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Numerical Solution of Linear Parabolic Equation With Rational Coefficients

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Abstract

In this paper, we present explicit scheme for solving rational coefficient (which depends only on space variable) parabolic equation. The explicit scheme is required some restriction on step size ratio $\frac{k}{h^2}, \to 0$ in stability, where k and h are step sizes for space and time respectively. In this paper, we will present the explicit scheme is sable without restriction on the step size ratio $\frac{k}{h^2}$. We also show the scheme converge to true solution under some conditions on coefficient.

Keywords: Runge-Kutta methods, method of lines, difference equation, parabolic equation

1. Introduction

A number of difference schemes for solving partial difference equations have been proposed. E. C. Du Fort and S. P. Frankel(reference.1953) and some others have proposed difference schemes based on methods of lines. However, in using the explicit lines methods, stability of algorithms is a serious problems for the step size ration of space and time. We(reference.2001,2002,2015) have proposed some explicit difference schemes by using the idea of methods of lines and overcome this problems for solving the parabolic equation. In this paper,we study the numerical method for solving the parabolic equation:

$$\frac{\partial u(x,t)}{\partial t} = a(x,t)\frac{\partial^2}{\partial x^2}u(x,t) \tag{1.1}$$

$$a(x,t) = \frac{cx+d}{ax+b} \tag{1.2}$$

$$\text{(x,t)} \in \Omega = \{(x,t); 0 \le x \le x_f, 0 \le t \le t_f\},$$

with the initial Dirichlet boundary condition

u(x,t) = { f(t) (0,t)$\in \partial\Omega \cup \Omega 0$. $(1,t) \in \partial\Omega \cup \Omega$.. (1.3)

In the usual schems, it is required the condition of step size ratio

$$\frac{k}{h^2} \to 0 \; as \; h,k \to 0,$$

in the convergence, where h and k for space and time respectively. In this paper, we propose the difference approximation to (1.1) where the step size ratio is defined by

$$\frac{k}{h^2} = c_0. \qquad (c_0 \text{ is any positive constant}) \tag{1.4}$$

The outline of this paper is as follows. In §2, by using idea of methods of lines, we present the explicit difference approximation to (1.1). In §3, we study the truncation errors of our scheme. In §4, we study the convergence of the scheme with the condition (1.4) and we will show that our scheme converges to the true solution of (1.1). In §5, we study stability of the scheme, and we will show that our scheme is stable for any step size k and h with the condition (1.4).

2. Difference Scheme

We will approximate (1.2) by replacing the derivative for space and time in the difference operator

$$\frac{\partial^2}{\partial x^2}u(x,t) \cong \frac{1}{h^2}\delta^2(u(x,t)),$$

$$\frac{\partial u(x,t)}{\partial t} \cong \frac{1}{k}\triangle u(x,t),\ \frac{1}{k}\nabla u(x,t), \tag{2.1}$$

where δ is the central difference operator , $\triangle$ forward difference operator,∇ backward difference operator. We divide $x-$space to N_1 points, $t-$space to N_2 points where h and k are the mesh size for $x-$space,$t-$space respectively. We denote the approximation to (1.1) at the mesh point $(x,t) = (jh, nk)$

$$u_j^n \cong u(jh, nk).$$

By using the idea proposed in (reference.2001,2002,2015),we define the difference approximation to (1.2) by the following scheme

$$u_j^{n+1} = u_j^n + \frac{c_0 a(j,n)}{(1+2\hat{c}_0 L_1)}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n). \tag{2.2}$$

where

$$\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) = \{u_{j-1}^n - 2\ u_j^n + u_{j+1}^n\},$$
$$L_1 = Max\{a(x,t); 0 < x \le x_f, 0 < t \le t_f\},$$
$$\hat{c}_0 = \frac{\hat{k}}{h^2}. \tag{2.3}$$

where the step size $\hat{k}$ in (2.3) is defined by

$$\hat{k} = k^{1+\tilde{\rho}}. \quad (0 < \tilde{\rho} \le 1) \tag{2.4}$$

3. Truncation Error

We define the truncation error $T(jh, nk)$ of (2.2),(2.3)

$$T(jh, nk) = u(jh, (n+1)k) - u(jh, nk)$$
$$- \frac{c_0}{(1+\hat{c}_0 L_1)}\Phi(u((j-1)h, nk), u(jh, nk), u((j+1)h, nk)). \tag{3.1}$$

We have used Taylor series expansion of (3.1). We have the following result.

Theorem [1] The truncation error of the difference approximation (2.2),(2.3) to (1.2) is given by

$$T(jh, nk) = k^{1+\tilde{\rho}} w(jh, nk), \tag{3.2}$$

where

$$w(jh, nk) = \frac{2c_0 L_1}{(1+2\hat{c}_0 L_1)} a(j,n) u_{xx}(j,n) + O(h^4) \tag{3.3}$$

4. Convergence

In this section, we study the convergence of the scheme (2.2). We set the approximation error by

$$e(jh, nk) = u(jh, nk) - u_j^n. \tag{4.1}$$

We use the abbreviation's

$$e_j^n = e(jh, nk),$$
$$T_j^n = T(jh, nk),$$
$$u(j,n) = u(jh, nk),$$
$$a(j,n) = a(jh, nk). \tag{4.2}$$

We set

$$p = \frac{k}{(1+2\hat{c}_0 L_1)}.$$
$$\rho = \tilde{\rho}(\frac{logk}{logh}).$$

From (2.2),(2.3),(3.1),(4.1), we have

$$e_j^{n+1} \;=\; e_j^n \;+\; p\, a(j,n)D^2[e_j^n], + T_j^{n+1}. \tag{4.3}$$

We set the initial conditions of (4.3)

$$e_j^0 = 0,$$

$$e_j^1 = T_j^1. \quad (0 < j < 1/h)$$

From (4.3), we have

$$e_j^2 = e_j^1 + pa(j,1)D^2[e_j^1] \;+\; T_j^2$$

$$= \sum_{l=1}^{2} T_j^l \;+ pa(j,1)D^2[T_j^1],$$

$$e_j^3 = \; e_j^2 + p(a,2)D^2[e_j^2] \;+\; T_j^3$$

$$= \sum_{l=1}^{3} T_j^l \;+ pa(j,2)D^2[\sum_{l=1}^{2} T_j^l] + p^2\{a(j,2)D^2[a(j.1)D^2[T_j^1]]\}.$$

$$e_j^n = \sum_{l=1}^{n} T_j^l \;+ p\{\sum_{l_1=1}^{n-1} pa(j,l_1)D^2[\sum_{l=1}^{l_1} T_j^{j_l}]\}$$

$$+\; p^2\{\sum_{l_1=1}^{n-2}\sum_{l_2=2}^{n-1}\{a(j,l_2)\; D^2[a(j,l_1)\sum_{l=1}^{l_1} T_j^l]\}\} + ...+$$

$$+\; p^s\{\sum_{l_1=1}^{n-s}\sum_{l_1<l_2<l_3<,,,<l_{s-1}<n-1} D^2[a(j,l_{s-1})D^2[a(j,l_{s-2})..D^2[a(j,l_1)])\sum_{l=1}^{l_1} T_j^l]]].]\} + ...+$$

$$p^{n-1}\{a(j,n-1)D^2[a(j,n-2)D^2[a(j,n-3)...D^2[a(j,1)D^2[T_j^1]...]]\}. \tag{4.4}$$

We set the propagation of $\sum_{l=1}^{m_t} T_j^l$ $(m_t = 1,2,3,4,..,n-1)$ by $e_{j,m_t}^n(\sum_{l=1}^{m_t} T_j^l)$

.

From (4.4), we have

$$e_j^n \;=\; e_{j,1}^n(T_j^1) \;+\; e_{j,2}^n(\sum_{l=1}^{2} T_j^1) \;+\; e_{j,3}^n(\sum_{l=1}^{3} T_j^1) \;+\; e_{j,n}^n(\sum_{l=1}^{n} T_j^1). \tag{4.5}$$

with

$$e_{j,1}^n(T_j^1) \;=\; e_{1,1}^n(p) + e_{2,1}^n(p^2) + ... + e_{n-1,1}^n(p^{n-1}).$$

and

$$e_{1,1}^n(p) = \sum_{l_1=1}^{n-1} a(j,l_1)D^2[T_j^1],$$

$$e_{2,1}^n(p^2) = \sum_{l_2=2}^{n-1} a(j,l_2)D^2[a(j,1)D^2[T_j^1]],$$

$$e_{j,1}^n(p^s) \;=\; e_{j,1}^{n-1}(p^s) \;+\; p(q + g(j,n-1))D^2[e_{j,1}^{n-1}(p^{(s-1)})]$$

$$=\; \sum_{l_p=s}^{n-1} a(j,l_p)\sum_{1<l_2<l_3<..<l_s<l_p} D^2[a(j,l_s)[D^2a(j,l_{ls-1})....[D^2[a(j,1)D^2[T_j^1]]..]]]. \tag{4.6}$$

We set

$$e_{j,m_t}^n \;=\; e_j^n(\sum_{l=1}^{m_t} T_j^l).$$

Then, we have

$$e^n_{j,m_t} = e^n_{1,m_t}(p) + e^n_{2,m_t}(p^2) + ... + e^n_{n-m_t,m_t}(p^{n-m_t}),$$

where

$$e^n_{j,m_t}(p) = \sum_{l_1=m_t}^{n-1} a(j,l_1)D^2[\sum_{l=1}^{m_t} T^1_j],$$

$$e^n_{j,m_t}(p^2) = \sum_{l_1=m_t}^{n-m_t}\sum_{l_1<l_2} a(j,l_2)D^2[a(j,l_1)D^2[\sum_{l=1}^{m_t} T^1_j]],$$

$$e^n_{j,m_t}(p^s) = e^{n-1}_{j,m_t}(p^s) + p(q+g(j,n-1))D^2[e^{n-1}_{j,m_t}(p^{(s-1)})]$$

$$= \sum_{l_p=s}^{n-1} a(j,m_t) \sum_{m_t<l_2<l_3<<l_p<s<n-1} D^2[a(,j,l_p)[D^2a(j,l_{lp-1})....[D^2[a(j,m)D^2[T^1_j]]..]]],$$

$$(n \geq m_t)(m_t = 2,3,..,n-1)$$

$$e^n_{j,n}(\sum_{l=1}^{n} T^l_j) = \sum_{l=1}^{n} T^l_j. \tag{4.7}$$

We have the coefficient of differential equation (1.2) in the following formula

$$a(j,l) = \frac{c(x+jh)+d}{a(x+jh)+b}$$

$$= \frac{cx+d_j}{ax+b_j}$$

with

$$b_j = b + ajh,\ d_j = d + cjh.$$

We define

$$q = \frac{c}{a}\ , g(j,l) = \frac{ad-bc}{a(ax+b_j)} \quad (r = |ad-bc|),$$

then we have

$$a(j,l) = q + g(j,l).(l = 1,2,3,..) \tag{4.8}$$

Through the paper, we study under the following hypotheses

$$(H1) \qquad o < q \leq 1, \quad 0 < a \leq 1,\ 1 \leq b,\ 0 \leq r \leq 1. \tag{4.9}$$

We define

$$D^2[\sum_{l=1}^{m_t} T^l_j] = Q[\sum_{l=1}^{m_t} T^l_j].$$

$$D^2[g(j,l_1)D^2[\sum_{l=1}^{m_t} T^l_j]] = Q[\sum_{l=1}^{m_t} T^l_j, g(j,l_1)].$$

$$D^2[g(j,l_2)D^2[g(j,l_1)D^2[\sum_{l=1}^{m_t} T^l_j]] = Q[\sum_{l=1}^{m_t} T^l_j, g(j,l_1), g(j,l_2)]. \tag{4.10}$$

From (4.6), we have the approximation errors $e^n_{j,1}(p)$ which consists of the factor $Q[T^1_j]$.

$$e^2_{j,1}(p) = p(q+g(j,1))Q[T^1_j].$$

$$e^3_{j,1}(p) = e^2_j(p) + p(q+g(j,1))D_2[e^{(2)}_{j,1}(p^{(0)})]$$

$$= e^2_{j,1}(p).$$

$$e^m_{j,1}(p) = p(q+g(j,1))D^2[T^1_{j,}]$$

$$= e^2_{j,1}(p). \qquad (m = 4,5,..,n) \tag{4.11}$$

From (4.11), we have

$$|e^n_{j,1}(p)| \le 2pQ[T^1_j]. \tag{4.12}$$

From (4.6), we have the approximation errors $e^n_{j,1}(p^2)$ which consists of the factor $Q[T^{(1)}_j, g(j,1)]$ or $Q[T^1_j, c_s]$,where we set

$$c_s = 1. \tag{4.13}$$

We study the value of $Q[T^1_j, g(j,1)]$, $Q[T^1_j, c_s]$ in $e^n_j(p^2)$. From (4.6), We have

$$\begin{aligned} e^3_{j,1}(p^2) &= e^2_{j,1}(p^2) + p(q + g(j,2))D^2[e^2_{j,1}(p)] \\ &= p(q + g(j,2))D^2[e^2_{j,1}(p)] \\ &= p^2(q + g(j,2))D^2[(q + g(j,1))D^2[T^1_j]] \\ &= p^2(q + g(j,2))(Q[T^{(1)}_j, g(j,1)] + qQ[T^1_j, c_s]). \end{aligned} \tag{4.14}$$

We assume

$$|Q[T^1_j, u(j,1)]| \le Q[1, s_1]. \tag{4.15}$$

where $u(j,1) = g(j,1)$ or c_s. From (4.14), we have

$$|e^3_{j,1}(p^2)| \le 2^2p^2Q[T^1_j, s_1].$$

Thruogh this paper, it is assumed

$$|Q[T^1_j, u(j,1), u(j,2), ..., u(j,m-1)]| \le Q[T^1_j, s_1, s_2, s_3, ..., s_{m-1}] \ (m = 1,2,..,n). \tag{4.16}$$

where $u(j,1) = g(j,1)$ or c_s. From (4.6), we have

$$\begin{aligned} e^4_{j,1}(p^2) &= e^3_{j,1}(p^2) + p(q + g(j,3))D^2[e^3_{j,1}(p)]. \\ &= e^3_{j,1}(p^2) + p(q + g(1,3))(qQ[T^1_j, c] + Q[T^1_j, g_1]. \end{aligned} \tag{4.17}$$

From (4.17),we have

$$|e^4_{j,1}(p^2)| \le 2^2p^2Q[T^1_j, s_1] + |p^2(q + g(j,3))(qQ[T^1_j, c] + Q[T^1_j, g_1])|.$$

If we assume (4.9),(4.16), we have

$$|e^4_{j,1}(p^2)| \le 2^2p^2(1 + 1)Q[T^1_j, s_1].$$

From (4.6), we have

$$e^m_{j,1}(p^2) = e^{m-1}_{j,1}(p^2) + p(q + g(j,m-1))D^2[e^{m-1}_{j,1}(p)]. \tag{4.18}$$

The approximation errors $e^m_{j,1}(p^2)$ consists of the factors $Q[T^1_j, g(j,1)]$ or $Q[T^1_j, c_s]$ with $(m-2)$ terms. If we assume (4.9),(4.16),then we have

$$|e^m_{j,1}(p^2)| \le 2^2p^2(m-2)Q[T^1_j, s_1]. \ (m = 3,4,..,n) \tag{4.19}$$

From (4.6),we have

$$\begin{aligned} e^4_{j,1}(p^3) &= e^3_{j,1}(p^3) + p(q + g(j,3))D^2[e^3_{j,1}(p_2)] \\ &= p(q + g(j,3))D^2[e^2_{j,1}(p^2)] \\ &= p^3(q + g(j,3))D^2[(q + g(j,2))D^2[(g(1) + q)[D^2[T^{(1)}_j]]] \\ &= p^3(q + g(j,3))(Q[T^1_j, g(1), g(2)] + q(Q[T^1_j, g(1), c_s] + Q[T^1_j, c_s, g(2)] + q^2Q[T^1_j, c_s, c_s]). \end{aligned}$$

If we assume (4.9) and (4.16), we have

$$|e^4_{j,1}(p^3)| \le 2^3p^3Q[T^1_j, s_1, s_1].$$

From (4.6), we have

$$e^5_{j,1}(p^3) = e^4_{j,1}(p^3) + p(q + g(j,4))D^2[e^4_{j,1}(p^2)]. \tag{4.20}$$

If we assume (4.9),(4.16),we have

$$|e^5_{j,1}(p^3)| \leq \ 2^3 p^3 Q[T^1_j, s_1, s_1)] \ + \ p|(q + g(j, 4))D^2[e^4_{j,1}(p^2)|. \tag{4.21}$$

From (4.21), we have

$$|e^5_{j,1}(p^3)| \leq \ 2^3 p^3 (1 + 1)Q[T^1_j, s_1, s_2]. \tag{4.22}$$

From (4.6), we have

$$e^m_{j,1}(p^3) \ = \ e^{m-1}_{j,1}(p^3) \ + \ p(q + g(j, (m - 1)))D^2[e^{m-1}_{j,1}(p^2)]. \tag{4.23}$$

In the methods same to (4.19),we have

$$|e^m_{j,1}(p^3)| \leq \ 2^3 p^3 (1 + 2 + .. + (m - 3))Q[T^1_j, s_1, s_2]. \ \ (m = 4, .., n) \tag{4.24}$$

We study the value of $e^n_{j,1}(p^4)$. From (4.6), we have

$$e^m_{j,1}(p^4) \ = \ e^{m-1}_{j,1}(p^4) \ + \ p(q + g(j, (m - 1)))D^2[e^{m-1}_j(p^3)], \tag{4.25}$$

From (4.25),we have

$$\begin{aligned} e^5_{j,1}(p^4) \ &= \ e^4_{j,1}(p^4) \ + \ p(q + g(j, 4))D^2[e^4_{j,1}(p^3)] \\ &= \ p(q + g(j, 4))D^2[e^4_{j,1}(p^3)] \\ &\leq \ 2^4 p^4 Q[T^1_j, s_1, s_2, s_3]. \end{aligned}$$

In the method same to (4.19),we have

$$|e^n_{j,1}(p^4)| \leq \ 2^4 p^4 (v_1 + v_2 + v_3 + .. + v_{n-4})Q[T^1_j, s_1, s_2, s_3]. \tag{4.26}$$

where we set

$$u_m = \sum_{k=1}^{m} k, \ \ v_m = \sum_{k=1}^{m} u_k, \ \ w_m = \sum_{k=1}^{m} v_k. \tag{4.27}$$

We study the value of $e^n_{j,1}(p^l)$ $(l \geq 5)$. From (4.6), we have

$$e^{m+1}_{j,1}(p^l) \ = \ e^m_{j,1}(p^l) \ + \ p(q + g(j, l))D^2[e^m_{j,1}(p^{l-1})]. \tag{4.28}$$

In the method same to (4.19), we have

$$|e^n_{j,1}(p^l)| \leq \ 2^l p^l (w_1 + w_2 + ... + w_{n-l})Q[T^1_j, s_1, s_2.s_3, s_4, .., s_{l-1}]. \tag{4.29}$$

where $w_i (i = 1, 2, ., n - l)$ are defined by (4.27).

We study the value of factor $Q[T^1_j, u_1, u_2, u_3, ..u_{m-1}]$ where $u_l = g(j, l)$ or c_s.

Through the paper,we set

$$g(j, t)^{(n)} = \frac{\partial^n}{\partial x^n} g(x, t), T^{1,(n)}_j = \frac{\partial^n}{\partial x^n} T^1_j(x, t), g(j, t)^0 = g(j, t), T^{1,(0)}_j = T^1_j.$$

We study the expansion of $Q[T^1_j]$, $Q[T^1_j, g(j, 1), c_s]$ and $Q[T^1_j, g(j, 1), g(j, 2)]$.

$$Q[T_j] = T^{1(2)}_j,$$

$$Q[T^1_j, g(1)] = D^2[g(j, 1)]T^1_j + 2D[g(j, 1)]T^{1,(1)}_j + g(j, 1)])T^{1,(2)}_j,$$

$$\begin{aligned} Q[T^1_j, g(j, 1), c_s] \ &= \ D^2[Q[T^{1,(2)}_j, g(j, 1)]] \\ &= D^2[D^2[[g(j, 1)]]T^{1,(2)}_j + 2D[g(j, 1)]T^{1,(3)}_j + g(j, 1)T^{1,(4)}_j, \end{aligned}$$

$$\begin{aligned} Q[T^1_j, g(j, 1), g(j, 2)] \ = \ & D^2[g(j, 2)]D^2[g(j, 1)]T^1_j \ + \ 2D^2[g(j, 2)]D[g(j, 1)]D[T^1_j] \\ &+D^2[g(j, 2)]g(j, 1)D^2[T^1_j] + 2D[g(j, 2)]\{D^3[g(j, 1)]T^1_j + D^2[g(j, 1)]D[T^1_j] + 2D^2[g(j, 1)]D[T^1_j] \\ &+2D[g(j, 1)]D^2[T^1_j] + D[g(j, 1)D^2[T^1_j] + g(j, 1)D^3[T^1_j]\} + g(j, 2)\{D^4[g(j, 1)]T^1_j \end{aligned}$$

$$+4D^3[g(j,1)]D[T_j^1] + 6D^2[g(j,1)]D^2[T_j^1] + 4D[g(j,1)]D^3[T_j^1] + g(j,1)D^4[T_j^1]\}. \tag{4.30}$$

From (4.30), we have the factor $Q[T_j]$ consists of terms

$$\{T_j^{1,(2)}\}.$$

The factor $Q[T_j^1, g(j,1)]$ consists of the terms

$$\{g(j,1)^{(n_1)}T_j^{1,(2+n_2)};\ n_1 + n_2 = 2, 0 \leq n_1, n_2 \leq 2\}.$$

The factor $Q[T_j^1, g(j,1), c_s]$ consists of terms

$$\{g(j,1)^{(n_1)}T_j^{1,(2+n_2)};\ n_1 + n_2 = 4, 0 \leq n_1, n_2 \leq 4\}.$$

The factor $Q[T_j^1, g(j,1), g(j,2)]$ consists of the terms

$$\{g(j,1)^{(n_1)}g(j,2)^{n_2}T_j^{1,(2+n_3)},\ n_1 + n_2 + n_3 = 4, 0 \leq n_1, n_2, n_3 \leq 4\}. \tag{4.31}$$

.

The factor $Q[T_j^1, u(j,1), u(j,2), u(j,3), ..., u(j,(m-1)) : u(j,k) = g(j,l) \text{ or } c_s]$ (4.32)
consists of terms

$$\{T_j^{1,(n_s+2)}g(j,l_1)^{(n_1)}g(j,l_2)^{(n_2)}, .., g(j,l_{s-1})^{(n_{s-1})}; n_1 + n_2 + .. + n_s = 2m, 0 \leq n_1, n_2, , n_s \leq 2m\},$$

or

$$\{T_j^{1,(2m)}\}. \tag{4.33}$$

We use the abbreviation of (4.32) by Q_m .

We set the number of term

$$T_j^{1,(n_s)}g(j,l_1)^{(n_1)}g(j,l_2)^{(n_2)}, , ., g(j,l_{s-1}^{(n_{s-1})}) \text{ in } Q_m$$

by

$$N[Q[T_j^1 g(j,l_1)g(j,l_2)g(j,l_3), .., g(j,l_{s_{(l-1)}})]].$$

From (4.31), we have

$$N[Q[T_j^1, g(j,1)]] = 2^2.$$

$$N[Q[T_j^1, g(j,1), c_s]] = 2^4.$$

$$N[Q[T_j^1, g(j,1), g(j,2)]] = 2^2 * 3^2.$$

If we assume

$$N[Q[T_j^1, g(j,1), g(j,2), .., g(j,m-1)]] \leq 2^2 * 3^{2(m-2)} < 3^{2(m-1)}.$$

Then, from (4.31), we have

$$N[Q[T_j^1, g(j,1), g(j,2), .., g(j,m)] = N[(D^2[g(j,m))][Q[T_j^{1,(2)}, g(j,1), .., g(j,m-1)]$$

$$+2N[D[g(j,m)]D[Q[T_j^{1,(2)}, g(j,1), .., g(j,m-1)]] + N[g(j,m)D^2[Q[T_j^{1,(2)}, g(j,1), .., g(j,m-1)]]$$

$$\leq 3^{2(m-1)} + 2^2.3^{2(m-1)} + 2^2 3^{2(m-1)}.$$

We have

$$N[Q[T_j^1, g(j,1), g(j,2), .., g(j,m)]] < 3^{2m}. \tag{4.34}$$

We study under the following hypotheses

$$(H2) \qquad |D^n(T_j^1)| \leq |D^n(\frac{1}{ax+b})|. \qquad (0 < b < b_j)$$

Then we have

$$|D^n(T_j^1)| \leq \frac{n!a^n}{(ax+b)^{n+1}}. \tag{4.35}$$

From (4.30),(4.35),we have

$$|Q[T_j^1, g(j,1)]| \leq \frac{2!a}{(ax+b_j)^3}\frac{2!a^2}{(ax+b)^3} + 2\frac{1}{(ax+b_j)^2}\frac{3!a^3}{(ax+b)^4} + \frac{1}{(ax+b_j)}\frac{4!a^3}{(ax+b_j)^5}.$$

we set

$$R[T_j^1, g(j,1)] = \frac{2!a}{(ax+b_j)^3}\frac{2!a^2}{(ax+b)^3} + 2\frac{1}{(ax+b_j)^2}\frac{3!a^3}{(ax+b)^4} + \frac{1}{(ax+b_j)}\frac{4!a^3}{(ax+b_j)^5}.$$

Then we have

$$Q[T_j^1, g(j,1)] \leq R[T_j^1, g(j,1)]. \tag{4.36}$$

The Factor $Q[T_j^1, g(j,1)]$ is bound by the rational function $R[T_j^1, g(j,1)]$.

From the inequality

$$|Q[T_j^1, c_s)| \leq \frac{4!a^4}{(ax+b)^5},$$

we set

$$R[T_j^1, c_s] = \frac{4!a^4}{(ax+b)^5}.$$

Then the Factor $Q[T_j^1, c_s]$ is bounded by the rational function $R[T_j^1, c_s]$.

We study the value of factor

$$g(j,l_1)^{n_1} g(j,l_2)^{n_2}, ., g(j,l_{s-1})^{n_{s-1}} T_j^{n_s}. \tag{4.37}$$

We use the abbreviation (4.37) by q_m .The value of q_m with

$$g(j,l_i) = \frac{1}{(ax+b_i)}(i = 1,2,.,(s-1)), n_1 = n_2 = ... = n_{s-1} = 0 \tag{4.38}$$

give maximum value in molecule of $R[q_m]$.

q_m with (4.38) is

$$\begin{aligned}|q_m| &\leq \frac{1}{a^m(ax+b_j)^m}\frac{(2m+2)!a^{2m+2}}{(ax+b)^{2m+3}} \\ &\leq \frac{(2m+2))!a^{m+2}}{(ax+b)^{3m+3}}.\end{aligned} \tag{4.39}$$

$R[q_m]$ with

$$g(j,l_i) = c_s \ (i = 1,2,..n_{l-1})) \tag{4.40}$$

give minimum value in denominator of $R[q_m]$.

q_m with (4.40) is

$$|q_m| \leq \frac{2(m+1)!a^{2(m+1)}}{(ax+b)^{2m+3}}. \tag{4.41}$$

From (4.39),(4.41),we set

$$R[T_j^{1,n_s}, g(j,l_1)\ n_1, g(j,l_2)^{n_2}, ., g(j,l_{l-1})^{n_{(s-1)}}] = \frac{(2m+2)!a^{m+2}}{(ax+b)^{2m+3}}. \tag{4.42}$$

Then,from (4.42), we have

$$|Q| \leq R[Q[T_j^1, g(j,1), g(j,2), ..g(j,m-1)]]. \tag{4.43}$$

We set $Q[T_j^{(1)}, s_1, s_2, s_3, ..., s_{m-1}]$ in (4.16) by

$$Q[T_j^{(1)}, s_1, s_2, s_3, ..., s_{m-1}] = 3^{2(m-1)}\frac{(2m)!a^{m+1}}{(ax+b)^{2m+1}}. \tag{4.44}$$

From (4.2), we set

$$p = \frac{k}{(1+2\hat{c}_0 L_1)} \leq c_0\ h^2. \tag{4.45}$$

From (4.12), (4.35),we have

$$\begin{aligned} |e^n_{j,1}(p)| &\leq |2pQ[T^1_j]| \\ &\leq 2h^2 c_0^2 \frac{2!a^2}{(ax+b)^3} h^{2+\rho} \\ &\leq C_{1,1} h^{2+\rho}, \end{aligned} \tag{4.46}$$

with

$$C_{1,1} = 2^2 h^2 c_0^2 (\tfrac{a^2}{b^3}).$$

We study under the following hypotheses

$$(H3) \qquad \begin{aligned} nk &\leq t_f, \\ nh &\leq x_f. \end{aligned} \tag{4.47}$$

From (4.19),(4.44), we have

$$\begin{aligned} |e^n_{j,1}(p^2)| &\leq p^2\ 2^2(n-2)R[T^1_j, s_1] \\ &\leq h^4 2^2 3^2 (n-2) c_0^3 \frac{4!a^3}{b^5} h^{2+\rho} \\ &\leq C_{2,1} h^{2+\rho}, \end{aligned} \tag{4.48}$$

with

$$C_{2,1} = 2^2 3^2 4! h^3 c_0^3 x_f (\frac{a^3}{b^5}).$$

From (4.24),(4.44),we have

$$\begin{aligned} |e^n_{j,1}(p^3) &\leq p^3\ 2^3(1+2+..+(n-3))Q[T^1_j, s_1, s_2] \\ &\leq 2^3 h^6 3^4 n^2 c_0^4 \frac{6!a^4}{b^7} h^{2+\rho} \\ &\leq C_{3,1} h^{2+\rho}, \end{aligned} \tag{4.49}$$

with

$$C_{3,1} = 2^3 3^4 6! h^4 c_0^4 x_f^2 \frac{a^4}{b^7}.$$

From (4.26),(4.44),we have

$$\begin{aligned} |e^n_{j,1}(p^4)| &\leq p^4\ 2^4(v_1+v_2+v_3+..+v_{n-4})Q[T^1_j, s_1, s_2, s_3] \\ &\leq 2^4 3^6 h^8 (n^3+n^2+O(n)) c_0^5 \frac{8!a^5}{b^9} h^{2+\rho} \\ &\leq C_{4,1} h^{2+\rho}, \end{aligned} \tag{4.50}$$

with

$$C_{4,1} = 2^4 3^6 8! h^5 c_0^5 (x_f^3 + x_f^2 + 0(h)) \frac{a^5}{b^9}.$$

We study $e^{(n)}_{j,1}(p^m)$. In the method same to (4.19),from (4.29),(4.44),we have

$$\begin{aligned} |e^n_{j,1}(p^m)| &< p^m\ 2^m\{w_1+w_2+w_3+..+w_{n-m}\} \\ & Q[T^1_j, s_1, s_2, s_3, s_4, ..s_{m-1}] \\ &\leq p^m 2^m c_0 \{\frac{1}{30}(n-m)(n-m+1)(2(n-m)+1)(3(n-m)^2+3(n-m)+1)+O((n-m)^4)\} \\ & 3^{2(m-1)} \frac{(2m)!a^{(m+1)}}{(ax+b)^{(2m+1)}} h^{2+\rho} \\ &< p^m (2)^m 3^{2(m-1)} c_0 \{(n-m)^5 + O((n-m)^4)\} \frac{(2m)!a^{(m+1)}}{b^{(2m+1)}} h^{2+\rho} \end{aligned}$$

$$< h^{2m}(c_0)^{m+1}2^m3^{2m}\{(n-m)^5+O((n-m)^4)\}\frac{(2m)!a^{(m+1)}}{b^{(2m+1)}}h^{2+\rho}$$

$$< h^{(2m-5)}(18)^m c^{m+1}x_f^5(2m)!\frac{a^{(m+1)}}{b^{(2m+1)}}h^{(2+\rho)}.(m=5,6,..,n-1) \quad (4.51)$$

Using the inequality

$$(2m!)h^{2m-5} = 5!(6h)(7h)..(mh)((m+1)h)((m+2)h)...(2mh)$$
$$\leq 5!x_f^{(2m-5)}2^{2m-5}.$$

From (4.51), we have

$$|e_{j,1}^{(n)}(p^m)| \leq \frac{5!ac_0}{b}\{\frac{72ac_0x_f^2}{b^2}\}^m h^{2+\rho}.$$

If we assume

$$|\frac{72ac_0x_f^2}{b^2}| \leq 1. \quad (4.52)$$

Then we have

$$|e_{j,1}^n(p^m)| \leq C_{m,1}h^{2+\rho}, (m=5,..,n-1) \quad (4.53)$$

with

$$C_{m,1} = \frac{5!ac_0}{b}.$$

We set

$$\tilde{C}_1 = \max_{1\leq m\leq n-1} C_{m,1}.$$

Then,from (4.46)-(4.53), we have we have

$$|e_{j,1}^n| \leq |e_{1,1}^n(p)| + |e_{2,1}^n(p^2)| + .. + |e_{n-1,1}^n(p^{n-1})|$$
$$\leq |C_{1,1}| + |C_{2,1}| + ... + |C_{n-1,1}|$$
$$\leq \tilde{C}_1\ x_f\ h^{1+\rho}. \quad (4.54)$$

We consider the propagation $e_j^n(\sum_{l=1}^{m_t} T_j^l)(m_t = 2,3,..,n-1)$.
We define

$$Q[\sum_{l=1}^{m_t} T_j^l, g_1, g_2, .., g_{m-1}] \leq m_t Q[T_j^1, g_1, g_2, .., g_{m-1}].\ (m_t \leq n-1) \quad (4.55)$$

From (4.12), (4.55),we have

$$|e_{j,m_t}^n(p)| \leq 2pQ(\sum_{l=1}^{m_t} T_j^l)$$
$$\leq 2pm_tQ[T_j^1]$$
$$\leq 2h^2c_0^2m_t\frac{2!a^2}{(ax+b)^3}h^{2+\rho}.(2\leq m_t\leq n-1)$$
$$\leq C_{1,m_t}h^{2+\rho}, \quad (4.56)$$

with

$$C_{1,m_t} = 2^2hx_fc_0^2(\frac{a^2}{b^3}).$$

From (4.19),(4.55), we have

$$|e_{j,m_t}^n(p^2)| \leq |p^2\ 2^2(n-2-(m_t-1))Q[(\sum_{l=1}^{m_t} T_j^1, s_1]|$$
$$\leq p^2 2^2(n-m_t-1)m_tQ[T_j^1, s_1]$$
$$\leq h^4 2^2 3^2(n-1)c_0^3m_t\frac{4!a^3}{b^5}h^{(2+\rho)}$$

$$\leq C_{2,m_t} h^{2+\rho}, \tag{4.57}$$

with

$$C_{2,m_t} = 2^2 3^2 4! h^2 c_0^3 x_f^2 \frac{a^3}{b^5}.$$

From (4.24),(4.55),we have

$$\begin{aligned}
|e^n_{j,1}(p^3)| &\leq p^3\ 2^3(1 + 2 + .. + (n - (m_t - 1) - 3))Q[(\sum_{l=1}^{m_t} T_j^1), s_1, s_2] \\
&\leq p^3\ 2^3(1 + 2 + .. + (n - 2))m_t Q[T_j^1, s_1, s_2] \\
&\leq h^6 2^3 3^4 n^2 m_t c_0^4 \frac{6!a^4}{b^7} h^{2+\rho} \\
&\leq C_{3,m_t} h^{2+\rho},
\end{aligned} \tag{4.58}$$

with

$$C_{3,m_t} = 2^3 3^4 6! h^3 c_0^4 x_f^3 \frac{a^4}{b^7}.$$

From (4.26),(4.55),we have

$$\begin{aligned}
|e^n_{j,m_t}(p^4)| &\leq 2^4 p^4\ (v_1 + v_2 +, + v_{n-m_t-3})Q[\sum_{l=1}^{m_t} T_j^1, s_1, s_2, s_3] \\
&\leq 2^4 p^4\ (v_1 + v_2 + .. + v_{n-m_t-3})m_t Q[T_j^1, s_1, s_2, s_3]| \\
&\leq 2^4 3^6 h^8 m_t((n - (m_t - 3))^3 + (n - (m_t - 3))^2 + O(n))c_0^5 \frac{8!a^5}{b^9} h^{2+\rho} \\
&\leq C_{4,m_t} h^{2+\rho},
\end{aligned} \tag{4.59}$$

with

$$C_{4,m_t} = 2^4 3^6 8! h^4 c_0^5 (x_f^4 + x_f^3 + 0(n^2)) \frac{a^5}{b^9}.$$

From (4.29),(4.55),we have

$$\begin{aligned}
|e^n_{j,m_t}(p^m)| &\leq |p^m\ 2^m\{w_1 + w_2 + w_3 + .. + w_{n-(m_t-1)}\} \\
& m_t Q[T_j^1, s_1, s_2, s_3, s_4, , ..s_{m-1}]| \\
&= p^m 2^m \{\frac{1}{30}(n - (m_t - 1) - m)(n - (m_t - 1) - m + 1)(2(n - (m_t - 1) - m) + 1) \\
&(3(n - (m_t - 1) - m)^2 + 3(n - (m_t - 1) - m) + 1) + O((n - (m_t - 1) - m)^4)\} \\
&3^{2(m-1)} c_0 m_t \frac{(2m)!a^{m+1}}{(ax + b)^{(2m+1)}} h^{2+\rho} \\
&< p^m 2^{3m} 3^{2m} c_0 \{(n - (m_t - 1) - m)^5 + O((n - m_t - m)^4)\} \frac{(2m)!a^{(m+1)}}{b^{(2m+1)}} h^{2+\rho}.(m = 5, 6, .., n - (m_t - 1) - 1)
\end{aligned} \tag{4.60}$$

From (4.51),(4.60) and the inequality,

$$\begin{aligned}
(2m!)h^{2m-7} &= 6!(7h)(8h)..(mh)((m + 1)h)((m + 2)h)...(2mh) \\
&< 6! x_f^{(2m-6)} 2^{2m-6}.
\end{aligned}$$

we have

$$\begin{aligned}
|p^m e^{(n)}_{j,m_t}(p^m)| &< 6! 2^{3m} 3^{2m} \{x_f^5 + O(n^4)\} x_f c_0^{m+1} \frac{a}{b} \{\frac{a}{b^2}\}^l (x_f)^{2m-7} h^{2+\rho}. \\
&\leq \frac{6!.c_0}{b} \{\frac{72ac_0 x_f^2}{b^2}\}^m h^{2+\rho}.
\end{aligned} \tag{4.61}$$

If we assume (4.52), Then we have

$$|e^{(n)}_{j,m_t}(p^m)| \leq C_{m,m_t} h^{2+\rho}, (m = 5, .., n - 1) \tag{4.62}$$

with

$$C_{m,m_t} = \frac{6!a}{b}.$$

We set

$$\tilde{C}_m = \max_{1\le n_1\le n-1} C_{n_1,m_t}.$$

Then,from (4.55)-(4.62), we have

$$|e^n_j(\sum_{l=1}^{m_t} T^l_j)| \le |e^n_{1,m_t}(p)| \ + \ |e^n_{2,m_t}(p^2)| +,..,+|e^n_{n-1,m_t}(p^{n-m_t})|$$

$$\le \ \tilde{C}_m x_f h^{1+\rho}. \tag{4.63}$$

We consider in the case $m_t = n-4, n-3, n-2, n-1, n$.

On the case $m_t = n-4$. From (4.12),(4.35),(4.55),we have

$$\begin{aligned}|e^n_{j,1}(p)| \ &\le 2pQ[\sum_{l=1}^{n-4} T^l_j]\\ &\le 2p(n-4)Q[T^1_j]\\ &\le 2h^2c_0^2n-4)\frac{2!a^2}{(ax+b)^3}h^{2+\rho}.\\ &\le C_{1,n-4}h^{2+\rho},\end{aligned} \tag{4.64}$$

with

$$C_{1,n-4} = 2^2hx_fc_0^2(\frac{a^2}{b^3}).$$

From (4.19),(4.44),(4.55),we have

$$\begin{aligned}|e^n_{j,n-4}(p^2)| \ &\le p^2 \ 2^2 3Q[\sum_{l=1}^{n-4} T^l_j, s_1]\\ &\le 2^2 3h^4c_0^2 3^2(n-4)Q[T^1_j, s_1]\\ &\le 2^2 3h^4 3^2 c_0^3(n-4)\frac{4!a^3}{b^5}h^{2+\rho}\\ &\le C_{2,n-4}h^{2+\rho},\end{aligned} \tag{4.65}$$

with

$$C_{2,n-4} = 2^2 3^3 4! h^3 c_0^3 x_f \frac{a^3}{b^5}.$$

From (4.24),(4.44),(4.55),we have

$$\begin{aligned}|e^n_{j,,n-4}(p^3)| \ &\le p^3 2^3(1+2)Q[\sum_{l=1}^{n-4} T^l_j, s_1, s_2]\\ &\le p^3 \ 2^3 3(n-4)Q[T^1_j, s_1, s_2]\\ &< 2^3 3^5 h^6 c_0^4(n-4)\frac{6!a^4}{b^7}h^{2+\rho}\\ &\le C_{3,n-4}h^{2+\rho},\end{aligned} \tag{4.66}$$

with

$$C_{3,n-4} = 2^3 3^5 6! h^5 c_0^4 x_f \frac{a^4}{b^7}.$$

From (4.26),(4.44),(4.55),we have

$$|e^n_{j,n-4}(p^4)| \ \le p^4 2^4 Q[\sum_{l=1}^{n-4} T^l_j, s_1, s_2]$$

$$\leq h^8 2^4 3^4 (n-4) c_0^4 Q[T_j^1, s_1, s_2]$$
$$\leq h^7 2^4 3^4 x_f c_0^5 \frac{8! a^5}{(ax+b)^9}$$
$$\leq C_{4,n-4} h^{2+\rho}, \tag{4.67}$$

with

$$C_{4,n-4} = 2^4 3^4 8! h^7 c_0^5 x_f \frac{a^5}{b^9}.$$

From (4.64)-(4.67),we have

$$|e^n_{j,n-4}| \leq |e^n_{j,n-4}(p)| + |e^n_{j,n-4}(p^2)| + |e^n_{j,,n-4}(p^3)| + |e^n_{j,n-4}(p^4)|$$
$$\leq (C_{1,n-4} + C_{2,n-4} + C_{3,n-4} + C_{4,n-4}) h^{2+\rho}.$$

If we set

$$\tilde{C}_{n-4} = \max_{1\leq n_1 \leq 4} C_{n_1,n-4}.$$

Then,we have

$$|e^n_{j,n-4}| \leq x_f \tilde{C}_{n-4} h^{1+\rho}. \tag{4.68}$$

On the case $m_x = n-3$. From (4.12),(4.35),(4.55),we have

$$|e^n_{1,n-3}(p)| \leq 2pQ[\sum_{l=1}^{n-3} T_j^l]$$
$$\leq 2p(n-3) Q[T_j^1]$$
$$\leq 2c_0^2 x_f \frac{2! a^2}{(ax+b)^3} h^{2+\rho}$$
$$\leq C_{1,n-3} h^{2+\rho}, \tag{4.69}$$

with

$$C_{1,n-3} = 2^2 x_f c_0^2 (\frac{a^2}{b^3}).$$

From (4.19),(4.44),(4.55),we have

$$|e^n_{j,n-3}(p^2)| \leq p^2\ 2^2 2(n-3) Q[T_j^1, s_1]$$
$$\leq 2^3 h^4 3^2 c_0^3 (n-3) \frac{4! a^3}{b^5} h^{2+\rho}$$
$$\leq C_{2,n-3} h^{2+\rho}, \tag{4.70}$$

with

$$C_{2,n-3} = 2^3 3^2 4! h^3 c_0^3 x_f \frac{a^3}{b^5}.$$

From (4.24),(4.44),(4.55),we have

$$|e^n_{j,,n-3}(p^3)| \leq p^3 2^3 Q[\sum_{l=1}^{n-3} T_j^l, s_1, s_2]$$
$$\leq p^3 2^3 (n-3) Q[T_j^1, s_1, s_2, s_3]$$
$$\leq 2^3 h^6 3^4 c_0^4 (n-3) \frac{6! a^4}{b^7} h^{2+\rho}$$
$$\leq C_{3,n-3} h^{2+\rho}, \tag{4.71}$$

with

$$C_{3,n-3} = 2^3 3^4 6! h^5 c_0^4 x_f \frac{a^4}{b^7}.$$

From (4.69)-(4.71),we have

$$|e^n_{j,n-3}| \leq |e^n_{j,n-3}(p)| + |e^n_{j,n-3}(p^2)| + |e^n_{j,,n-3}(p^3)|$$

$$\leq (C_{1,n-3} + C_{2,n-3} + C_{3,n-3})h^{2+\rho}.$$

If we set

$$\tilde{C}_{n-3} = \max_{1\leq n_1\leq 3} C_{n_1,n-3}.$$

Then we have

$$|e^n_{j,n-3}| \leq x_f \tilde{C}_{n-3} h^{1+\rho}. \tag{4.72}$$

On the case $m_t = n-2$. From (4.12),(4.36),(4.55),we have

$$\begin{aligned}|e^n_{j,n-2}(p)| &\leq 2pQ[\sum_{l=1}^{n-2} T^l_j] \\ &\leq 2p(n-2)Q[T^1_j] \\ &\leq 2h^2(n-2)c_0^2 \frac{2!a^2}{(ax+b)^3} h^{2+\rho} \\ &\leq C_{1,n-2} h^{2+\rho},\end{aligned} \tag{4.73}$$

with

$$C_{1,n-2} = 2^2 h x_f c_0^2 (\frac{a^2}{b^3}).$$

From (4.19),(4.44),(4.55),we have

$$\begin{aligned}|e^n_{j,n-3}(p^2)| &\leq p^2 Q[\sum_{l=1}^{n-2} T^l_j, s_1] \\ &\leq p^2\ 2^2(n-2)Q[T^1_j, s_1] \\ &\leq 2^2 3^2 h^4 c_0^3 (n-2) \frac{4!a^3}{b^5} h^{2+\rho} \\ &\leq C_{2,n-2} h^{2+\rho},\end{aligned} \tag{4.74}$$

with

$$C_{2,n-2} = 2^2 3^2 4! h^3 c_0^3 x_f \frac{a^3}{b^5}.$$

From (4.73),(4.74), we have

$$\begin{aligned}|e^n_{j,n-2}| &\leq |e^n_{j,n-2}(p)| + |e^n_{j,n-2}(p^2)| \\ &\leq (C_{1n-2} + C_{2,n-2})h^{2+\rho}.\end{aligned} \tag{4.75}$$

If we set

$$\tilde{C}_{n-2} = \max_{1\leq n_1\leq 2} C_{n_1,n-2},$$

Then, we have

$$|e^n_{j,n-2}| \leq x_f \tilde{C}_{n-2} h^{1+\rho}. \tag{4.76}$$

On the case $m_t = n-1$. From (4.12),(4.36),(4.55),we have

$$\begin{aligned}|e^n_{1,n-1}| &\leq 2pQ\sum_{l=1}^{n-1} T^l_j] \\ &\leq 2p(n-1)Q[T^1_j] \\ &\leq 2h^2 c_0^2 (n-1) \frac{2!a^2}{(ax+b)^3} h^{1+\rho} \\ &\leq \tilde{C}_{n-1} h^{2+\rho},\end{aligned} \tag{4.77}$$

with

$$\tilde{C}_{n-1} = 2^2 h x_f c_0^2 (\frac{a^2}{b^3}).$$

From (4.54),(4.63),(4.68),(4.72),(4.76),(4.77),we have

$$\sum_{n_1=1}^{n-1}[e^n_{j,n_1}(T^{n_1}_j)| \le |e^n_{j,1}(T^1_j)| + |e^n_{j,2}(\sum_{l=1}^{2}T^l_j)| + |e^n_{j,3}(\sum_{l=1}^{3}T^l_j)| + |e^n_{j,n-1}(\sum_{l=1}^{n-1}T^l_j)|$$
$$\le (\tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 + ... + \tilde{C}_{n-1})h^{2+\rho}.$$

We set

$$\tilde{C}_x = \max_{1\le m\le n-1}\tilde{C}_m.$$

Then, we have

$$\sum_{n_1=1}^{n-1}[e^n_{j,n_1}(T^{n_1}_j)| \le \tilde{C}_x\ x_f\ h^{1+\rho}. \quad (4.78)$$

On the case $m = n$, we have

$$e^n_{j,n} = \sum_{1\le m\le n}T(j,m).$$

we have

$$|e^n_{j,n}| \le \sum_{1\le m\le n}|w(j,n)|k^{1+\tilde{\rho}}.$$

If we assume

$$|w(jh,nk)| \le \tilde{C}_n. \quad (4.79)$$

for some constant $\tilde{C}_n$. Then we have

$$|e_{j,n}| \le \sum_{1\le m\le n}|w(j,n)|k^{1+\tilde{\rho}}$$
$$\le \tilde{C}_n t_f k^{\rho}. \quad (4.80)$$

From (4.78),(4.80), we have

$$E^n = \sum_{n_1=1}^{n-1}||[e^n_{j,n_1}(\sum_{n_2=1}^{n_1}T^{n_2}_j)]|| + ||[e^n_{j,n}]||$$
$$\le (\tilde{C}_x x_f h^{\rho} + \tilde{C}_n t_f k^{\tilde{\rho}}).$$

which leads to

$$lim_{h,k->0}||E^n|| = 0.$$

Theorem [2] Suppose that for step size space and time h and k with the condition (1.4), there exists positive numbers $j(h)$ and $n(k)$

$$j(h)h_i \to x \in [0,x_f](i\to\infty) \quad n(k)k_i \to t \in [0,t_f].$$

and the conditions (4.9),(4.35),(4.47),(4.52),(4.79) are satisfied and $|u(x,t)|, |u_x(x,t)|, |u_{xx}(x,t)|$ are bounded in $[0,x_f] * [0,t_f]$.

Then the scheme (2.2) converge to the solution $u(x,t)$ of the differential equation (1.1) uniformly.

5. Stability

In this section, we study the stability of the numerical process (2.2),(2.3) and define as follows.

Definition: The numerical processes $\{Y^n \in R_n\}$ is stable if there exists a positive constant K_2 such that

$$||Y^n|| \le K_1, \quad (5.1)$$

where $||.||$ denotes some norm and the constant K_1.

We use the following result.

Lemma If the conditions (4.9),(4.35),(4.47),(4.52) are satisfied. Then we have

$$||D^2[e^n_j]|| \le K_2, \quad (5.2)$$

for some constant K_2

Proof

From (4.11), we have

$$\begin{aligned}|D^2[e^n_{j,1}(p)]| &= |D^2[p(q+g(j,1)Q[T^1_j]]| \\ &\leq |2pQ[T^1_j, s_1]| \\ &\leq 2h^2 3^2 c_0^2 \frac{4!a^3}{(ax+b)^5} h^{2+\rho} \\ &\leq L_{1,1}h^2, \end{aligned} \tag{5.3}$$

with

$$L_{1,1} = 2.3^2 h^{2+\rho} c_0^2 .4!.\tfrac{a^3}{b^5}.$$

From (4.18), we have

$$D^2[e^m_{j,1}(p^2)] = D^2[e^{m-1}_j(p^2)] + pD^2[(q+g(j,m-1)D^2[e^{m-1}_j(p)]].(m=2,3,..,n)$$

In the method same to (4.19), we have

$$\begin{aligned}|D^2[e^n_{j,1}(p^2)]| &\leq 2^2p^2Q[T^1_j, s_1, s_2] \\ &\leq h^4 2^2 3^4 (n-2) c_0^3 \frac{6!a^4}{b^7} h^{2+\rho} \\ &\leq L_{2,1}h^2, \end{aligned} \tag{5.4}$$

with

$$L_{2,1} = 2^2 3^4 .6!.h^{3+\rho} c_0^3 x_f \frac{a^4}{b^7}.$$

From (4.23), we have

$$D^2[e^m_{j,1}(p^3)] = D^2[e^{m-1}_j(p^3)] + pD^2[(q+g(j,m-1)D^2[e^{m-1}_j(p^2)]].(m=3,4,.,n)$$

In the method same to (4.19), we have

$$\begin{aligned}|D^2[e^n_{j,1}(p^3)]| &\leq 2^3p^3(1+2+..,+(n-3))Q[T^1_j, s_1, s_2, s_3] \\ &\leq h^6 c_0^4 2^3 3^6 (n-3)(n-2)Q[T^n_j, s_1, s_2, s_3] \\ &\leq L_{3,1}h^2, \end{aligned} \tag{5.5}$$

with

$$L_{3,1} = 2^3 3^6 .8!.c_0^4 x_f^2 h^{4+\rho} \frac{a^5}{b^9}.$$

From (4.25), we have

$$D^2[e^m_{j,1}(p^4)] = D^2[e^{m-1}_{j,1}(p^4)] + pD^2[(q+g(j,(m-1)))D^2[e^{m-1}_{j,1}(p^3)]], (m=4,5,..,n)$$

In the method same to (4.19), we have

$$\begin{aligned}|D^2[e^n_{j,1}(p^4)]| &\leq p^4\, 2^4(v_1+v_2+v_3+..+v_{n-4})Q[T^1_j, s_1, s_2, s_3, s_4] \\ &\leq 2^4 3^6 h^8 (n^3+n^2+O(n)) c_0^5 \frac{10!a^5}{b^9} h^{2+\rho} \\ &\leq L_{4,1}h^2, \end{aligned} \tag{5.6}$$

with

$$L_{4,1} = 2^4 3^8 .10!.c_0^5 (x_f^3 + x_f^2 + O(x_f)) h^{5+\rho} \frac{a^6}{b^{11}}.$$

From (4.28), we have

$$D^2[e^m_{j,1}(p^l)] = D^2[e^{m-1}_{j,1}(p^l)] + pD^2[(q+g(j,(m-1)))D^2[e^{m-1}_{j,1}(p^3)]],$$

In the method same to (4.19), we have

$$|D^2[e^{(n)}_{j,1}(p^l)]| \leq p^l\, 2^l(w_1 + w_2 + .. + w_{n-l})Q[T^1_j, s_1, s_2, .., s_l],$$

$$\leq 2^l p^l((n-l)(n-l+1)((n-l)^3 + 5(n-l)^2 + O(n-l)))Q[T^n_j, s_1, s_2, .., s_l]$$

$$< 2^l.7!.c_0^{l+1}2^l3^{2l}2^{l+2}x_f^4x_f^{2l-4}\frac{a^{l+2}}{b^{2l+3}}h^{2+\rho}$$

$$\leq L_{l,1}h^2,\ \ (l = 5, 6, .., n-1) \tag{5.7}$$

with

$$L_{l,1} = 4.7!.c_0\frac{a^2}{b^3}(\frac{72c_0x_f^2a}{b^2})^l h^{2+\rho},$$

If we assume (4.51),(4.52). Then we have

$$|D^2[e^{(n)}_{j,1}(p^l)]| \leq L_{l,1}h^2,$$

with

$$L_{l,1} = 4.6!.c_0\frac{a^2}{b^3}h^{\rho}.(l = 5, 6, .., n-1)$$

If we assume (4.52),we set

$$\tilde{L}_1 = \max_{1\leq n_1\leq n-1} L_{n_1,1}. \tag{5.8}$$

From (5.3)-(5.8),we have

$$|D^2[e^n_{j,1}]| < |D^2[e^n_{1,1}(p)]| + |D^2[e^n_{2,1}(p^2)]| + .. + |D^2[e^n_{n-1,1}(p^{n-1})]|$$

$$\leq |L_{1,1}|\ + |L_{2,1}| + ... + |L_{n-1,1}|$$

$$\leq \tilde{L}_1\ x_f\ h. \tag{5.9}$$

We study the value of $D^2[e^n_j](\sum_{l=1}^{m_t} T^l_j)(m_t = 2, 3, .., n-1)$. From (4.11), we have

$$|D^2[e^n_{j,1}(p)]| = |D^2[p(q + g(j,1)Q[\sum_{l=1}^{m_t} T^l_j]|$$

$$\leq |2pQ[\sum_{l=1}^{m_t} T^l_j, s_1]|$$

$$\leq 2h^2c_0^2m_t\frac{4!a^2}{(ax+b)^3}h^{2+\rho}$$

$$\leq L_{1,m_t}h^2, \qquad (m_t = 2, 3, ., n-5) \tag{5.10}$$

with

$$L_{1,m_t} = 2.3^2h^{1+\rho}x_fc_0^2.4!.\tfrac{a^3}{b^5}.$$

From (4.18), we have

$$D^2[e^m_{j,m_t}(p^2)] = D^2[e^{m-1}_{jm_t,}(p^2)] + pD^2[(q + g(j, m-1)D^2[e^{m-1}_{j,m_t}(p)]].(m = 2, 3, .., n)$$

In the method same to (4.19), we have

$$|D^2[e^n_{j,m_t}(p^2)]| \leq\ 2^2p^2Q[\sum_{l=1}^{m_t} T^l_j, s_1, s_2]$$

$$\leq h^4 2^2 3^4 m_t(n-(m_t-1)-2)c_0{}^3\frac{6!a^3}{b^5}h^{2+\rho}$$

$$\leq L_{2,m_t}h^2, \tag{5.11}$$

with

$$L_{2,m_t} = 2^2 3^4.6!.h^{2+\rho}c_0^3x_f^2\frac{a^3}{b^5}.$$

From (4.23), we have

$$D^2[e^m_{j,m_t}(p^3)] = D^2[e^{m-1}_{j,m_t}(p^3)] + pD^2[(q + g(j, m-1)D^2[e^{m-1}_{j,m_t}(p^2)]].(m = 3, 4, .., n)$$

In the method same to (4.19), we have

$$|D^2[e^n_{j,m_t}(p^3)]| \leq\ 2^3 p^3(1 + 2 + .., +(n-(m_t-1)-3)Q[\sum_{l=1}^{m_t} T^l_j, s_1, s_2, s_3] \tag{5.12}$$

$$\leq h^6 2^3 3^6 c_0^3 m_t(n-(m_t-1)-2)(n-(m_t-1)-3)Q[T^n_j, s_1, s_2, s_3]$$

$$\leq L_{3,m_t} h^2,$$

with

$$L_{3,m_t} = 2^3 3^6.8!.c_0^4 {x_f}^3 h^{3+\rho}\frac{a^5}{b^9}.$$

From (4.25), we have

$$D^2[e^m_{j,m_t}(p^4)] \ =\ D^2[e^{m-1}_{j,m_t}(p^4)] \ +\ pD^2[(q + g(j,(m-1)))D^2[e^{m-1}_{j,m_t}(p^3)]].(m = 4, 5, .., n)$$

In the method same to (4.19), we have

$$|D^2[e^n_{j,m_t}(p^4)]| \ \leq 2^4 p^4 m_t((n-m_t-1)^3 + (n-(m_t-1))^2 + O(n))Q[T^1_j, s_1, s_2, s_3, s_4]$$

$$\leq C_{4,m_t} h^2, \tag{5.13}$$

with

$$L_{4,m_t} = 2^4 3^8.10!.c_0^5 x_f(x_f^3 + x_f^2 + O(x_f))h^{4+\rho}\frac{a^6}{b^{11}}.$$

From (4.28), we have

$$D^2[e^m_{j,m_t}(p^l)] \ =\ D^2[e^{m-1}_{j,1}(p^l)] \ +\ pD^2[(q + g(j,(m-1)))D^2[e^{m-1}_{j,m_t}(p^{l-1})]].$$

In the method same to (4.19), we have

$$|D^2[e^n_{j,m_t}(p^l)]| \ \leq p^l\ 2^l(w_1 + w_2 + .. + w_{n-l})Q[\sum_{l=1}^{m_t} T^l_j, s_1, s_2, .., s_l],$$

$$\leq 2^l p^l((n-(m_t-1)-l)(n-(m_t-1)-l+1)((n-(m_t-1)-l)^3+5(n-(m_t-1)-l)^2+O(n-(m_t-1)-l)))m_t Q[T^n_j, s_1, s_2, .., s_l]$$

$$< 2^l.8!.c_0^{l+1} 2^l 3^{2l} 2^{l+2} x_f^5 (x_f)^{(2l-5)}\frac{a^{l+2}}{b^{2l+3}} h^{2+\rho}$$

$$\leq L_{l,1} h^2,\ (l = 5, 6, .., n-(m_t-1)-1) \tag{5.14}$$

with

$$L_{l,m_t} = 4.8!.c_0\frac{a^2}{b^3}(\frac{72c_0 x_f^2 a}{b^2})^l h^\rho.$$

If we assume (4.51),(4.52),we set

$$\tilde{L}_{m_t} = \max_{1\leq n_1\leq n-1} L_{n_1,m_t}. \tag{5.15}$$

If we assume (4.52), then from (5.10)-(5.15),we have

$$|D^2[e^n_j](\sum_{l=1}^{m_t} T^l_j)| \leq |D^2[e^n_{1,m_t}(p)]| \ +\ |D^2[e^n_{2,m_t}(p^2)]| \ +\ +... + |D^2[e^n_{n-m_t,m_t}(p^{n-m_t})]|$$

$$\leq\ \tilde{L}_{m_t} x_f h. \qquad (m_t = 2, 3, .., n-5) \tag{5.16}$$

We study the case ; $m_t = n-4, n-3, n-2, n-1, n$.

On the case $m_t = n-4$. In the method same to (5.3), we have

$$|D^2[e^n_{j,m_t}(p)]| \leq |2pQ[\sum_{l=1}^{n-4} T^l_j, s_1]|$$

$$\leq 2h^2 3^2 c_0^2 (n-4)\frac{4!a^3}{(ax+b)^5} h^{2+\rho}$$

$$\leq L_{1,n-4}h^2, \tag{5.17}$$

with

$$L_{1,n-4} = 2.3^2 h^{1+\rho} c_0^2 4! x_f \tfrac{a^3}{b^5}.$$

In the method same to (5.4), we have

$$|D^2[e^n_{j,m_t}(p^2)]| \leq 2^2 p^2 Q[\sum_{l=1}^{n-4} T^l_j, s_1, s_2]$$

$$\leq h^4 2^2 3^4 (n-(m_t-1)-2)(n-4)c_0^3 \frac{6!a^4}{b^7} h^{2+\rho}$$

$$\leq L_{2,n-4}h^2, \tag{5.18}$$

with

$$L_{2,n-4} = 2^2 3^5 6! h^{3+\rho} c_0^3 x_f (\frac{a^4}{b^7}).$$

In the method same to (5.5), we have

$$|D^2[e^n_{j,m_t}(p^3)]| \leq 2^3 p^3 (1+2+..,+(n-(m_t-1)-3))Q[\sum_{l=1}^{n-4} T^l_j,, s_1, s_2, s_3]$$

$$\leq h^6 c_0^3 2^3 3^7 (n-4) Q[T^n_j, s_1, s_2, s_3],$$

which lead to

$$|D^2[e^n_{j,1}(p^3)]| \leq L_{3,n-2}h^2, \tag{5.19}$$

with

$$L_{3,n-2} = 2^3 3^7.8!.c_0^4 x_f h^{5+\rho}\frac{a^5}{b^9}.$$

In the method same to (5.6), we have

$$|D^2[e^{(n)}_{j,m_t}(p^4)]| \leq p^4\, 2^4 (v_1+v_2+..+v_{n-(m_t-1)-4}) Q[\sum_{l=1}^{n-4} T^l_j, s_1, s_2, s_3, s_4]$$

$$\leq 2^4 p^4 (n-4) Q[T^n_j, s_1, s_2, s_3, s_4] \tag{5.20}$$

$$\leq C_{4,1}h^2,$$

with

$$L_{4,n-4} = 2^4 3^8.10!.c_0^5 x_f h^{7+\rho}\frac{a^6}{b^{11}}.$$

From (5.17)-(5.20),we have

$$|D^2[e^n_{j,n-4}]| < |D^2[e^n_{j,n-4}(p)]| + |D^2[e^n_{j,n-4}(p^2)]| + |D^2[e^n_{j,n-4}(p^3)]| + |D^2[e^n_{j,n-4}(p^4)]|$$

$$< (L_{1,n-4} + L_{2,n-4} + L_{3,n-4} + L_{4,n-4})h^2. \tag{5.21}$$

If we set

$$\tilde{L}_{n-4} = \max_{1\leq n_1 \leq 4} L_{n_1,n-4}.$$

Then, from (5.21),we have

$$|D^2[e^n_{j,n-4}]| \ < \ x_f \tilde{L}_{n-4} h. \tag{5.22}$$

On the case $m_t = n-3$. In the method same to (5.3), we have

$$\begin{aligned}|D^2[e^n_{j,1}(p)]| &\leq \ |2pQ[\sum_{l=1}^{n-3} T^l_j, s_1]| \\ &\leq 2h^2 3^2 c_0^2 (n-3)\frac{4!a^3}{(ax+b)^5} h^{2+\rho} \\ &\leq L_{1,n-3} h^2,\end{aligned} \tag{5.23}$$

with

$$L_{1,n-3} = 2.3^2 h^{1+\rho} c_0 4! x_f (\tfrac{a^3}{b^5}).$$

In the method same to (5.4), we have

$$\begin{aligned}|D^2[e^n_{j,m_t}(p^2)]| &\leq \ 2^2 p^2 Q[\sum_{l=1}^{n-4} T^l_j, s_1, s_2] \\ &\leq h^4 2^2 3^4 2(n-4) c_0^3 \frac{6!a^4}{b^7} h^{2+\rho} \\ &\leq L_{2,n-3} h^2,\end{aligned} \tag{5.24}$$

with

$$L_{2,n-3} = 2^3 3^4 6! h^{3+\rho} c_0^3 x_f \frac{a^4}{b^7}.$$

In the method same to (5.5),we have

$$\begin{aligned}|D^2[e^n_{j,m_t}(p^3)]| &\leq \ 2^3 p^3 (1+2+..,+(n-(m_t-1)-3)) Q[\sum_{l=1}^{n-3} T^l_{j,}, s_1, s_2, s_3] \\ &\leq h^6 c_0^3 2^3 3^6 (n-3) Q[T^n_j, s_1, s_2, s_3], \\ &\leq L_{3,n-2} h^2,\end{aligned} \tag{5.25}$$

with

$$L_{3,n-2} = 2^3 3^6 .8!. c_0^4 x_f h^{5+\rho} \frac{a^5}{b^9}.$$

From (5.23)-(5.25), we have

$$\begin{aligned}|D^2[e^n_{j,n-3}]| \ &< \ |D^2[e^n_{j,n-3}(p)]| \ + \ |D^2[e^n_{j,n-3}(p^2)]| \ + |D^2[e^n_{j,n-4}(p^3)]| \\ &< (L_{1,n-3} + L_{2,n-3} + L_{3,n-3}).\end{aligned} \tag{5.26}$$

If we set

$$\tilde{L}_{n-3} = \max_{1 \leq n_1 \leq 3} L_{n_1,n-3}.$$

Then, from (5.28), we have

$$|D^2[e^n_{j,n-3}]| \ < \ x_f \tilde{L}_{n-3} h. \tag{5.27}$$

On the case $m_t = n-2$. In the method same to (5.3), we have

$$\begin{aligned}|D^2[e^n_{j,m_t}(p)]| &\leq \ |2pQ[\sum_{l=1}^{n-2} T^l_j, s_1]| \\ &\leq 2h^2 3^2 c_0^2 (n-2)\frac{4!a^3}{(ax+b)^5} h^{2+\rho}. \\ &\leq L_{1,n-2} h^2,\end{aligned} \tag{5.28}$$

with

$$L_{1,n-2} = 2.3^2 h^{1+\rho} c_0^2 4! x_f \tfrac{a^3}{b^5}.$$

In the method same to (5.4), we have

$$
\begin{aligned}
|D^2[e^n_{j,m_t}(p^2)]| \leq \ & 2^2 p^2 Q[\sum_{l=1}^{n-2} T^l_j, s_1, s_2]. \\
& \leq h^4 2^2 3^4 (n-2) c_0^3 \frac{6! a^4}{b^7} h^{2+\rho} \\
& \leq L_{2,n-2} h^2, \qquad (5.29)
\end{aligned}
$$

with

$$L_{2,n-2} = 2^2 3^4 6! h^{3+\rho} c_0^3 x_f \frac{a^4}{b^7}.$$

From (5.28),(5.29), we have

$$
\begin{aligned}
|D^2[e^n_{j,n-2}]| \ &< \ |D^2[e^n_{j,n-2}(p)]| \ + \ |D^2[e^n_{j,n-2}(p^2)]| \\
&< (L_{1,n-2} + L_{2,n-2}). \qquad (5.30)
\end{aligned}
$$

If we set

$$\tilde{L}_{n-2} = \max_{1 \leq n_1 \leq 2} L_{n_1, n-2}.$$

Then, from (5.30), we have

$$|D^2[e^n_{j,n-2}]| \ < \ x_f \tilde{L}_{n-2} h. \qquad (5.31)$$

On the case $m_t = n - 1$. In the method same to same (5.3), we have

$$D^2[e^n_{j,n-1}(p)] \leq \tilde{L}_{n-1} h^2, \qquad (5.32)$$

with

$$\tilde{L}_{n-1} \ = \ 2.3^2.4!.c_0^2 x_f h^{1+\rho} \frac{a^3}{b^5}.$$

From (5.9),(5.16),(5.22),(5.27),(5.31),(5.32), we have

$$
\begin{aligned}
|D^2[e^n_j]| \ \leq \ & |D^2[e^n_{j,1}(T^1_j)]| \ + \ |D^2[e^n_{j,2}(\sum_{l=1}^{2} T^l_j)]| \ + \ |D^2[e^n_{j,3}(\sum_{l=1}^{3} T^l_j)]| \ + \ |D^2[e^n_{j,n-1}(\sum_{l=1}^{n-1} T^l_j)]|, \\
& \leq (\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + ... + \tilde{L}_{n-1}) h. \qquad (5.33)
\end{aligned}
$$

If we set

$$\tilde{L}_x = \max_{1 \leq l \leq n-1} \tilde{L}_l.$$

Then,from (5.33),we have

$$|D^2[e^n_j]| \leq \tilde{L}_x \ x_f. \qquad (5.34)$$

On the case $m = n$, we have

$$e^n_{j,n} \ = \ \sum_{1 \leq m \leq n} T(j, m).$$

we have

$$|D^2[e^n_{j,n}]| = \sum_{1 \leq m \leq n} |D^2[w(j, m)]|. \qquad (5.35)$$

From (4.35), we have

$$D^2[w(jh, nk)]| \leq L_n,$$

for some constant L_n, then, from (5.35), we have

$$
\begin{aligned}
|D^2[e_{j,n}]| \ &= \ \sum_{1 \leq m \leq n} |D^2[w(j, m)]| k^{1+\tilde{\rho}} \\
&\leq \tilde{L}_n, \qquad (5.36)
\end{aligned}
$$

with

$$\tilde{L}_n = L_n t_f k^{\tilde{\rho}}.$$

From (5.34),(5.36), we have

$$|D^2[e_j^n]| \leq \sum_{n_1=1}^{n-1} |D^2[e_j^{n_1}(\sum_{n_2=1}^{n_1} T_j^{n_2})]| + |D^2[e_{j,n}^n]|$$

$$\leq (\tilde{L}_x + \tilde{L}_n)(= K_2).$$

We study stability of numerical processes (2.4). From (2.1), we have

$$u_j^{n+1} \;=\; u_j^n + \frac{c_0 a(j,n)}{(1+2\hat{c}_0 L_1)}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)$$

$$= \; u_j^n \;+\; \frac{c_0 a(j,n)}{(1+2c_0 L_1)}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) + \frac{c_0 a(j,n)}{(1+2\hat{c}_0 L_1)}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)$$

$$-\frac{c_0 a(j,n)}{(1+2c_0 L_1)}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)). \tag{5.37}$$

From (5.53), we have

$$|u_j^n \;+\; \frac{c_0 a(j,n)}{(1+2c_0 L_1)}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)|$$

$$\leq |(1 - \frac{2c_0 a(j,n)}{(1+2c_0 L_1)})|u_j^n|| + |\frac{c_0 a(j,n)}{(1+2c_0 L_1)})|u_{j-1}^n||| \;+\; |\frac{c_0 a(j,n)}{(1+2c_0 L_1)})||u_{j+1}^n||. \tag{5.38}$$

We set the vector $||U^n||$ by

$$||U^n|| = Max\{|u_j^n|; 0 \leq j \leq 1/h\}.$$

From the inequality

$$1 - \frac{2c_0 a(j,n)}{(1+2c_0 L_1)} \;\geq 0,$$

From (5.38), we have

$$|u_j^n \;+\; \frac{c_0 a(j,n)}{(1+2c_0 L_1)}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)| \leq ||U^n||. \tag{5.39}$$

From (5.37), we have

$$\frac{c_0 a(j,n)}{1+2\hat{c}_0 L_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) - \;\frac{c_0 a(j,n)}{1+2c_0 L_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)$$

$$= c_0 a(j,n)\{\frac{1}{1+2\hat{c}_0 L_1} - \frac{1}{1+2c_0 L_1}\}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n).$$

$$= \frac{2ka(j,n)L_1(c_0 - \hat{c}_0)}{(1+2\hat{c}_0 L_1)(1+2c_0 L_1)}.\frac{1}{h^2}\{u_{j-1}^n - 2u_j^n + u_{j+1}^n\}.$$

From the equation

$$\{u_{j-1}^n - 2u_j^n + u_{j+1}^n\} = (u(j-1,n) - e_{j-1}^n) - 2(u(j,n) - e_j^n) + (u(j+1,n) - e_{j+1}^n)$$

$$= h^2\{u_{xx}(j,n) - e_{xx}(j,n)) + O(h^4)\},$$

We have

$$|\frac{c_0 a(j,n)}{1+2\tilde{c}_0 L_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) - \;\frac{c_0 a(j,n)}{1+2kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)| = |\frac{2c_0 a(j,n)L_1(k-\tilde{k}))}{(1+2\tilde{c}_0 L_1)(1+2kL_1)}\{u_{xx} - e_{xx} + O(h^2)\}|, \tag{5.40}$$

$$\leq |2kc_0 a(j,n)L_1(1-h^{\rho})\{u_{xx} - e_{xx} + O(h^2)\}|.$$

If we assume

$$||u_{xx}(x,t)|| \leq K_3.$$

then, from (5.39),(5.40), we have

$$u_{n+1} \;\leq\; ||U^n|| + wh, \tag{5.41}$$

with

$$w = 2kc_0L_1^2|(1 - h^\rho)|(K_2 + K_3).$$

From (5.41), we have

$$||U^n|| \leq ||U^0|| + x_f\tilde{w},$$

with

$$\tilde{w} = 2L_1^2c_0(1 + h^\rho)(K_2 + K_3).$$

We have the result.

Theorem [3] For any given step size h, k which satisfy (1.4). The difference processes (2.1) applied to the differential equation (1.2) is stable with the conditions (4.9),(4.47),(4.52).

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