beta \|J\|_{L^2(\Omega)} \|\operatorname{curl} \vec{v}_h\|_{L^2(\Omega)} \le C \|\operatorname{curl} \vec{v}_h\|_{\mathrm{Y}_h}.$$

(3) For $\vec{u}_h \in \mathrm{Y}_h$, there exists $\eta > 0$ such that when $\theta = 0$,

$$\begin{aligned} \Re e(e^{i\theta} a(\vec{u}_h, \vec{u}_h)) &= \Re e(\int_\Omega \varepsilon^{-1} \operatorname{curl}(\vec{u}_h)^2 + \lambda \operatorname{div}(\mu \vec{u}_h)^2 + \kappa^2 \mu \vec{u}_h^2) \\ &\ge \min(\varepsilon^{-1}, \lambda, \kappa^2 \mu) \|\vec{u}_h\|^2_{\mathrm{Y}_h} \\ &= \eta \|\vec{u}_h\|^2_{\mathrm{Y}_h}. \end{aligned}$$

So we proved the well-posedness of this discretized problem. □

Remark 3.19. *Throughout this paper, we do not actually compute a numerical solution to (3.24), but we can still deduce an expected estimate of error for it.*

Definition 3.20. *To approximate the variational formulation via the form of polynomials, we define an operator $r_h : \mathrm{Y} \to \boldsymbol{V}_h$*

$$r_h \vec{v} = \textstyle\sum_j \vec{v}_j \phi_j(x), \quad \forall \vec{v} \in \mathrm{Y}.$$

where $\phi_j(x)$ is a finite basis of $\boldsymbol{V}_h$.

Lemma 3.21. ***(Cea's Lemma)*** *Let $\boldsymbol{V}$ be a Hilbert space and $\boldsymbol{V}_h$ be a finite dimensional space in $\boldsymbol{V}$. If $\vec{u} \in \boldsymbol{V}$ and $\vec{u}_h \in \boldsymbol{V}_h$, then*

$$\|\vec{u} - \vec{u}_h\|_{\boldsymbol{V}} \le C \inf_{\vec{v}_h \in \boldsymbol{V}_h} \|\vec{u} - \vec{v}_h\|_{\boldsymbol{V}}.$$

Proof. We first construct two variational formulations with $\vec{u}$ and $\vec{u}_h$:

$$\begin{aligned} a(\vec{u}_h, \vec{\omega}_h) &= \ell(\vec{\omega}_h), \quad \forall \vec{\omega}_h \in \boldsymbol{V}_h \\ a(\vec{u}, \vec{\omega}) &= \ell(\vec{\omega}), \quad \forall \vec{\omega} \in \boldsymbol{V}. \end{aligned}$$

Observing that $\boldsymbol{V}_h \subset \boldsymbol{V}$, we can substitute $\vec{\omega}$ with $\vec{\omega}_h$. By subtracting the two equations, we have

$$|a(\vec{u} - \vec{u}_h, \vec{\omega}_h)| = 0.$$

When $\vec{\omega}_h = \vec{u}_h - \vec{v}_h$, we can further establish that $|a(\vec{u} - \vec{u}_h, \vec{u}_h - \vec{v}_h)| = 0$. By the continuity and coercivity of $a(\vec{u}, \vec{v})$, there exists $M > 0$ and $\alpha > 0$ such that

$$|a(\vec{u}, \vec{v})| \le M \|\vec{u}\|_{\boldsymbol{V}} \|\vec{v}\|_{\boldsymbol{V}} \text{ and } a(\vec{u}, \vec{u}) \ge \alpha \|\vec{u}\|^2_{\boldsymbol{V}}.$$

Substituting $\vec{u} = \vec{u} - \vec{u}_h$ into the coercivity equation, we have

$$a(\vec{u}-\vec{u}_h, \vec{u}-\vec{u}_h) \geq \alpha\|\vec{u}-\vec{u}_h\|_{\boldsymbol{V}}^2.$$

Since $|a(\vec{u}-\vec{u}_h, \vec{u}_h-\vec{v}_h)| = 0$, we deduce that

$$\begin{aligned}|a(\vec{u}-\vec{u}_h, \vec{u}-\vec{v}_h)| &= |a(\vec{u}-\vec{u}_h, \vec{u}-\vec{u}_h)| + |a(\vec{u}-\vec{u}_h, \vec{u}_h-\vec{v}_h)| \\ &\geq \alpha\|\vec{u}-\vec{u}_h\|_{\boldsymbol{V}}^2.\end{aligned}$$

Given $|a(\vec{u}-\vec{u}_h, \vec{u}-\vec{v}_h)| \leq M\|\vec{u}-\vec{u}_h\|_{\boldsymbol{V}}\|\vec{u}-\vec{v}_h\|_{\boldsymbol{V}}$, we have

$$\alpha\|\vec{u}-\vec{u}_h\|_{\boldsymbol{V}}^2 \leq M\|\vec{u}-\vec{u}_h\|_{\boldsymbol{V}}\|\vec{u}-\vec{v}_h\|_{\boldsymbol{V}}.$$

Reducing the like terms, we obtain that

$$\|\vec{u}-\vec{u}_h\|_{\boldsymbol{V}} \leq \frac{M}{\alpha}\|\vec{u}-\vec{v}_h\|_{\boldsymbol{V}}, \quad \forall \vec{v}_h \in \boldsymbol{V}_h.$$

□

Theorem 3.22. *For $\vec{u} \in H^2(\Omega) \times H^2(\Omega)$, the estimate of error, which we denote as h, is first-order and we have*

$$\|\vec{u}-\vec{u}_h\|_{\mathrm{Y}} \leq Ch\|\vec{u}\|_{\boldsymbol{H}^2(\Omega)}.$$

Furthermore, we can generalize this inequality to when $\vec{u} \in H^{k+1}(\Omega) \times H^{k+1}(\Omega)$

$$\|\vec{u}-\vec{u}_h\|_{\mathrm{Y}} \leq Ch^k\|\vec{u}\|_{\boldsymbol{H}^{k+1}(\Omega)},$$

where $k > 0$ and $k+1 > \frac{N}{2}$.

Proof. By Cea's inequality 3.21, we deduce that

$$\begin{aligned}\|\vec{u}-\vec{u}_h\|_{\mathrm{Y}} &\leq C \inf_{\vec{v}_h \in \mathrm{Y}_h} \|\vec{u}-\vec{v}_h\|_{\mathrm{Y}} \\ &\leq C\|\vec{u}-r_h\vec{u}\|_{\mathrm{Y}} \\ &\leq Ch\|\vec{u}\|_{\mathrm{Y}}, \quad \forall\, \vec{u} \in H^2(\Omega) \times H^2(\Omega),\end{aligned}$$

where the last inequality comes from (Ciarlet, 2014). Similarly from (Ciarlet, 2014), when $\vec{u} \in H^{k+1}(\Omega) \times H^{k+1}(\Omega)$, we have

$$\|\vec{u}-\vec{u}_h\|_{\mathrm{Y}} \leq Ch^k\|\vec{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}.$$

□

Acknowledgements

After reading H. Brezis' book on functional analysis (Brezis, 2010), I was deeply inspired by the uniqueness and elegance of the variational point of view to solve partial differential equations. I became aware of this Maxwell problem while conducting research in my high school physics class; this seemingly familiar elliptic PDEs under the name of "Maxwell's equations" reminded me of the variational approach. Driven by a strong mathematical curiosity, I delved deeper into this research and wrote this paper.

I would like to first appreciate my advisor, William Hou, for guiding me through this remarkable research journey. Specifically, he supported me with the selection of topic, provided me with helpful resources for my research, and accompanied me to troubleshoot various difficulties along the way.

I would also like to express my gratitude to my family members who thoroughly supported, empowered, and encouraged me during my inquiry and research process. Without them, this research would be impossible to complete.

References

Assous, F., Ciarlet, P., Labrunie, S. (2018). *Analyses of Time-Harmonic Problems*. In: Mathematical Foundations of Computational Electromagnetism. Applied Mathematical Sciences. vol 198. Springer, Cham. https://doi.org/10.1007/978-3-319-70842-3_8

Bombieri, E. (1977). [1] *Variational problems and elliptic equations, in Mathematical Developments Arising from Hilbert Problems* (Browder, F., ed.), Proc. Sympos. Pure Math., Vol. 28, Part 2;, American Mathematical Society, pp. 525C536. https://doi.org/10.1090/pspum/028.2

Brezis, H. (2010). *Functional Analysis, Sobolev spaces and partial differential equations*. Springer, universitext ed. https://doi.org/10.1007/978-0-387-70914-7

Ciarlet, P. G. (1978). *The finite element methods for elliptic problems*. North-Holland, Amsterdam. https://doi.org/10.1137/1.9780898719208

Ciarlet, P. (1979). *The Finite Element Method for Elliptic Problems*. (2nd ed.), North-Holland. https://doi.org/10.1137/1.9780898719208

Ciarlet, P. Jr., Wu H., & Zou J. (2014). *Edge element methods for Maxwells equations with strong convergence for Gauss laws*. SIAM J. Numer. Anal., 52, pp. 779C807. https://doi.org/10.1137/120899856

Ciarlet, P. Jr. (2020). *On the approximation of electromagnetic fields by edge finite elements. Part 3 : sensitivity to coefficients*. SIAM J. Math. Anal., 52, pp. 3004C3038. https://doi.org/10.1137/19M1275383

Ciarlet, P. Jr. (2020). *Mathematical and numerical analyses for the div-curl and div-curlcurl problems with a sign-changing coefficient*. Preprint HAL, hal-02567484. https://hal.archives-ouvertes.fr/hal-02567484

Evans, L. C. (2010). *partial differential equations*. vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed. www.ams.org/bookpages/gsm-19

Han Q., & Lin F. H. (1997). *Elliptic Partial Differential Equations*. American Mathematical Society. https://doi.org/10.1090/cln/001

Lax, P. (2002). *Functional Analysis*, Wiley. www.ams.org/bookpages/gsm-156

Le Dret, H., Lucquin, B. (2016). *The Variational Formulation of Elliptic PDEs. In: Partial Differential Equations: Modeling, Analysis and Numerical Approximation*. International Series of Numerical Mathematics, vol 168. Birkh?user, Cham. https://doi.org/10.1007/978-3-319-27067-8

Pinchover, Y., & Rubinstein, J. (2005). *An Introduction to Partial Differential Equations*. Cambridge University Press. https://doi.org/10.1017/CBO9780511801228

Sophie, A., Dhia, B., Chesnel, L., & Ciarlet, P. (2014). *Two-dimensional Maxwell's equations with sign-changing coefficients*. Applied Numerical Mathematics. Vol 79. pp. 29-41. https://doi.org/10.1016/j.apnum.2013.04.006

Sophie, A., Dhia, B., Chesnel, L., & Rihani, M. (2022). *Maxwell's equations with hypersingularities at a conical plasmonic tip*. Journal de Mathmatiques Pures et Appliques. Vol 161. pp. 70-110. https://doi.org/10.1016/j.matpur.2022.03.001

Copyrights