A Generalization of Projective Module

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Abstract

Let V be a submodule of a direct sum of some elements in $\mathcal{U}$, and X be a submodule of a direct sum of some elements in $\mathcal{N}$, where $\mathcal{U}$ and $\mathcal{N}$ are families of R-modules. A $\mathcal{U}$-free module is a generalization of a free module. According to the definition of $\mathcal{U}$-free module, we define three kinds of projective$_{\mathcal{U}}$ in this research, i.e., projective$_{\underline{\mathcal{U}}}$, projective$_{\mathcal{U}}$ module, and strictly projective$_{\mathcal{U}}$ module. The notion of strictly projective$_{\mathcal{U}}$ is a generalization of the projective module. In this research, we discuss the relationship between projective modules and the three types of modules. Furthermore, we show that the properties of $\mathcal{U}$ impact the properties of the projective$_{\mathcal{U}}$ module so that we can determine some properties of the projective$_{\mathcal{U}}$ module based on the properties of the family of $\mathcal{U}$ of R-modules.

Keywords: projective module, projective$_{\mathcal{U}}$ module, free module

1. Introduction

A notion of a U-exact sequence as a generalization of an exact sequence (Davvaz and Parnian-Garamaleky, 1999). A V-coexact sequence is a dual of U-exact sequence. Furthermore, the concept of U-exact sequences is used to generalize the Schanuel Lemma (Anvariyeh and Davvaz, 2005) and homological algebra (Davvaz and Shabani-Solt, 2002). Then, Anvariyeh and Davvaz study the connections between projective modules and U-split sequences (Anvariyeh and Davvaz, 2002). Moreover, Madanshekaf gives some results about quasi-exact sequences related to finitely presented modules and torsion functor (Madanshekaf, 2008). Furthermore, Aminizadeh et al. establish a quasi-exact sequence of S-acts. They use the quasi-exact sequence to give the properties of S-acts such as torsion freeness and principal weak flatness (Aminizadeh, 2019).

Motivated by the definition of U-exact sequence, Fitriani et al. introduce a sub-exact sequence as the generalization of an exact sequence (Fitriani et al., 2016). Moreover, Fitriani et al. use the concept of sub-exact sequence to generalize linearly independent sets in R-modules (Fitriani et al., 2017). First, they establish the notion of an X-sub-linearly independent module. Then, by using the V-coexact sequence, Fitriani et al. introduce a $\mathcal{U}_V$-generated module (Fitriani et al., 2018a). The $\mathcal{U}_V$-generated module is a dual of X-sub-linearly independent. This concept is motivated by the definition of $\mathcal{U}$-generated module (Anderson and Fuller, 2018), (Clark et al., 2006). A class of $\mathcal{U}_V$-generated modules form a pre-additive category (Fitriani and Faisol, 2020).

The concept of $\mathcal{U}_V$-generated module and X-sub-linearly independent module are used to construct a $\mathcal{U}$-basis and a $\mathcal{U}$-free module, which is a basis and a free module relative to a family $\mathcal{U}$ of R-modules (Fitriani et al., 2018b). There are there types of $\mathcal{U}$-basis, i.e., $\underline{\mathcal{U}}$-basis, $\mathcal{U}$-basis, and strictly $\mathcal{U}$-basis. According to these basis, there are three types of $\mathcal{U}$-free module, i.e., $\underline{\mathcal{U}}$-free module, $\mathcal{U}$-free module, and strictly $\mathcal{U}$-free module. By taking the family $\mathcal{U}$ of R-modules consisting of only one element, i.e., a module R as an R-module, we have a free module is a special case of a strictly $\mathcal{U}$-free module. Besides that, the concept of X-sub-exact sequence is used to determine the Noetherian property of the submodule of the generalized power series module (Faisol et al., 2021). Furthermore, Fitriani et al. use the X-sub-exact sequence to construct the category of the submodule of a uniserial module. This category is a pre-additive category (Fitriani et al., 2021).

We know that the free module is closely related to the projective module. The projective module is defined as a direct summand of a free module (Adkins and Weintraub, 1992). The projective module concept has an important role in Schanuel's Lemma which states that if $0 \to K \to P \to M \to 0$ dan $0 \to K' \to P' \to M \to 0$ are short exact sequences of R-modules and P and P' are projective, then $K \oplus P'$ is isomorphic to $K' \oplus P$. Besides, a projective module can be used to characterize semisimple modules, semisimple rings, and left coherent rings (Wisbauer, 1991).

Related to the projective module, Bennis and Mahdou study the strongly Gorenstein projective (Bennis and Mahdou, 2006), Wang and Kim give the generalization of projective modules relative to the w-operation (Wang and Kim, 2015). They define w-projective module and w-invertible modules. Furthermore, Popescu and Popescu give the equivalence of the two categories of projective modules (Popescu and Popescu, 2022)

There is an opportunity to define a new concept, namely projective$_{\mathcal{U}}$ module, i.e., a projective module relative to a family $\mathcal{U}$ of R-modules by using the concept of the $\mathcal{U}$-free module. In this research, we define three kinds of projective$_{\mathcal{U}}$, i.e., projective$_{\underline{\mathcal{U}}}$, projective$_{\mathcal{U}}$ module, and strictly projective$_{\mathcal{U}}$ module. Then, we discuss the relationship between projective modules and the three types of modules. Furthermore, we give the properties of the projective module that are still satisfied by the projective$_{\mathcal{U}}$ module. Moreover, we show that the properties of $\mathcal{U}$ impact the properties of projective$_{\mathcal{U}}$ module. With this result, we can determine some properties of projective$_{\underline{\mathcal{U}}}$, projective$_{\mathcal{U}}$ module, and strictly projective$_{\mathcal{U}}$ module according to the properties of the family $\mathcal{U}$ of R-modules. Some of the properties we discuss are related to the injective module, projective module, Noetherian module, and Artinian Module. Besides that, we study the connection between projective$_{\underline{\mathcal{U}}}$, projective$_{\mathcal{U}}$ module, strictly projective$_{\mathcal{U}}$ module, duo module, quasi-injective module, and quasi-projective module.

2. Main Results

The concepts of X-sub-linearly independent module, $\mathcal{U}_V$-generated module, and $\mathcal{U}$-free module are defined as follows.

Given two families of R-modules $\mathcal{U} = \{U_\lambda\}_\Lambda$ and $\mathcal{N} = \{N_\lambda\}_\Lambda$, V is a submodule of $\oplus_\Lambda U_\lambda$, and X is a submodule of $\coprod_\Lambda N_\lambda$. If we can define an epimorphism f:

$$V \xrightarrow{f} N,$$

then N is generated by $\mathcal{U}_V$ (Fitriani et al., 2018a). Furthermore, if there exist a monomorphism g:

$$X \xrightarrow{g} M,$$

we say $\mathcal{N} = \{N_\lambda\}_\Lambda$ is an X-sublinearly independent to M (Fitriani et al., 2017). Then, we define two sets of X-sub-linearly independent modules and $\mathcal{U}_V$-generated modules as follows. The set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is a set of all $X \subseteq \oplus_\Lambda U_\lambda$ such that $\mathcal{U}$ is X-sub-linearly independent to M, and

$$\mathcal{U}(M) = \{V \subseteq \oplus_\Lambda U_\lambda | M \text{ is } \mathcal{U}_V - \text{generated}\}.$$

Since $0 \in \sigma(0, \oplus_\Lambda U_\lambda, M)$, $\sigma(0, \oplus_\Lambda U_\lambda, M) \neq \emptyset$.

If we have a pair of submodules (X, V), then this pair is said to be an $\underline{\mathcal{U}}$-basis of R-module M if (X, V) fulfill two conditions, i.e. M is generated by $\mathcal{U}_V$, and $\mathcal{U}$ is an X-sublinearly independent to M (Fitriani et al., 2018a). Furthermore, in case V is a minimal element of the set $\mathcal{U}(M)$, and X is a maximal element of the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$, we say (X, V) is a $\mathcal{U}$-basis of M. If X is isomorphic to V in a specific situation, the pair (X, V) is said to be a $\mathcal{U}$-strictly basis of M (Fitriani et al., 2018a).

Definition 1 *Given a family of modules over ring R, i.e.* $\mathcal{U} = \{U_\lambda\}_\Lambda$, *and X, V are two submodules of* $\oplus_\Lambda U_\lambda$. *A module K is said to be:*

1. $\underline{\mathcal{U}}$*-free if K has* $\underline{\mathcal{U}}$*-basis;*
2. $\mathcal{U}$*-free if K has* $\mathcal{U}$*-basis;*
3. $\mathcal{U}$*-strictly free if K has* $\mathcal{U}$*- strictly basis.*

(Fitriani et al., 2018a)

We construct the definition of three kinds of projective modules, i.e., projective$_{\underline{\mathcal{U}}}$, projective$_{\mathcal{U}}$, and strictly projective$_{\mathcal{U}}$ modules as follows, inspired by the definition of the projective module as a direct summand of a free module.

Definition 2 *Given a family of modules over ring R, i.e.* $\mathcal{U} = \{U_\lambda\}_\Lambda$, *and X, V are two submodules of* $\oplus_\Lambda U_\lambda$.

1. *If K is a a direct summand of M, where M is a* $\underline{\mathcal{U}}$*-free module, then K is said to be a projective*$_{\underline{\mathcal{U}}}$.
2. *If K is a direct summand of M, where M is a* $\mathcal{U}$*-free module, then K is said to be a projective*$_{\mathcal{U}}$.
3. *If K is a direct summand of M, where M is a strictly* $\mathcal{U}$*-free module, then K is said to be a strictly projective*$_{\mathcal{U}}$.

In the following example, we give the example of the strictly projective$_{\mathcal{U}}$ module.

Example 1 Given a family $\mathcal{U}$ of $\mathbb{Z}$-modules, where $\mathcal{U}$ is a set of $\mathbb{Z}_p$, p prime. We can write

$$\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime number}\}.$$

Then, we take two different prime numbers, say k and l. We have a pair $(\mathbb{Z}_k \oplus \mathbb{Z}_l, \mathbb{Z}_k \oplus \mathbb{Z}_l)$ is a $\mathcal{U}$-basis of $\mathbb{Z}_{kl}$ as a $\mathbb{Z}$-module. Consequently, $\mathbb{Z}_k \oplus \mathbb{Z}_l$ is a strictly $\mathcal{U}$-basis of $\mathbb{Z}_{kl}$. Therefore $\mathbb{Z}_{kl}$ is a strictly $\mathcal{U}$-free module. Then, all direct summands of $\mathbb{Z}_{kl}$ as a module over $\mathbb{Z}$ is a strictly projective$_{\mathcal{U}}$ module. As an example, we choose $k = 2$ and $l = 3$. We have all direct summands of $\mathbb{Z}_6$ as a $\mathbb{Z}$-module is a strictly projective$_{\mathcal{U}}$ module.

In case a $\mathcal{U}$-free module M is a simple module, we can construct only two projective$_{\mathcal{U}}$ modules, i.e., $\{0\}$ and module M itself. We have the following proposition according to Definition 2.

Proposition 1 *Every strictly projective$_{\mathcal{U}}$ module is projective$_{\mathcal{U}}$, and every projective$_{\mathcal{U}}$ module is projective$_{\underline{\mathcal{U}}}$.*

Proof. We assume that a module K is a strictly projective$_{\mathcal{U}}$ module. Consequently, K is a direct summand of a strictly $\mathcal{U}$-free module M. Module has a strictly $\mathcal{U}$-basis, say (X', V'). From (Fitriani et al., 2018a), we have X' is a maximal element of the set $0 \in \sigma(0, \oplus_\Lambda U_\lambda, M)$, $\sigma(0, \oplus_\Lambda U_\lambda, M)$, and V' is the minimal element of the set $\mathcal{U}(M)$. Therefore, (X', V') is a $\mathcal{U}$-basis of M. So, M is a $\mathcal{U}$-free module, and hence K is a projective$_{\mathcal{U}}$ module.

Now, assume that a module L is a projective$_{\mathcal{U}}$ module. Furthermore, L is a direct summand of a $\mathcal{U}$-free module N. Module N has a $\mathcal{U}$-basis, say (X_1, V_1). We have (X_1, V_1) is a $\underline{\mathcal{U}}$-basis of N. Therefore, N is a $\underline{\mathcal{U}}$-free module and hence L is a projective$_{\underline{\mathcal{U}}}$ module. □

In the following proposition, we give the connection between the projective module and strictly projective$_{\mathcal{U}}$ module.

Proposition 2 *Every projective module is a $\mathcal{U}$-strictly projective.*

Proof. We know that if a free module F has a basis X, then $F \cong \oplus_{x \in X} R_x$, for every $R_x \cong R$. We can select $\mathcal{U} = \{R\}$. As a result, $\oplus_{x \in X} R_x = R^X$ is a $\mathcal{U}$-strictly basis of F. It implies that F is strictly $\mathcal{U}$ free. Based on this, every free R -module F is a $\mathcal{U}$-strictly free module. By treating $\mathcal{U} = \{R\}$ as a family of R-modules, module projective is a special case of $\mathcal{U}$-strictly projective. □

Furthermore, we investigate the properties of the projective module that are still satisfied by the $\mathcal{U}$-projective module.

Now, we consider the properties of projective$_{\underline{\mathcal{U}}}$ module.

Proposition 3 *Every projective$_{\underline{\mathcal{U}}}$ module is a $\mathcal{U}_V$-generated module.*

Proof. Let M be a projective$_{\underline{\mathcal{U}}}$ module. It implies that M is a direct summand of N, where N is a $\underline{\mathcal{U}}$-free module. Based on Definition 1, N has a $\underline{\mathcal{U}}$-basis. Let (X, V) is a $\underline{\mathcal{U}}$-basis of N, where X and V are submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. So, we have the exact sequence:

$$V \xrightarrow{f} N \to 0.$$

Since M is a direct summand of N, there is an epimorphism π from N to M or the following sequence:

$$N \xrightarrow{\pi} M \to 0$$

is exact. Hence, $\pi \circ f$ is a surjective homomorphism from V to M. It implies that M is a $\mathcal{U}_V$-generated module. □

Based on (Fitriani et al., 2020), every projective$_{\underline{\mathcal{U}}}$ module is in category of $\mathcal{U}_V$-generated modules. As a direct result of Proposition , we have the following characteristic of projective$_{\mathcal{U}}$ and strictly projective$_{\mathcal{U}}$ module.

Corrolary 1 *Every projective$_{\mathcal{U}}$ module is a $\mathcal{U}_V$-generated module.*

Proof. Let K be a projective$_{\mathcal{U}}$ module. It implies that K is a direct summand of L, where L is a $\mathcal{U}$-free module. Based on Definition , L has a $\mathcal{U}$-basis. Let (X', V') is a $\mathcal{U}$-basis of L, where X' and V' are submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. So, we have the exact sequence:

$$V' \xrightarrow{f} L \to 0.$$

Since K is a direct summand of L, there is an epimorphism π from L to K or the following sequence:

$$L \xrightarrow{\pi} K \to 0$$

is exact. Hence, $\pi \circ f$ is a surjective homomorphism from V' to K. So, K is a $\mathcal{U}_V$-generated module. □

Corrolary 2 *Every strictly projective$_{\mathcal{U}}$ module is a $\mathcal{U}_V$-generated module.*

Proof. Let P be a strictly projective$_{\mathcal{U}}$ module. Therefore, P is a direct summand of Q, where Q is a strictly free module. Based on Definition , Q has a strictly $\mathcal{U}$-basis. Let (X'', V'') is a strictly $\mathcal{U}$-basis of N, where X'' and V'' are submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. So, we have the exact sequence:

$$V'' \xrightarrow{f} Q \to 0.$$

Since P is a direct summand of Q, there is an epimorphism π from Q to P or the following sequence:

$$Q \xrightarrow{\pi} P \to 0$$

is exact. Hence, $\pi \circ f$ is a surjective homomorphism from V'' to P. So, P is a $\mathcal{U}_V$-generated module. □

The direct sum of some projective$_{\underline{\mathcal{U}}}$ module is a projective$_{\underline{\mathcal{U}}}$ module, as shown in the following proposition.

Proposition 4 *Given a family of R-modules, $\mathcal{U} = \{U_\lambda\}_\Lambda$, and V is a submodules of $\oplus_\Lambda U_\lambda$. If $M_1, M_2, ...M_k$ are projective$_{\underline{\mathcal{U}}}$ modules, then $\bigoplus_{i=1}^k M_i$ is a projective$_{\underline{\mathcal{U}}}$ module.*

Proof. Let $M_1, M_2, ...M_k$ be projective$_{\underline{\mathcal{U}}}$ modules, for $i = 1, 2, ..., k$. Hence, there are $\underline{\mathcal{U}}$-free modules $N_1, ...N_k$ and M_i is a direct summand of N_i, for $i = 1, 2, ..., k$. For $i = 1, 2, ..., n$, we have

$$V_i \xrightarrow{f_i} N_i \to 0$$

and

$$0 \to X_i \xrightarrow{g_i} N_i$$

are exact, where X_i and V_i are submodules of $\oplus_\Lambda U_\lambda$. Now, we define $f = \prod_{i=1}^n f_i$, and $g = \prod_{i=1}^n g_i$. Therefore the sequences

$$\oplus_i^n V_i \xrightarrow{f} \oplus_i^n N_i \to 0$$

and

$$0 \to \oplus_i^n X_i \xrightarrow{g} \oplus_i^n N_i$$

are exact. Hence $\oplus_i^n N_i$ is a $\underline{\mathcal{U}}$-free module and $\oplus_i^n M_i$ is a direct summand of $\oplus_i^n N_i$, implying that $\oplus_i^n M_i$ is a projective$_{\underline{\mathcal{U}}}$ module. □

As a consequence of Proposition 4, we have the following property of projective$_{\mathcal{U}}$ and strictly projective$_{\mathcal{U}}$ module.

Corrolary 3 *Given a family of R-modules, $\mathcal{U} = \{U_\lambda\}_\Lambda$, and V_i are submodules of $\oplus_\Lambda U_\lambda$, for $i = 1, 2, ..., k$. If $M_1, M_2, ...M_k$ are projective$_{\mathcal{U}}$ modules, then $\bigoplus_{i=1}^k M_i$ is a projective$_{\mathcal{U}}$ modules.*

Proof. Let $M_1, M_2, ...M_k$ be projective$_{\mathcal{U}}$ modules, for $i = 1, 2, ..., k$. Hence, there are $\mathcal{U}$-free modules $N_1, ...N_k$ and M_i is a direct summand of N_i, for $i = 1, 2, ..., k$. For $i = 1, 2, ..., n$, we have

$$V_i \to N_i \to 0$$

and

$$0 \to X_i \to N_i$$

are exact, where X_i and V_i are submodules of $\oplus_\Lambda U_\lambda$. Therefore the sequences

$$\oplus_i^n V_i \to \oplus_i^n N_i \to 0$$

and

$$0 \to \oplus_i^n X_i \to \oplus_i^n N_i$$

are exact. Hence $\oplus_i^n N_i$ is a $\mathcal{U}$-free module and $\oplus_i^n M_i$ is a direct summand of $\oplus_i^n N_i$, implying that $\oplus_i^n M_i$ is a projective$_{\mathcal{U}}$ module. □

Corrolary 5 *Given a family of R-modules, $\mathcal{U} = \{U_\lambda\}_\Lambda$, and V_i are submodules of $\oplus_\Lambda U_\lambda$, for $i = 1, 2, ..., k$. If $M_1, M_2, ...M_k$ are strictly projective$_\mathcal{U}$ modules, then $\bigoplus_{i=1}^k M_i$ is a strictly projective$_\mathcal{U}$ modules.*

Proof. Let $M_1, M_2, ...M_k$ be strictly projective$_\mathcal{U}$ modules, for $i = 1, 2, ..., k$. Hence, there are strictly $\mathcal{U}$-free modules $N_1, ...N_k$ and M_i is a direct summand of N_i, for $i = 1, 2, ..., k$. For $i = 1, 2, ..., n$, we have

$$V_i \to N_i \to 0$$

and

$$0 \to X_i \to N_i$$

are exact, where X_i and V_i are submodules of $\oplus_\Lambda U_\lambda$. Therefore the sequences

$$\oplus_i^n V_i \to \oplus_i^n N_i \to 0$$

and

$$0 \to \oplus_i^n X_i \to \oplus_i^n N_i$$

are exact. Hence $\oplus_i^n N_i$ is a strictly $\mathcal{U}$-free module and $\oplus_i^n M_i$ is a direct summand of $\oplus_i^n N_i$, implying that $\oplus_i^n M_i$ is a strictly projective$_\mathcal{U}$ module. □

Furthermore, we show that the properties of the family $\mathcal{U}$ of R-modules impact the properties of $\mathcal{U}$-projective module. Before that, we review the definition of the V-injective module. If given two modules over ring R, say V and L. The module L is V-injective if and only if for every monomorphism f from K to V, and for every homomorphism g from K to L, there exists a homomorphism h from V to L such that $h \circ f = g$. We can see this condition in the following commutative diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow[f]{} & V \\ & & \Big\downarrow g & & \\ & & L & & \end{array}$$

A module L is V-injective if and only if for every monomorphism f and homomorphism g, we can find a homomorphism h such that the diagram is commutative, i.e., $g = h \circ f$ (Wisbauer, 1991).

Proposition 5 *Let M be a direct summand of a $\underline{\mathcal{U}}$-free module N with $\underline{\mathcal{U}}$-basis (X, V), so that M is a projective$_{\underline{\mathcal{U}}}$ module. If L is V-injective, then L is M-injective.*

Proof. Assume that M is a direct summand of N, where N is a $\underline{\mathcal{U}}$-free module with basis (X, V). So, we have the following exact sequences:

$$0 \to X \to N$$

and

$$V \xrightarrow{h} N \to 0. \tag{1}$$

Let L be a module over a ring R. Assume that L is V-injective. We will show that L is M-injective.

Since M is a direct summand of N, there exist an epimorphism $p : N \to M$. From (1), we have the following diagram:

$$\begin{array}{ccc} V & & \\ \Big\downarrow h & & \\ N & \xrightarrow[p]{} & M \end{array}$$

Based on the diagram, we can define a homomorphism $g : V \to M$, where $g = p \circ h$. Since p and h are epimorphisms, g is an epimorphism. Therefore, we have the following exact sequence:

$$0 \to \ker(g) \to V \xrightarrow{g} M \to 0.$$

If L is V-injective, then L is M-injective (Wisbauer, 1991). So, for every monomorphism ψ from K to L and homomorphism μ from K to M, we can find a homomorphism δ from M to L such that the diagram is commutative, i.e., $\mu = \delta \circ \psi$

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow[\psi]{} & M \\ & & \Big\downarrow \mu & & \\ & & L & & \end{array}$$

□

Proposition 6 *Let M be a direct summand of a $\underline{\mathcal{U}}$-free module N with $\underline{\mathcal{U}}$-basis (X, V), so that M is a projective$_{\underline{\mathcal{U}}}$ module. If V is semisimple, then the module M is semisimple.*

Proof. Assume that M is a direct summand of N, where N is a $\underline{\mathcal{U}}$-free module with basis (X, V). So, we have the following exact sequences:

$$0 \to X \to N$$

and

$$V \xrightarrow{f} N \to 0. \tag{2}$$

Let L be a module over a ring R. Assume that L is V-injective. We will show that L is M-injective.

Since M is a direct summand of N, there exist an epimorphism $\pi : N \to M$. From (2), we have the following diagram:

$$\begin{array}{ccc} V & & \\ \downarrow f & & \\ N & \xrightarrow[\pi]{} & M \end{array}$$

Based on the diagram, we can define a homomorphism $k : V \to M$, where $k = \pi \circ f$. Since π and f are epimorphisms, k is an epimorphism. Therefore, we have the following exact sequence:

$$0 \to \ker(k) \to V \xrightarrow{k} M \to 0$$

is exact. Based on (Wisbauer, 1991), if V is semisimple, then M is semisimple. □

Proposition 7 *Let M be a direct summand of a $\underline{\mathcal{U}}$-free module N with $\underline{\mathcal{U}}$-basis (X, V), so that M is a projective$_{\underline{\mathcal{U}}}$ module. If V is Noetherian, then M is Noetherian.*

Proof. Assume that M is a direct summand of N, where N is a $\underline{\mathcal{U}}$-free module with basis (X, V). So, we have the following exact sequences:

$$0 \to X \to N$$

and

$$V \xrightarrow{f} N \to 0. \tag{3}$$

Let L be a module over a ring R. Assume that L is V-injective. We will show that L is M-injective.

Since M is a direct summand of N, there exist an epimorphism $\pi : N \to M$. From (3), we have the following diagram:

$$\begin{array}{ccc} V & & \\ \downarrow f & & \\ N & \xrightarrow[\pi]{} & M \end{array}$$

Based on the diagram, we can define a homomorphism $k : V \to M$, where $k = \pi \circ f$. Since π and f are epimorphisms, k is an epimorphism. Therefore, we have the following exact sequence:

$$0 \to \ker(k) \to V \xrightarrow{k} M \to 0.$$

We have assumed that module V is Noetherian, hence based on (Wisbauer, 1991), the module M is Noetherian. □

Proposition 8 *Let M be a direct summand of a $\underline{\mathcal{U}}$-free module N with $\underline{\mathcal{U}}$-basis (X, V), so that M is a projective$_{\underline{\mathcal{U}}}$ module. If V is Artinian, then M is Artinian.*

Proof. Assume that M is a direct summand of N, where N is a $\underline{\mathcal{U}}$-free module with basis (X, V). So, we have the following exact sequences:

$$0 \to X \to N$$

and

$$V \xrightarrow{f} N \to 0. \tag{4}$$

Let L be a module over a ring R. Assume that L is V-injective. We will show that L is M-injective.

Since M is a direct summand of N, there exist an epimorphism $\pi : N \to M$. From (4), we have the following diagram:

$$\begin{array}{ccc} V & & \\ \downarrow{\scriptstyle f} & & \\ N & \xrightarrow[\pi]{} & M \end{array}$$

Based on the diagram, we can define a homomorphism $k : V \to M$, where $k = \pi \circ f$. Since π and f are epimorphisms, k is an epimorphism. Therefore, we have the following exact sequence:

$$0 \to \ker(k) \to V \xrightarrow{k} M \to 0.$$

Since V is Artinian, based on (Wisbauer, 1991), we have the module M over a ring R is Artinian. □

We recall that a submodule N of R-module M is called fully invariant if $f(N)$ is contained in N for every R-endomorphism f of M. M is called a duo module provided every submodule of M is fully invariant (Özcan et al., 2006).

Proposition 9 *Let M be a direct summand of a $\underline{\mathcal{U}}$-free module N with $\underline{\mathcal{U}}$-basis (X, V), so that M is a projective$_{\underline{\mathcal{U}}}$ module. If V is a duo module, quasi-injective and quasi-projective, then M is a duo module and V-injective.*

Proof. Assume that V is a duo module and a quasi-injective. Hence, according to the properties of duo modules in (Özcan, 2006), we get every submodule of V is a duo module. Furthermore, since V is a duo module and quasi-projective, we get every homomorphic image of V is a duo module. Using the properties in Proposition 5, we can conclude that M is the V-injective module. □

Based on Proposition 5, we can determine the properties of a M-injective module from the projective$_{\mathcal{U}}$ module concept. Then, by Proposition 5-9, we can determine the properties of M-injective module, Noetherian module, Artinian module, and duo module by using the projective$_{\mathcal{U}}$ module concept.

3. Conclusions

The projective$_{\mathcal{U}}$ module is a generalization of the projective module. The direct sum of some projective$_{\mathcal{U}}$ modules is projective$_{\mathcal{U}}$. Some properties of projective$_{\underline{\mathcal{U}}}$, projective$_{\mathcal{U}}$ module, and strictly projective$_{\mathcal{U}}$ module according to the properties of the family $\mathcal{U}$ of R-modules. The properties are related to the injective module, projective module, Noetherian module, Artinian Module, and duo module.

For further research, the concept of projective$_{\underline{\mathcal{U}}}$ module can be applied to generalize Schanuel's Lemma. For this generalization, we need to characterize the projective$_{\underline{\mathcal{U}}}$ module by using the split exact sequence.

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