Journal of Mathematics Research; Vol. 15, No. 1; February 2023
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Solving Some Fractional Ordinary Differential Equations by SBA Method

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Received: October 29, 2022 Accepted: January 25, 2023 Online Published: February 1, 2023

doi:10.5539/jmr.v15n1p47 URL: https://doi.org/10.5539/jmr.v15n1p47

Abstract

In this paper we have solved some temporal fractional functional equations in the sense of Caputo by a numerical method called SOME BLAISE ABBO(SBA). Unlike classical numerical methods, this method bypasses discretization. Despite its youth, it has already proven itself. Indeed, its accuracy and efficiency have already been proven in the solution of ODEs and PDEs with integer derivative. Its application to fractional functional equations constitutes an important scientific contribution. On the one hand, we demonstrate the efficiency of the SBA method to find exact solutions, when they exist, of some rather complicated problems, due to their nonlinearity. On the other hand, through these results, we bring essential information allowing the analysis of a given phenomenon in order to help the best decision making.

Keywords: adomian decompositional scheme, SOME BLAISE ABBO (SBA) method, fractional functional equations, Burgers equation

AMS classification codes : 65Nxx, 65Lxx, 65Mxx, 65Qxx

1. Introduction

During the last three decades the notion of fractional calculus is an emerging theme, one of the main reasons being its numerous applications in many scientific disciplines such as physics, fluid mechanics, electrochemistry, etc (A KADEM & D.BALEANU.(2010), A KADEM & D.BALEANU.(56(3):332-338 2011), A KADEM & D.BALEANU.(56(5):629-635 2011), A.KHALOUTA & A KADEM (2019)), A.R NABULSI (2011), H. JAFARI, et al (2012), H. JADARIi, et al (2014), J. SINGH, D. KUMAR & A. KILICAM (2014), J.T KATSIKADELIS (2012),J.R.WANG & Y. ZHOU (2011) K. B OLDHAM (2010), Y. SHANG (2014), Z. DAHMANI, A. ANBER (2010)).

But until now, the construction of analytical or numerical solutions of these equations, which are often derived from the modeling of physical phenomena, remains a difficult problem to solve. Many attempts to solve these equations can be found in the literature, for example in (M. H. CHERIF (2016)), the HPM, HPTM methods have been used as well as the FNDM, NHPM methods in (A. KHALOUTA (2019)). But most of these methods, although very complex, give only an approximate solution of the problem.

In this paper, we innovate by using the **SOME BLAISE ABBO (SBA)** method to construct, when they exist, the exact solutions of the fractional temporal functional equations in the sense of Caputo. In general, the exact solution is difficult to reach because of the complexity of the nonlinearity. The strength of this method is that with the Picard principle which it uses in combination with the Adomian methods and successive approximations, one can easily dispense with the nonlinearity and reduce the complex problem to a simple linear equation. This avoids the calculation of Adomian polynomials. The particularity of this method is also that it does not discretize like classical numerical methods and therefore the physical properties of the modeled phenomena are preserved . This method has been used by many researchers to solve EDOs, EDPs etc. (see in B.ABBO, O. SO, G. BARRO & B. SOME (2007) & B.ABBO (2007), B. SOME (2018)) with integer order derivatives. It is also used by NEBIE & al in (A. W. NEBIE, F. BERE, B. ABBO, Y. PARE (2021)) to solve fractional PDEs in the Caputo sense.

After having recalled, in Section 2, some basic notions on fractional computations and described the SBA method in Section 3, we devoted Section 4 to the illustration of the efficiency of the method on some examples of fractional functional equations where the time derivative is in the Caputo sense. Section 5 is the conclusion.

2. Basics of Fractional Calculus Theory

We give basic definitions and notations of the theory of fractional calculus used in this article. For more details see

(F. NOROUZI and G. M. N' GURKATA (2020), A. KHALOUTA (2019), M. H. CHERIF, Z. DAHMANI, A. ANBER (2010)).

Definition 1 : Fractional Riemann-Liouville integral

Let $f \in C([0;+\infty[)$. The fractional Riemann-Liouville integral (on the left) of order $\alpha \geq 0$ of the function f denoted $\mathcal{I}^{\alpha} f$ is defined by:

$$\begin{cases} \mathcal{I}^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, t > 0. \\ \mathcal{I}^0 f(t) = f(t) \end{cases} \tag{1}$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2 : Fractional derivative in the sense of Caputo

Let $f \in C^m([0;+\infty[)$, $m \in \mathbb{N}$ $\alpha > 0$ and $n = [\alpha] + 1$.
The fractional derivative in the Caputo sense (on the left) of order α of the function f noted ${}^c\mathcal{D}^{\alpha} f$ is defined by:

$$\begin{cases} {}^c\mathcal{D}^{\alpha} f(t) = \mathcal{I}^{n-\alpha} \circ \dfrac{d^n}{dt^n} f(t) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ si \ n-1 < \alpha < n \\ {}^c\mathcal{D}^{\alpha} f(t) = \dfrac{d^n}{dt^n} f(t), \ si \ \alpha = n \end{cases} \tag{2}$$

where $[\alpha]$ denotes the integer part of α.

Definition 3 :Fractional differential equation of Caputo type

Let $\alpha > 0$, α not belong to $\mathbb{N}$, $n = [\alpha] + 1$, $y \in C^m([0;+\infty[)$ and $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, then:

$${}^c\mathcal{D}^{\alpha} y(t) = f(t, y(t)), \tag{3}$$

where t is an independent variable and y is an unknown real function of variable t is called a fractional differential equation of Caputo type with the following initial conditions:

$$y^{(k)}(0) = b_k, (k = 0, 1, 2, ..., n-1). \tag{4}$$

3. Bacground of SBA Method

3.1 Adomian Decompositional Scheme

(B.Abbo (2007), B. SOME (2018) , G. ADOMIAN (1990), K. ABBAOUI (1995), Y. CHERRUAULT, G. SACCOMANDI and B. SOME (1992), Y. CHERRUAUT (1989).

Let us take again here the description made in our book (B. SOME (2018)). Consider the general functional equation

$$F(u(t)) = g(t) \tag{5}$$

where F is any nonlinear operator of a Banach space E in E having linear and nonlinear terms, g a given function with values in E and $u \in E$ the unknown function.

We decompose the operator F in linear and nonlinear terms. This can be expressed as:

$$F = L + R + N \tag{6}$$

where $L + R$ is the linear part of the operator with L "invertible" in the Adomian sense, R the remainder of the linear part and N denotes the nonlinear part of F.

By injecting (6) into (5) we get :

$$Lu + Ru + Nu = g. \tag{7}$$

By applying L^{-1}, the inverse of L in the Adomian sense, to (7) we obtain the following relation called Adomian canonical form:

$$u = \theta + L^{-1} g - L^{-1} Ru - L^{-1} Nu, \tag{8}$$

because $L^{-1} Lu = u - \theta$ where θ verifies $L\theta = 0$, i.e. θ is the integration constant if F is a differential.

The Adomian method consists in looking for the solution when it exists, u of (8) in the form of a series. Let us then pose

$$u = \sum_{n=0}^{+\infty} u_n \tag{9}$$

and decompose the term Nu into a series of the form:

$$Nu = \sum_{n=0}^{+\infty} A_n \tag{10}$$

where A_n are special polynomials called "Adomian polynomials", which depend exclusively on $u_0, u_1, ..., u_n$ and are obtained from the following theorem.

Theorem 1 (B. SOME (2018), Y. CHERRUAULT, G. SACCOMANDIS and B. SOME (1992)):

The A_n depend only on $u_0, u_1, ..., u_n$ and are obtained by the following formula:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{+\infty} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{11}$$

where λ is a parameter introduced by convenience.

Assuming that the series (9) and (10) are convergent, by bringing them in (8), we obtain the expression:

$$\sum_{n=0}^{+\infty} u_n = \theta + L^{-1}g - L^{-1}\left(R\left(\sum_{n=0}^{+\infty} u_n\right)\right) - L^{-1}\left(\sum_{n=0}^{+\infty} A_n\right) \tag{12}$$

By identification, we obtain the terms u_n of the series $\sum_{n=0}^{+\infty} u_n$ by the following algorithm:

$$\begin{cases} u_0 = \theta + L^{-1}g \\ u_1 = -L^{-1}Ru_0 - L^{-1}A_0 \\ \vdots \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n \end{cases} \tag{13}$$

called Adomian algorithm.

In practice, the calculation of Adomian polynomials becomes very difficult when the nonlinearity, N, of the equation is strong. To get around this difficulty, researchers from the team of Professor Blaise SOME at the Laboratoire d'Analyse Numerique d'Informatique et de BIOmathematique (L.A.N.I.BIO) of the University of Ouagadougou developed in 2006 the method called SOME BLAISE ABBO (SBA). We give below a brief description of this method after a brief review of its history.

3.2 History of the SBA Method

The SOME BLAISE ABBO (SBA) method is a method that was born from the idea of getting around the difficulties encountered in the calculation of Adomian polynomials. It was developed at the Laboratory of Numerical Analysis of Informatics and BIOmathematics (L.A.N.I.BIO) of the University of Ouagadougou, in 2006 by Professor SOME Blaise, Director of the Laboratory, and his doctoral student, B. ABBO. This method, which has already shown its effectiveness with nonlinear functional equations of integer order, is the combination of two methods and one principle. These are the method of successive approximations, the Adomian decompositional method and the Picard principle.

3.3 General Solution of EDFs by the SBA Method

We adopt here the description proposed by the inventor of the method in his book (B. SOME(2018). Thus the description is made on a non linear fractional ODE and the technique (SBA) is adapted to non linear fractional PDEs.

Let us solve the following fractional PDE problem:

$$\begin{cases} \dfrac{d^\alpha u}{dt^\alpha} = -R(u) - N(u),\ 0 < t < T \\ u^{(i)}(0) = f_i, (i = 0, 1, 2, ..., h-1) \end{cases} \tag{14}$$

in a suitable functional space $V = C^m([0; +\infty[), m \in \mathbb{N}$ with
R a linear operator of V in V;
N a nonlinear term of V in V;
$u \in V$ the unknown function;
$\frac{d^\alpha u}{dt^\alpha}$ the fractional derivative of order α of u in the sense of Caputo;

$h = [\alpha] + 1$ where $[\alpha]$ is the integer part of α.

Using the method of successive approximations, the above problem (14) can be approximated by the following iterative scheme:

$$\begin{cases} \frac{d^\alpha u^k}{dt^\alpha} = -R(u^k) - N(u^{k-1}),\ 0 < t < T \\ u^{(i)^k}(0) = f_i, (i = 0, 1, 2, ..., h-1) \end{cases} \tag{15}$$

The solution of the scheme (15) by the method of approximations consists in determining at each iteration $(k = 1, 2, ...)$ approximated solutions $u^1, u^2, ..., u^n$ which are series. But this requires a judicious choice of the initial condition u^0. And the solution u of the problem (14) we are looking for, if it exists, is obtained by

$$u = \lim_{k \to +\infty} u^k \tag{16}$$

if the sequence $(u^k)_k$ is convergent in V. Now we propose a new technique using the Adomian decompositional method at each step k, under an assumption of "good" choice of u^0.

Consider the approximate problem (15) above:

$$\begin{cases} \frac{d^\alpha u^k}{dt^\alpha} = -R(u^k) - N(u^{k-1}) \\ u^{(i)^k}(0) = f_i, (i = 0, 1, 2, ..., h-1) \end{cases} , k = 1, 2, ... \tag{17}$$

We will use the Adomian method described in section (3.1) to solve the problem at each iteration step. After a choice of u^0, the problem at step k consists in determining u^k solution of

$$\begin{cases} \frac{d^\alpha u^k}{dt^\alpha} = -R(u^k) + g_{k-1} \\ u^{(i)^k}(0) = f_i, (i = 0, 1, 2, ..., h-1) \end{cases} , k = 1, 2, ... \tag{18}$$

where we have formally posed $g_{k-1} = -N(u^{k-1})$. Indeed with the choice of operators

$$L(.) = \frac{d^\alpha(.)}{dt^\alpha} \ \ et \ \ L^{-1}(.) = \mathcal{I}_0^\alpha(.) \tag{19}$$

where L^{-1} is the inverse of L in the Adomian sense.

We can formally write

$$u^k = \theta^k(t) - L^{-1}\left(R(u^k)\right) + L^{-1}g_{k-1}, \quad avec \ \ \theta^k(t) = \sum_{i=0}^{h-1} \frac{t^i}{i!} f_i \tag{20}$$

which is a canonical form of Adomian.

Remark 1

$$L^{-1}Lu^k(t) = u^k(t) - \sum_{i=0}^{h-1} \frac{t^i}{i!} u^{(i)^k}(0). \tag{21}$$

We can therefore deduce the following Adomian algorithm for a fixed k:

$$\begin{cases} u_0^k = \theta^k(t) + L^{-1}(g_{k-1}) \\ u_{n+1}^k = -L^{-1}\left(R\left(u_n^{k-1}\right)\right), n \geq 0 \end{cases} , k \geq 1. \tag{22}$$

NB:

- $u^{(i)^k}(0) = f_i, (i = 0, 1, 2, ..., h-1)$ is the Cauchy condition for each step k.
- u_0^k is the first term of the Adomian series for the step k.
- Perhaps it should be said that the (20) scheme is not quite a fixed scheme even if it is similar to it; nor even to the Adomian scheme in the classical sense. Here we combine ideas borrowed from these two classical techniques to come up with an approximate scheme.

Then the solution at each step k will be calculated by:

$$u^k = \sum_{n=0}^{+\infty} u_n^k,\ k = 1, 2, ...$$

The exact solution u of (14), if it exists, is

$$u = \lim_{k\to+\infty} u^k = \lim_{k\to+\infty} \left(\sum_{n=0}^{+\infty} u_n^k \right). \tag{23}$$

The details of the calculation of u^k will be explained in the applications.

An important step in the SBA numerical method is the choice of the first iteration term u^0 of the scheme of successive approximations which must not be arbitrary as usual. We propose the Picard principle:

Picard's principle: possible choice of u^0 It is very fundamental to start the development of the SBA algorithm. The principle consists in choosing for the first iteration, the value u^0 which cancels the non-linearity . Indeed when we choose u^0 such that $N(u^0) = 0$, we reduce the non linear problem to a linear one, which is easy and allows to avoid the computation of Adomian polynomials. We will check for the following iterations if $Nu^1 = 0, Nu^2 = 0, Nu^3 = 0, ...$

4. Solving Fractional ODEs

Example 1: Consider the following nonlinear fractional ODE:

$$\begin{cases} \dfrac{d^\alpha u(t)}{dt^\alpha} - \theta u(t) = \theta u^2(t) - u(t)\dfrac{d^\alpha u(t)}{dt^\alpha} & ,0 < \alpha \leq 1 \\ u(0) = \beta \end{cases} \tag{24}$$

with $u \in C^m([0;T]), m \in \mathbb{N}^*, t \in [0;T]$, $\theta, \beta \in \mathbb{R}$ and $\dfrac{d^\alpha(.)}{dt^\alpha}$ the derivative in the sense of Caputo.

By posing

$L(.) = \dfrac{d^\alpha(.)}{dt^\alpha}, L^{-1}(.) = \mathcal{I}_t^\alpha(.), Ru = -\theta u\ et\ Nu = -\theta u^2(t) + u(t)\dfrac{d^\alpha u}{dt^\alpha}$

where L^{-1} is the inverse of L in the Adomian sense we obtain the following SBA algorithm:

$$\left\{ u_0^k = u(0) - L^{-1}\left(N(u^{k-1})\right) u_{n+1}^k = -L^{-1}\left(R(u_n^k)\right), n \geq 0 \quad ,k \geq 1 \right. \tag{25}$$

Calculation of $u^k, k \geq 1$. First iteration: calculation of u^1. For $k = 1$, the algorithm (25) becomes:

$$\left\{ u_0^1 = u(0) - L^{-1}\left(N(u^0)\right) u_{n+1}^1 = -L^{-1}(R(u_n^1)), n \geq 0. \right. \tag{26}$$

Let's take u^0 *such as* $Nu^0 = 0$. Per iteration, we calculate:

- $u_0^1 = \beta$;
- $u_1^1 = \dfrac{\theta\beta t^\alpha}{\Gamma(\alpha+1)}$;
- $u_2^1 = \dfrac{\theta^2\beta t^{2\alpha}}{\Gamma(2\alpha+1)}$.

We then conjecture that

$$u_n^1(t) = \frac{\beta(\theta t^\alpha)^n}{\Gamma(n\alpha+1)}, n \geq 0.$$

Let's show this result by recurrence.

It is clear that the result is true for $n = 0$. Suppose it is true up to order h, $h \geq 0$ and show that it is true to order $h + 1$. We have:

$$\begin{aligned} u_{h+1}^1 &= -L^{-1}\left(R(u_h^1)\right) \\ &= \frac{\theta^{h+1}\beta}{\Gamma(\alpha)\Gamma(h\alpha+1)}\int_0^t (t-x)^{\alpha-1}x^{h\alpha}dx \\ &= \frac{\beta(\theta t^\alpha)^{h+1}}{\Gamma((h+1)\alpha+1)}. \end{aligned}$$

So the result is true to the order $h + 1$.

So for all $n \geq 0$,

$$u_n^1(t) = \frac{\beta(\theta t^\alpha)^n}{\Gamma(n\alpha+1)}.$$

Therefore the approximate solution at the first iteration is :

$$\begin{aligned} u^1(t) &= \sum_{n=0}^{+\infty} u_n^1(t) \\ &= \beta\sum_{n=0}^{+\infty}\left(\frac{(\theta t^\alpha)^n}{\Gamma(n\alpha+1)}\right) \\ &= \beta\mathbb{E}_\alpha(\theta t^\alpha). \end{aligned}$$

Second iteration: calculation of u^2.

For $k = 2$, the algorithm (26) becomes:

$$\begin{cases} u_0^2 = u(0) - L^{-1}\left(N(u^1)\right) \\ u_{n+1}^2 = -L^{-1}\left(R(u_n^2)\right), n \geq 0 \end{cases}. \tag{27}$$

Let's check that $N(u^1) = 0$. We have:

$$\begin{aligned} N(u^1(t)) &= -\theta\left(u^1(t)\right)^2 + u^1(t)\frac{d^\alpha u^1(t)}{dt^\alpha} \\ &= \theta\left(\beta\mathbb{E}_\alpha(\theta t^\alpha)\right)^2 - \theta\left(\beta\mathbb{E}_\alpha(\theta t^\alpha)\right)^2 \\ &= 0. \end{aligned}$$

By developing the algorithm (27) as before we obtain:

$$\begin{cases} u_0^2 = \beta \\ u_1^2 = \dfrac{\theta\beta t^\alpha}{\Gamma(\alpha+1)} \\ u_2^2 = \dfrac{\beta(\theta t^\alpha)^2}{\Gamma(2\alpha+1)} \\ \vdots \\ u_n^2(t) = \dfrac{\beta(\theta t^\alpha)^n}{\Gamma(n\alpha+1)}. \end{cases} \tag{28}$$

Therefore the approximate solution to the second iteration is :

$$u^2(t) = \beta\mathbb{E}_\alpha(\theta t^\alpha).$$

Recursively, we obtain for $k \geq 2$:

$$u^k(t) = \sum_{n=0}^{+\infty} u_n^k(t,x) = \beta \mathbb{E}_\alpha(\theta t^\alpha).$$

Therefore the exact solution of the problem is

$$u(t) = \lim_{k \to +\infty} (u^k(t)) = \beta \mathbb{E}_\alpha(\theta t^\alpha).$$

Example 2: Time fractional Burgers model

$$\begin{cases} \left(\dfrac{d^\alpha u(t)}{dt^\alpha}\right)^2 + \dfrac{d^\alpha u(t)}{dt^\alpha} - u^2(t) + u(t) = 0 \\ u(0) = \theta \\ \dfrac{du(0)}{dt} = \beta \end{cases}, 1 < \alpha \leq 2 \tag{29}$$

with $u \in C^m([0;T]), m \in \mathbb{N}, t \in [0;T]$, $\beta \in \mathbb{R}$ and $\dfrac{d^\alpha(.)}{dt^\alpha}$ the derivative in the sense of Caputo.

By positing

$L(.) = \dfrac{d^\alpha(.)}{dt^\alpha}, L^{-1}(.) = \mathcal{I}_t^\alpha(.), Ru = u \; et \; Nu = -u^2 + \left(\dfrac{d^\alpha u}{dt^\alpha}\right)^2,$

where L^{-1} is the inverse of L in the Adomian sense.

We obtain the following SBA algorithm:

$$\begin{cases} u_0^k = \theta + t\beta - L^{-1}\left(N(u^{k-1})\right) \\ u_{n+1}^k = -L^{-1}\left(R(u_n^k)\right), n \geq 0 \end{cases}, k \geq 1. \tag{30}$$

Calculation of $u^k, k \geq 1$. First iteration: calculation of u^1.

For $k = 1$, the algorithm (30) becomes:

$$\begin{cases} u_0^1 = \theta + t\beta - N(u^0) \\ u_{n+1}^1 = -L^{-1}(R(u_n^1)), n \geq 0. \end{cases} \tag{31}$$

Let us take u^0 such that $Nu^0 = 0$.

By developing the algorithm (31) as in the previous example we obtain:

$$\begin{cases} u_0^1 = \theta + t\beta \\ u_1^1 = -\dfrac{\theta t^\alpha}{\Gamma(\alpha+1)} - \dfrac{\beta t^{\alpha+1}}{\Gamma(\alpha+2)} \\ u_2^1 = \dfrac{\theta t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{\beta t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ u_3^1 = -\dfrac{\theta t^{3\alpha}}{\Gamma(3\alpha+1)} - \dfrac{\beta t^{3\alpha+1}}{\Gamma(3\alpha+2)} \\ \vdots \\ u_n^1(t) = \dfrac{\theta(-t^\alpha)^n}{\Gamma(n\alpha+1)} + \dfrac{\beta t(-t^\alpha)^n}{\Gamma(n\alpha+2)}. \end{cases} \tag{32}$$

Let us show by recurrence that $u_n^1(t) = \dfrac{\theta(-t^\alpha)^n}{\Gamma(n\alpha+1)} + \dfrac{\beta t(-t^\alpha)^n}{\Gamma(n\alpha+2)}, n \geq 0.$

For $n = 0$, we have:

$$u_0^1(t) = \frac{\theta(-t^\alpha)^0}{\Gamma(0\times\alpha+1)} + \frac{\beta t(-t^\alpha)^0}{\Gamma(0\times\alpha+2)} = \theta + t\beta.$$

Suppose that for $h \geq 0$,

$$u_h^1(t) = \frac{\theta(-t^\alpha)^h}{\Gamma(h\alpha+1)} + \frac{\beta t(-t^\alpha)^h}{\Gamma(h\alpha+2)}.$$

For $n = h+1$, we have:

$$\begin{aligned}
u_{h+1}^1 &= -L^{-1}\left(R(u_h^1)\right) \\
&= \frac{(-1)^{h+1}\theta}{\Gamma(h\alpha+1)\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}x^{h\alpha}dx + \frac{(-1)^{h+1}\beta}{\Gamma(h\alpha+2)\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}x^{h\alpha+1}dx \\
&= \frac{\theta(-t^\alpha)^{h+1}}{\Gamma((h+1)\alpha+1)} + \frac{\beta t(-t^\alpha)^{h+1}}{\Gamma((h+1)\alpha+2)}.
\end{aligned}$$

Therefore the approximate solution at the first iteration is

$$\begin{aligned}
u^1(t) &= \sum_{n=0}^{+\infty} u_n^1(t) \\
&= \sum_{n=0}^{+\infty}\left(\frac{\theta(-t^\alpha)^n}{\Gamma(n\alpha+1)} + \frac{\beta t(-t^\alpha)^n}{\Gamma(n\alpha+2)}\right) \\
&= \theta\mathbb{E}_\alpha(-t^\alpha) + \beta t\mathbb{E}_{\alpha,2}(-t^\alpha).
\end{aligned}$$

Second iteration: calculation of u^2.

For $k=2$, the algorithm (30) becomes:

$$\begin{cases} u_0^2 = u(0) - L^{-1}\left(N(u^1)\right) \\ u_{n+1}^2 = -L^{-1}\left(R(u_n^2)\right), n \geq 0. \end{cases} \tag{33}$$

Let's check that $N(u^1) = 0$.

We have:

$$\begin{aligned}
N(u^1(t)) &= -\left(u^1(t)\right)^2 + \left(\frac{d^\alpha u^1(t)}{dt^\alpha}\right)^2 \\
&= -\Big(\theta\mathbb{E}_\alpha(-t^\alpha) + \beta t\mathbb{E}_{\alpha,2}(-t^\alpha)\Big)^2 + \Big(-\theta\mathbb{E}_\alpha(-t^\alpha) - \beta t\mathbb{E}_{\alpha,2}(-t^\alpha)\Big)^2 \\
&= \Big(\theta\mathbb{E}_\alpha(-t^\alpha) + \beta t\mathbb{E}_{\alpha,2}(-t^\alpha)\Big)^2 - \Big(\theta\mathbb{E}_\alpha(-t^\alpha) + \beta t\mathbb{E}_{\alpha,2}(-t^\alpha)\Big)^2 \\
&= 0.
\end{aligned}$$

We get the same:

$$\begin{cases} u_0^2 = \theta + t\beta \\ u_1^2 = -\dfrac{\theta t^\alpha}{\Gamma(\alpha+1)} - \dfrac{\beta t^{\alpha+1}}{\Gamma(\alpha+2)} \\ u_2^2 = \dfrac{\theta t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{\beta t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ u_3^2 = -\dfrac{\theta t^{3\alpha}}{\Gamma(3\alpha+1)} - \dfrac{\beta t^{3\alpha+1}}{\Gamma(3\alpha+2)} \\ \vdots \\ u_n^2(t) = \dfrac{\theta(-t^\alpha)^n}{\Gamma(n\alpha+1)} + \dfrac{\beta t(-t^\alpha)^n}{\Gamma(n\alpha+2)}. \end{cases} \tag{34}$$

Thus the approximate solution at the second iteration is

$$u^2(t) = \theta\mathbb{E}_\alpha(-t^\alpha) + \beta t\mathbb{E}_{\alpha,2}(-t^\alpha).$$

Recursively, we obtain for $k \geq 2$:

$$u^k(t) = \sum_{n=0}^{+\infty} u_n^k(t, x) = \theta \mathbb{E}_\alpha(-t^\alpha) + \beta t \mathbb{E}_{\alpha,2}(-t^\alpha).$$

Therefore the exact solution of the problem is

$$u(t) = \lim_{k \to +\infty} (u^k(t)) = \theta \mathbb{E}_\alpha(-t^\alpha) + \beta t \mathbb{E}_{\alpha,2}(-t^\alpha).$$

5. Conclusion

The results obtained in this paper show that the SBA method is a good method for the numerical solution of temporal fractional functional equations in the sense of Caputo. We have explained without difficulty the exact solutions of two non-linear fractional ODEs. For our part the method is very promising to find a wide application in the search for solutions of nonlinear fractional evolution equations in the sense of Caputo. We also believe that fractional equations in the Riemann-Liouville sense could be solved by the method. But this will require a change in the value of θ given in the equation (20).

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