Journal of Mathematics Research; Vol. 14, No. 6; December 2022
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Optimal Control of a Nonlinear Elliptic Problem With Missing Data

Thomas Tindano[1], Sadou Tao[2], Somdouda Sawadogo[3]

[1]Universit Joseph KI-ZERBO, Dpartement de Mathmatiques, 03 BP 7021 Ouagadougou 03, Burkina Faso

[2]Universit Joseph KI-ZERBO, Dpartement de Mathmatiques, 03 BP 7021 Ouagadougou 03, Burkina Faso

[3]Ecole Normale Suprieure,Dpartement des sciences exactes, 01 BP 1757 Ouagadougou 01, Burkina Faso

Correspondence: Thomas Tindano, Universit Joseph KI-ZERBO,Dpartement de Mathmatiques, 03 BP 7021 Ouagadougou 03, Burkina Faso

Received: October 16, 2022 Accepted: November 26, 2022 Online Published: November 29, 2022

doi:10.5539/jmr.v14n6p46 URL: https://doi.org/10.5539/jmr.v14n6p46

Abstract

In this work, we give a characterization of the control for ill-posed problems. We propose the regularization method which consists in improving the data in order to obtain a well-posed problems. As the problem is nonlinear, we will use the adapted low-regret control method in order to be able to respectively determine the charaterization of the low-regret and no-regret control of the problem.

Keywords: No-regret control, low-regret, control, adapted low-regret control, nonlinear

AMS SUBJECT CLASSIFICATIONS: 35Q93; 49J20; 93C05; 93C41

1. Statement of the Problem

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \in \mathbb{N}^*$ with a boundary $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$.
we consider, the state $z \in L^2(\Omega)$ and the control $v = \{v_0, v_1\} \in L^2(\Gamma_0) \times L^2(\Gamma_0)$ linked by:

$$\begin{cases} -\triangle z - z^3 = 0 \quad in \quad \Omega, \\ z = v_0, \dfrac{\partial z}{\partial \nu} = v_1 \quad on \quad \Gamma_0. \end{cases} \tag{1}$$

Problem (1) is a cauchy problem. In general, problem (1) does not admit a solution and there is instability of the solution when it exists, see for instance (J. L. Lions, 1983), (J. Smoler, 1983).

Consider the space U defined by: $U = \left\{((v_0, v_1), z) \in (L^2(\Gamma_0))^2 \times L^2(\Omega), -\triangle z - z^3 = 0 \quad in \quad \Omega, z = v_0 \ \ and \ \ \dfrac{\partial z}{\partial \nu} = v_1 \ \ on \ \ \Gamma_0\right\}$,
and suppose that U is convex and not empty ($U \neq \emptyset$).

The couples $((v_0, v_1), z) \in U$ are admissible couples and U is the admissible set.
Let J be a strictly convex cost functional, defined for all admissible control-state couples (v_0, v_1, z) by:

$$J(v_0, v_1, z) = ||z - z_d||^2_{L^2(\Omega)} + N_0||v_0||^2_{L^2(\Gamma_0)} + N_1||v_1||^2_{L^2(\Gamma_0)}, \tag{2}$$

where $(N_0, N_1) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$ and $z_d \in L^2(\Omega)$ the desired state. We are then interested in the problem:

$$\inf \{J(v_0, v_1, z), (v_0, v_1, z) \in U\}. \tag{3}$$

According to the properties of J and those of U, problem (3) admits a unique solution (u_0, u_1, z) that we should characterize. To obtain a singular optimality system (SOS) associated with (u_0, u_1, z), (J. L. Lions, 1983), (J. L. Lions, 1968) has proposed a method of approximation by penalization-adapted. He obtained SOS under the supplementary hypothesis of slater type:

$$The \quad admissible \quad set \quad of \quad controls \quad v \quad has \quad a \quad nonempty \quad interior \ (\mathring{U} \neq \emptyset). \tag{4}$$

So, regularization methods may be considered. Theoretical concepts and also computational implementation related to the Cauchy problem have been discussed by many authors.

In the parabolic and hyperbolic cases, we can quote (M. Barry, O. Nakoulima, & G. B. Ndiaye, 2013), (M. Barry, & G. B. Ndiaye, 2014),(O. Nakoulima, A. Omrane, & J. Velin, 2003), (J. P. Kernevez, 1980) and (G. Mophou, R. G. Foko Tiomela, & A. Seibou, 2020).

In the elliptic case, we can cite (J. L. Lions, 1983), (J. Velin, 2004), (O. Nakoulima, 1994), (S. Sougalo, & O. Nakoulima, 1998). (C. Kenne, G. Leugering, & G. Mophou, 2020), consider a model of population dynamics with age dependence and spatial structure but unknown birth rate and use the notion of low-regret to prove that we can bring the state of the system to desired state by acting on the system via a distributed control.

Lions used the notions of Pareto control (Lions, 1986) and equivalently the no-regret control (Lions, 1992) in application to the control of systems having missing data.

In this paper, we will do the regularization of the problem and use the control method to be able to characterize the control. Thus, as z is unknown on Γ_1 we define the fonction $g = \{g_0, g_1\}$, such as $z = g_0$ and $\dfrac{\partial z}{\partial \nu} = g_1$ on Γ_1 and we consider the following systems :

$$\begin{cases} -\triangle z - z^3 = 0 & in \quad \Omega, \\ z = v_0; \quad \dfrac{\partial z}{\partial \nu} = v_1 & on \quad \Gamma_0, \\ z = g_0; \quad \dfrac{\partial z}{\partial \nu} = g_1 & on \quad \Gamma_1, \end{cases} \tag{5}$$

where g_0 and g_1 belong to G and represents "the pollution" that is unknown (incomplete data), with G a closed space of $L^2(\Gamma_1)$ endowed with the norm of $L^2(\Gamma_1)$.

We use here a method that we find well adapted: the low-regret control concept, introduced by Lions. Lions was the first to use it to control distributed systems of incomplete data, motivated by a number of applications in economics and ecology. Lions in (J. L. Lions, 1983) proposed a method of approximation by penalization and obtained a singular optimality system, under a supplementary hypothesis of slater type. In (T. Tindano, S. Tao, & S. Sawadogo, 2022), T. Tindano, S. Tao and S. Sawadogo use the regularization method to control a time-dependent missing data problem. In (O. Nakoulima, & G. M. Mophou, 2009), O. Nakoulima and G. M. Mophou use a regularization method which consists in viewing a singular problem as a limit of a family of well-posed problems. They have obtained a singular optimality system for the considered control problem, also assuming the slater condition.

In (A. Berhail, & A. Omrane, 2015), A. Berhail and A. Omrane use a regularization approach which generates incomplete information. They get a singular optimality system characterizing the no-regret control for a Cauchy elliptic problem but in linear case.

In (O. Nakoulima, A. Omrane, & J. Velin, 2002), O. Nakoulima, A. Omrane and J. Velin study No-regret control for nonlinear distributed systems with incomplete data using the adapted low-regret control method.

In (LOUISON Lo?c, 2015), LOUISON Lo?c uses the adapted low-regret control method to study nonlinear Nye-Tinker-Barber systems.

In the present paper, we shall study the control of a nonlinear problem where the control v is a couple ($v = (v_0, v_1)$) and g is also a couple ($g = (g_0, g_1)$) using the adapted low-regret control method. As far as we know, this problem has not yet been treated, therefore this work is a contribution.

The rest of this paper is organized as follows. In section 2, we will give the characterization of the low-regret and no-regret control. So, in the subsection 2.3-2.4 and subsection 2.5, the optimal control of the regularized system is discussed and the characterization of adapted low-regret control is determined. In the subsection 2.6 and subsection 2.7, we obtain a singular optimality system for the low-regret and when $\gamma \to 0$ we obtain no-regret control to the original problem, where γ is strictly positive parameters. In section 3, we will end with a conclusion.

2. The Low-Regret and No-Regret Control

The problem being with incomplete data, it is impossible to solve it directly, this is how we use the regularization technique which consists in transforming the problem (1) into a complete data problem.

Therefore, we consider the following regularized problem:

$$\begin{cases} -\triangle^2 z_\varepsilon - z_\varepsilon^3 - \varepsilon z_\varepsilon = 0 & in \quad \Omega, \\ z_\varepsilon - \dfrac{\partial \triangle z_\varepsilon}{\partial \nu} = v_0; \quad \dfrac{\partial z_\varepsilon}{\partial \nu} + \triangle z_\varepsilon = v_1 & on \quad \Gamma_0, \\ \varepsilon z_\varepsilon - \dfrac{\partial \triangle z_\varepsilon}{\partial \nu} = \varepsilon g_0; \quad \varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu} + \triangle z_\varepsilon = \varepsilon g_1 & on \quad \Gamma_1, \end{cases} \tag{6}$$

where ε is a strictly positive parameter($\varepsilon > 0$); $v = (v_0, v_1) \in (L^2(\Gamma_0))^2$ and $g = (g_0, g_1) \in (L^2(\Gamma_1))^2$.

Remark 2.1. *For any fixed εg_0 and εg_1, we assume the existence of solution to (6). In The rest of work, εg_0 and εg_1 are considered as data perturbations.*

Indeed, if $\varepsilon \to 0$, $z_\varepsilon \to z$ and by making the change of variable $\eta = \triangle z$, the system (6) becomes:

$$\begin{cases} -\triangle\eta - z^3 = 0 & in \quad \Omega, \\ z - \dfrac{\partial \eta}{\partial \nu} = v_0; \quad \dfrac{\partial z}{\partial \nu} + \eta = v_1 & on \quad \Gamma_{0,} \\ \dfrac{\partial \eta}{\partial \nu} = 0; \quad \eta = 0 & on \quad \Gamma_1. \end{cases} \tag{7}$$

From (7) we have: $\dfrac{\partial \eta}{\partial \nu} = \eta = 0$ on Γ_1.

By using the unique continuation theorem of Mizohata (S. Mizohata, 1958), we deduce from (7): $\dfrac{\partial \eta}{\partial \nu} = \eta = 0$ on Γ_0.
In substitute in Γ_0, we obtain:

$$z = v_0; \qquad \frac{\partial z}{\partial \nu} = v_1 \qquad on \quad \Gamma_0, \tag{8}$$

that is, the same conditions of the original problem (1).

2.1 Cost Function and Low-Regret Control

Consider the cost functional J_ε defined by:

$$J_\varepsilon(v,g) = \|z_\varepsilon(v,g) - z_d\|^2_{L^2(\Omega)} + N_0\|v_0\|^2_{L^2(\Gamma_0)} + N_1\|v_1\|^2_{L^2(\Gamma_0)}. \tag{9}$$

Lemma 2.1. *Consider the cost functional J_ε given by (9). For any $v \in (L^2(\Gamma_0))^2$ and for any $g \in (L^2(\Gamma_1))^2$, we have:*

$$J_\varepsilon(v,g) - J_\varepsilon(0,g) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + 2\left\langle z_\varepsilon(v,0) - z_d, \frac{\partial z_\varepsilon}{\partial g_0}(v,0) + \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\right\rangle_{L^2(\Omega)} -$$

$$-2\left\langle z_\varepsilon(0,0) - z_d, \frac{\partial z_\varepsilon}{\partial g_0}(0,0) + \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\right\rangle_{L^2(\Omega)}. \tag{10}$$

Proof : we can refer to (T. Tindano, S. Tao, & S. Sawadogo, 2022) by replacing $z_\varepsilon(v,g)$ by $z_\varepsilon(v,0) + \dfrac{\partial z_\varepsilon}{\partial g_0}(v,0) + \dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)$.

■

Lemma 2.2. *We consider the fonction J_ε defined by (9). For any $v \in (L^2(\Gamma_0))^2$ and for any $g \in (L^2(\Gamma_1))^2$, we have:*

$$J_\varepsilon(v,g) - J_\varepsilon(0,g) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + 2\varepsilon\left(\langle \zeta_\varepsilon(v) - \zeta_\varepsilon(0), g_0\rangle_{L^2(\Gamma_1)} + \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v) - \frac{\partial \zeta_\varepsilon}{\partial \nu}(0), g_1\right\rangle_{L^2(\Gamma_1)}\right), \tag{11}$$

where $\zeta_\varepsilon(v)$ solution of:

$$\begin{cases} -\triangle^2\zeta_\varepsilon - 3\zeta_\varepsilon (z_\varepsilon)^2 - \varepsilon\zeta_\varepsilon = -(z_\varepsilon(u_\varepsilon,0) - z_d) & in \quad \Omega, \\ \zeta_\varepsilon - \dfrac{\partial \triangle\zeta_\varepsilon}{\partial \nu} = 0; \quad \dfrac{\partial \zeta_\varepsilon}{\partial \nu} + \triangle\zeta_\varepsilon = 0 & on \quad \Gamma_0, \\ \varepsilon\zeta_\varepsilon - \dfrac{\partial \triangle\zeta_\varepsilon}{\partial \nu} = 0; \quad \varepsilon\dfrac{\partial \zeta_\varepsilon}{\partial \nu} + \triangle\zeta_\varepsilon = 0 & on \quad \Gamma_1. \end{cases} \tag{12}$$

Let us give some preliminary results which will be used for the proof of Lemma 2.2 :
$z_\varepsilon(v,g)$ is differentiable on $(L^2(\Gamma_1))^2$:
$z_\varepsilon(v,g+h) - z_\varepsilon(v,g) = h_0.\dfrac{\partial z_\varepsilon}{\partial g_0}(v,g) + h_1\dfrac{\partial z_\varepsilon}{\partial g_1}(v,g) + \| (h_0,h_1) \|_{(L^2(\Gamma_1))^2}\ \epsilon(h_0,h_1)$,

Proposition 2.1. *$z_\varepsilon(v,0)$ is solution of systems:*

$$\begin{cases} -\triangle^2 \dfrac{\partial z_\varepsilon}{\partial g_0}(v,0) - 3\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) z_\varepsilon^2(v,0) - \varepsilon \dfrac{\partial z_\varepsilon}{\partial g_0}(v,0) = 0 & in \quad \Omega, \\ \dfrac{\partial z_\varepsilon}{\partial g_0}(v,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) = 0; \quad \dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) = 0 & on \quad \Gamma_0, \\ \varepsilon\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) = \varepsilon g_0; \quad \varepsilon\dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) = 0 & on \quad \Gamma_1, \end{cases} \tag{13}$$

$$\begin{cases} -\triangle^2 \dfrac{\partial z_\varepsilon}{\partial g_1}(v,0) - 3\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)\right) z_\varepsilon^2(v,0) - \varepsilon \dfrac{\partial z_\varepsilon}{\partial g_1}(v,0) = 0 & in \quad \Omega, \\ \dfrac{\partial z_\varepsilon}{\partial g_1}(v,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)\right) = 0; \quad \dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)\right) = 0 & on \quad \Gamma_0, \\ \varepsilon\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)\right) = 0; \quad \varepsilon\dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(v,0)\right) = \varepsilon g_1 & on \quad \Gamma_1. \end{cases} \tag{14}$$

Proof :

⊗ Let us show that $z_\varepsilon(v,0)$ is a solution of (13):

$$\bullet - \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) - 3\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) z_\varepsilon^2(v,0) - \varepsilon\frac{\partial z_\varepsilon}{\partial g_0}(v,0) = \frac{\partial}{\partial g_0}\left(-\triangle^2 z_\varepsilon(v,0) - z_\varepsilon^3(v,0) - \varepsilon z_\varepsilon(v,0)\right) = 0,$$

because, $-\triangle^2 z_\varepsilon(v,0) - (z_\varepsilon(v,0))^3 - \varepsilon z_\varepsilon(v,0) = 0 \quad$ in $\quad \Omega$. Thus,

$$-\triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) - 3\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) z_\varepsilon^2(v,0) - \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) = 0 \quad in \quad \Omega.$$

$\bullet$*For boundary conditions* Γ_0 :

$$\begin{aligned} \frac{\partial z_\varepsilon}{\partial g_0}(v,0) - \frac{\partial}{\partial \nu}\triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) &= \left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) - \frac{\partial}{\partial \nu}\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right), \\ &= \left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - \frac{\partial}{\partial \nu}\triangle z_\varepsilon(v,\lambda g_0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,0) - \frac{\partial}{\partial \nu}\triangle z_\varepsilon(0,0)}{\lambda}\right) = 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) + \triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) &= \frac{\partial}{\partial \nu}\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) + \triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right), \\ &= \left(\lim_{\lambda\longrightarrow 0}\frac{\frac{\partial z_\varepsilon}{\partial \nu}(v,\lambda g_0) - \triangle z_\varepsilon(v,\lambda g_0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{\frac{\partial z_\varepsilon}{\partial \nu}(0,0) - \triangle z_\varepsilon(0,0)}{\lambda}\right) = 0. \end{aligned}$$

$\bullet$*For boundary conditions* Γ_1 :

$$\begin{aligned} \varepsilon\frac{\partial z_\varepsilon}{\partial g_0}(v,0) - \frac{\partial}{\partial \nu}\triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) &= \varepsilon\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) - \frac{\partial}{\partial \nu}\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right), \\ &= \left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon z_\varepsilon(v,\lambda g_0) - \frac{\partial}{\partial \nu}\triangle z_\varepsilon(v,\lambda g_0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon z_\varepsilon(0,0) - \frac{\partial}{\partial \nu}\triangle z_\varepsilon(0,0)}{\lambda}\right) = \varepsilon g_0, \end{aligned}$$

$$\begin{aligned} \varepsilon\frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) + \triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(v,0)\right) &= \varepsilon\frac{\partial}{\partial \nu}\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) + \triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(v,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right), \\ &= \left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon\frac{\partial z_\varepsilon}{\partial \nu}(v,\lambda g_0) - \triangle z_\varepsilon(v,\lambda g_0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon\frac{\partial z_\varepsilon}{\partial \nu}(0,0) - \triangle z_\varepsilon(0,0)}{\lambda}\right) = 0. \end{aligned}$$

⊗ Using similar reasoning as before, we show that $z_\varepsilon(v,0)$ is solution of (14).

■

Proposition 2.2. *$z_\varepsilon(0,0)$ is solution of systems:*

$$\begin{cases} -\triangle^2 \dfrac{\partial z_\varepsilon}{\partial g_0}(0,0) - 3\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) z_\varepsilon^2(0,0) - \varepsilon \dfrac{\partial z_\varepsilon}{\partial g_0}(0,0) = 0 & in \quad \Omega, \\ \dfrac{\partial z_\varepsilon}{\partial g_0}(0,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = 0; \quad \dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = 0 & on \quad \Gamma_0, \\ \varepsilon\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = \varepsilon g_0; \quad \varepsilon\dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = 0 & on \quad \Gamma_1, \end{cases} \tag{15}$$

$$\begin{cases} -\triangle^2 \dfrac{\partial z_\varepsilon}{\partial g_1}(0,0) - 3\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)\right) z_\varepsilon^2(0,0) - \varepsilon \dfrac{\partial z_\varepsilon}{\partial g_1}(0,0) = 0 & in \quad \Omega, \\ \dfrac{\partial z_\varepsilon}{\partial g_1}(0,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)\right) = 0; \quad \dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)\right) = 0 & on \quad \Gamma_0, \\ \varepsilon\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0) - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)\right) = 0; \quad \varepsilon\dfrac{\partial}{\partial \nu}\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)\right) + \triangle\left(\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)\right) = \varepsilon g_1 & on \quad \Gamma_1. \end{cases} \tag{16}$$

Proof :

⊗ Let us show that $z_\varepsilon(0,0)$ is a solution of (15):

$$\bullet -\triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) - 3\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) z_\varepsilon^2(0,0) - \varepsilon\frac{\partial z_\varepsilon}{\partial g_0}(0,0) = \frac{\partial}{\partial g_0}\left(-\triangle^2 z_\varepsilon(0,0) - z_\varepsilon^3(0,0) - \varepsilon z_\varepsilon(0,0)\right) = 0.$$

because, $-\triangle^2 z_\varepsilon(0,0) - (z_\varepsilon(0,0))^3 - \varepsilon z_\varepsilon(0,0) = 0 \quad$ in $\quad \Omega$, thus,

$$-\triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) - 3\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) z_\varepsilon^2(0,0) - \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = 0 \quad in \quad \Omega.$$

•*For boundary conditions* Γ_0 :

$$\frac{\partial z_\varepsilon}{\partial g_0}(0,0) - \frac{\partial}{\partial \nu}\triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = \left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) - \frac{\partial}{\partial \nu}\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right),$$

$$= \left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - \dfrac{\partial}{\partial \nu}\triangle z_\varepsilon(0,\lambda g_0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,0) - \dfrac{\partial}{\partial \nu}\triangle z_\varepsilon(0,0)}{\lambda}\right) = 0,$$

$$\frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) + \triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = \frac{\partial}{\partial \nu}\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) + \triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right),$$

$$= \left(\lim_{\lambda\longrightarrow 0}\frac{\dfrac{\partial z_\varepsilon}{\partial \nu}(0,\lambda g_0) - \triangle z_\varepsilon(0,\lambda g_0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{\dfrac{\partial z_\varepsilon}{\partial \nu}(0,0) - \triangle z_\varepsilon(0,0)}{\lambda}\right) = 0.$$

•*For boundary conditions* Γ_1 :

$$\varepsilon\frac{\partial z_\varepsilon}{\partial g_0}(0,0) - \frac{\partial}{\partial \nu}\triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = \varepsilon\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) - \frac{\partial}{\partial \nu}\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right),$$

$$= \left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon z_\varepsilon(0,\lambda g_0) - \dfrac{\partial}{\partial \nu}\triangle z_\varepsilon(0,\lambda g_0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon z_\varepsilon(0,0) - \dfrac{\partial}{\partial \nu}\triangle z_\varepsilon(0,0)}{\lambda}\right) = \varepsilon g_0,$$

$$\varepsilon\frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) + \triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\right) = \varepsilon\frac{\partial}{\partial \nu}\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right) + \triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(0,\lambda g_0) - z_\varepsilon(0,0)}{\lambda}\right),$$

$$= \left(\lim_{\lambda \longrightarrow 0} \frac{\varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu}(0, \lambda g_0) - \triangle z_\varepsilon(0, \lambda g_0)}{\lambda} \right) - \left(\lim_{\lambda \longrightarrow 0} \frac{\varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu}(0,0) - \triangle z_\varepsilon(0,0)}{\lambda} \right) = 0.$$

⊗ Using similar reasoning as before, we show that $z_\varepsilon(0,0)$ is solution of (16).

■

Proof of Lemma 2.2:

• By multiplying the first equation of (12) by $\frac{\partial z_\varepsilon}{\partial g_0}(v,0)$, we have:

$$\langle -\triangle^2 \zeta_\varepsilon(v) - 3\zeta_\varepsilon(v)(z_\varepsilon)^2 - \varepsilon\zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega = \langle -\left(z_\varepsilon(u_\varepsilon^\gamma, 0) - z_d\right), \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega,$$

$$\textit{and that } \langle \triangle^2 \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega + 3\langle \zeta_\varepsilon(v), (z_\varepsilon)^2 \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega + \langle \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega = \langle (z_\varepsilon(u_\varepsilon,0) - z_d), \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega,$$

Applying Green's formulation to the first membrer, we have:

$$\begin{aligned}
\langle \triangle^2 \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega &= \langle \frac{\partial(\triangle \zeta_\varepsilon)}{\partial \nu}(v), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_\Gamma - \langle \triangle \zeta_\varepsilon(v), \frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Gamma + \langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Gamma - \langle \zeta_\varepsilon(v), \frac{\partial}{\partial \nu}\triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Gamma + \\
&\quad + \langle \zeta_\varepsilon(v), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega, \\
&= \langle \frac{\partial(\triangle \zeta_\varepsilon)}{\partial \nu}(v), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_{\Gamma_1} - \langle \triangle \zeta_\varepsilon(v), \frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} - \langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon\frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} - \langle \zeta_\varepsilon(v), \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} + \\
&\quad + \langle \zeta_\varepsilon(v), \varepsilon g_0\rangle_{\Gamma_1} + \langle \zeta_\varepsilon(v), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega, \\
&= \langle \frac{\partial(\triangle \zeta_\varepsilon)}{\partial \nu}(v) - \varepsilon\zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_{\Gamma_1} - \langle \triangle \zeta_\varepsilon(v) + \varepsilon\frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} + \langle \zeta_\varepsilon(v), \varepsilon g_0\rangle_{\Gamma_1} + \\
&\quad + \langle \zeta_\varepsilon(v), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega, \\
&= \langle \zeta_\varepsilon(v), \varepsilon g_0\rangle_{\Gamma_1} + \langle \zeta_\varepsilon(v), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega,
\end{aligned}$$

$$\Rightarrow \langle \triangle^2 \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_\Omega + 3\langle \zeta_\varepsilon(v), (z_\varepsilon)^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega + \langle \zeta_\varepsilon(v), \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega = \langle \zeta_\varepsilon(v), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right) + 3(z_\varepsilon)^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right) + \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega + \langle \zeta_\varepsilon(v), \varepsilon g_0\rangle_{\Gamma_1}.$$

$$\textit{hence} \quad \langle (z_\varepsilon(u_\varepsilon,0) - z_d), \frac{\partial z_\varepsilon}{\partial g_0}(v,0)\rangle_\Omega = \langle \zeta_\varepsilon(v), \varepsilon g_0\rangle_{\Gamma_1}.$$

• By multiplying the first equation of (12) by $\frac{\partial z_\varepsilon}{\partial g_1}(v,0)$, we have:

$$\langle -\triangle^2 \zeta_\varepsilon(v) - 3\zeta_\varepsilon(v)(z_\varepsilon)^2 - \varepsilon\zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\rangle_\Omega = \langle -(z_\varepsilon(u_\varepsilon,0) - z_d), \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\rangle_\Omega,$$

$$\Rightarrow \langle \triangle^2 \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\rangle_\Omega + 3\langle \zeta_\varepsilon(v), (z_\varepsilon)^2 \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\rangle_\Omega + \langle \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\rangle_\Omega = \langle (z_\varepsilon(u_\varepsilon,0) - z_d), \frac{\partial z_\varepsilon}{\partial g}(v,0)\rangle_\Omega.$$

By again applying Green's formulation to the first membrer as before, we obtain:

$$\langle \triangle^2 \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\rangle_\Omega = \langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g_1\rangle_{\Gamma_1} + \langle \zeta_\varepsilon(v), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_1}\right)\rangle_\Omega.$$

As a result:

$$\langle (z_\varepsilon(u_\varepsilon,0) - z_d), \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\rangle_\Omega = \langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g_1\rangle_{\Gamma_1}.$$

• By multiplying the first equation of (12) for the case $v = 0$ by $\dfrac{\partial z_\varepsilon}{\partial g_0}(0,0)$, we have:

$$\langle -\triangle^2\zeta_\varepsilon(0) - 3\zeta_\varepsilon(0)\,(z_\varepsilon)^2 - \varepsilon\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega = \langle -\left(z_\varepsilon(u_\varepsilon,0) - z_d\right), \frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega,$$

$$\Rightarrow \langle \triangle^2\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega + 3\langle\zeta_\varepsilon(0),(z_\varepsilon)^2\frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega + \langle\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega = \langle\left(z_\varepsilon(u_\varepsilon,0) - z_d\right), \frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega.$$

Applying Green's formulation to the first membrer, we have:

$$\begin{aligned}
\langle \triangle^2\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega &= \langle \frac{\partial(\triangle\zeta_\varepsilon)}{\partial\nu}(0), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_\Gamma - \langle\triangle\zeta_\varepsilon(0), \frac{\partial}{\partial\nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Gamma + \langle\frac{\partial\zeta_\varepsilon}{\partial\nu}(0), \triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Gamma - \langle\zeta_\varepsilon(0), \frac{\partial}{\partial\nu}\triangle\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Gamma + \\
&\quad + \langle\zeta_\varepsilon(0), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega, \\
&= \langle \frac{\partial(\triangle\zeta_\varepsilon)}{\partial\nu}(0), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_{\Gamma_1} - \langle\triangle\zeta_\varepsilon(0), \frac{\partial}{\partial\nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} - \langle\frac{\partial\zeta_\varepsilon}{\partial\nu}(0), \varepsilon\frac{\partial}{\partial\nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} - \langle\zeta_\varepsilon(0), \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} + \\
&\quad + \langle\zeta_\varepsilon(0), \varepsilon g_0\rangle_{\Gamma_1} + \langle\zeta_\varepsilon(0), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega, \\
&= \langle \frac{\partial(\triangle\zeta_\varepsilon)}{\partial\nu}(0) - \varepsilon\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_{\Gamma_1} - \langle\triangle\zeta_\varepsilon(0) + \varepsilon\frac{\partial\zeta_\varepsilon}{\partial\nu}(0), \frac{\partial}{\partial\nu}\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_{\Gamma_1} + \langle\zeta_\varepsilon(0), \varepsilon g_0\rangle_{\Gamma_1} + \\
&\quad + \langle\zeta_\varepsilon(0), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega, \\
&= \langle\zeta_\varepsilon(0), \varepsilon g_0\rangle_{\Gamma_1} + \langle\zeta_\varepsilon(0), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega,
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \langle \triangle^2\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_0}\rangle_\Omega + 3\langle\zeta_\varepsilon(0),(z_\varepsilon)^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega + \langle\zeta_\varepsilon(0), \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega &= \langle\zeta_\varepsilon(0), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right) + 3\,(z_\varepsilon)^2\left(\frac{\partial z_\varepsilon}{\partial g_0}\right) + \varepsilon\left(\frac{\partial z_\varepsilon}{\partial g_0}\right)\rangle_\Omega + \\
&\quad + \langle\zeta_\varepsilon(0), \varepsilon g_0\rangle_{\Gamma_1},
\end{aligned}$$

$$\Longrightarrow \langle\left(z_\varepsilon(u_\varepsilon,0) - z_d\right), \frac{\partial z_\varepsilon}{\partial g_0}(0,0)\rangle_\Omega = \langle\zeta_\varepsilon(0), \varepsilon g_0\rangle_{\Gamma_1}.$$

• By multiplying the first equation of (12) by $\dfrac{\partial z_\varepsilon}{\partial g_1}(0,0)$, we have:

$$\langle -\triangle^2\zeta_\varepsilon(0) - 3\zeta_\varepsilon^\gamma(0)\,(z_\varepsilon)^2 - \varepsilon\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\rangle_\Omega = \langle -\left(z_\varepsilon(u_\varepsilon,0) - z_d\right), \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\rangle_\Omega,$$

$$\Rightarrow \langle \triangle^2\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\rangle_\Omega + 3\langle\zeta_\varepsilon(0),(z_\varepsilon)^2\frac{\partial z_\varepsilon}{\partial g_1}(0,0)\rangle_\Omega + \langle\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\rangle_\Omega = \langle\left(z_\varepsilon(u_\varepsilon,0) - z_d\right), \frac{\partial z_\varepsilon}{\partial g}(0,0)\rangle_\Omega.$$

By again Applying Green's formulation to the first membrer as before, we obtain:

$$\langle \triangle^2\zeta_\varepsilon(0), \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\rangle_\Omega = \langle\frac{\partial\zeta_\varepsilon}{\partial\nu}(0), \varepsilon g_1\rangle_{\Gamma_1} + \langle\zeta_\varepsilon(0), \triangle^2\left(\frac{\partial z_\varepsilon}{\partial g_1}\right)\rangle_\Omega.$$

As a result:

$$\langle\left(z_\varepsilon(u_\varepsilon,0) - z_d\right), \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\rangle_\Omega = \langle\frac{\partial\zeta_\varepsilon}{\partial\nu}(0), \varepsilon g_1\rangle_{\Gamma_1}.$$

In summary:

$$\begin{aligned}
\left\langle z_\varepsilon(v,0) - z_d, \frac{\partial z_\varepsilon}{\partial g_0}(v,0) + \frac{\partial z_\varepsilon}{\partial g_1}(v,0)\right\rangle_\Omega - \left\langle z_\varepsilon(0,0) - z_d, \frac{\partial z_\varepsilon}{\partial g_0}(0,0) + \frac{\partial z_\varepsilon}{\partial g_1}(0,0)\right\rangle_\Omega &= \langle\zeta_\varepsilon(v) - \zeta_\varepsilon(0), \varepsilon g_0\rangle_{\Gamma_1} + \\
&\quad + \langle\frac{\partial\zeta_\varepsilon}{\partial\nu}(v) - \frac{\partial\zeta_\varepsilon}{\partial\nu}(0), \varepsilon g_1\rangle_{\Gamma_1}.
\end{aligned}$$

Thus, $J_\varepsilon(v,g) - J_\varepsilon(0,g) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + 2\langle \zeta_\varepsilon(v) - \zeta_\varepsilon(0), \varepsilon g_0\rangle_{L^2(\Gamma_1)} + 2\left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v) - \frac{\partial \zeta_\varepsilon}{\partial \nu}(0), \varepsilon g_1\right\rangle_{L^2(\Gamma_1)}$.

Hence the result.

■

Remark 2.2.

$$\sup_{g\in(L^2(\Gamma_1))^2} (J(v,g) - J(0,g)) = J_\varepsilon(v,0) - J_\varepsilon(0,0) +$$

$$+2 \sup_{g\in(L^2(\Gamma_1))^2}\left(\langle \zeta_\varepsilon(v) - \zeta_\varepsilon(0), \varepsilon g_0\rangle_{L^2(\Gamma_1)} + \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v) - \frac{\partial \zeta_\varepsilon}{\partial \nu}(0), \varepsilon g_1\right\rangle_{L^2(\Gamma_1)}\right),$$

$$= J_\varepsilon(v,0) - J_\varepsilon(0,0) + 2\varepsilon \sup_{g\in(L^2(\Gamma_1))^2}\left(\langle \zeta_\varepsilon(v) - \zeta_\varepsilon(0), g_0\rangle_{L^2(\Gamma_1)} + \left\langle \frac{\partial}{\partial \nu}\left(\zeta_\varepsilon(v) - \zeta_\varepsilon(0)\right), g_1\right\rangle_{L^2(\Gamma_1)}\right),$$

it is deduced that:

$$\sup_{g\in(L^2(\Gamma_1))^2} (J(v,g) - J(0,g)) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + 2\varepsilon \sup_{g\in(L^2(\Gamma_1))^2}\left(\langle S(v), g_0\rangle_{L^2(\Gamma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu}, g_1\right\rangle_{L^2(\Gamma_1)}\right), \tag{17}$$

with $S(v) = \zeta(v) - \zeta(0)$ and,

$$\sup_{g\in(L^2(\Gamma_1))^2}\left(\langle S(v), g_0\rangle_{L^2(\Gamma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu}, g_1\right\rangle_{L^2(\Gamma_1)}\right) = \begin{cases} +\infty \quad if \quad \left(\langle S(v), g_0\rangle_{L^2(\Gamma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu}, g_1\right\rangle_{L^2(\Gamma_1)}\right) \neq 0, \\ 0 \quad if \quad S(v) \perp g_0 \ and \ \frac{\partial S(v)}{\partial \nu} \perp g_1 \ \forall \ g \in (L^2(\Gamma_1))^2. \end{cases}$$

To give meaning to the following minimization problem:

$$\inf_{v\in(L^2(\Gamma_0))^2}\left(\sup_{g\in(L^2(\Gamma_1))^2} (J(v,g) - J(0,g))\right), \tag{18}$$

we consider:

$$O = \left\{ v \in (L^2(\Gamma_0))^2 \ such\ as\ \langle S(v), g_0\rangle_{L^2(\Gamma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu}, g_1\right\rangle_{L^2(\Gamma_1)} = 0, \quad \forall g \in (L^2(\Gamma_1))^2 \right\}. \tag{19}$$

2.2 Low-Regret Control

However, to make the resolution of (18) simpler as, J.L.Lions we introduce the parameter $-\gamma \parallel g_0 \parallel^2_{L^2(\Gamma_1)}$ and $-\gamma \parallel g_1 \parallel^2_{L^2(\Gamma_1)}$ where γ is a positive parameter and means a relaxation parameter ($\gamma > 0$). Thus, we obtain :

$$\inf_{v\in(L^2(\Gamma_0))^2}\left(\sup_{g\in(L^2(\Gamma_1))^2} (J_\varepsilon(v,g) - J_\varepsilon(0,g) - \gamma \parallel g_0 \parallel^2_{L^2(\Gamma_1)} - \gamma \parallel g_1 \parallel^2_{L^2(\Gamma_1)})\right), \tag{20}$$

the control u solution of (20) is a low-regret control, see (Nakoulima O., Omrane A.,& Dorville R, 2004).

Indeed, the concept of ≪ low-regret control ≫ depends on γ and the norm $\parallel g \parallel$. It is interpreted as an approximation no-regret control.

With low-regret control, we admit the possibility to make a choice of control u slightly catastrophic than the ground state with a margin of error that must not exceed $\gamma\|g\|^2_{L^2(\Gamma_1)}$.

Lemma 2.3. *For any $v \in (L^2(\Gamma_0))^2$, relaxed problem (20) becomes:*

$$\inf_{v\in(L^2(\Gamma_0))^2}\left(J_\varepsilon(v,0) - J_\varepsilon(0,0) + \frac{\varepsilon^2}{\gamma} \parallel S(v) \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(v)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)}\right). \tag{21}$$

Proof :

From (11),

$$J_\varepsilon(v,g)-J_\varepsilon(0,g)=J_\varepsilon(v,0)-J_\varepsilon(0,0)+2\left(\left\langle\zeta_\varepsilon^\gamma(v)-\zeta_\varepsilon^\gamma(0),\varepsilon g_0\right\rangle_{L^2(\Gamma_1)}+\left\langle\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(v)-\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(0),\varepsilon g_1\right\rangle_{L^2(\Gamma_1)}\right),$$

so,

$$\sup_{g\in(L^2(\Gamma_1))^2}(J_\varepsilon(v,g)-J_\varepsilon(0,g)-\gamma\parallel g_0\parallel^2_{L^2(\Gamma_1)}-\gamma\parallel g_1\parallel^2_{L^2(\Gamma_1)})=J_\varepsilon(v,0)-J_\varepsilon(0,0)+$$

$$+2\sup_{g\in(L^2(\Gamma_1))^2}\left(\left\langle\zeta_\varepsilon^\gamma(v)-\zeta_\varepsilon^\gamma(0),\varepsilon g_0\right\rangle_{L^2(\Gamma_1)}+\left\langle\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(v)-\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(0),\varepsilon g_1\right\rangle_{L^2(\Gamma_1)}-\frac{\gamma}{2}\parallel g_0\parallel^2_{L^2(\Gamma_1)}-\frac{\gamma}{2}\parallel g_1\parallel^2_{L^2(\Gamma_1)}\right).$$

According to Fenchel transformation,

$$\sup_{g\in(L^2(\Gamma_1))^2}\left(\left\langle\zeta_\varepsilon^\gamma(v)-\zeta_\varepsilon^\gamma(0),\varepsilon g_0\right\rangle_{L^2(\Gamma_1)}+\left\langle\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(v)-\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(0),\varepsilon g_1\right\rangle_{L^2(\Gamma_1)}-\frac{\gamma}{2}\parallel g_0\parallel^2_{L^2(\Gamma_1)}-\frac{\gamma}{2}\parallel g_1\parallel^2_{L^2(\Gamma_1)}\right)$$

$$=\frac{\varepsilon^2}{2\gamma}\parallel\zeta_\varepsilon^\gamma(v)-\zeta_\varepsilon^\gamma(0)\parallel^2_{L^2(\Gamma_1)}+\frac{\varepsilon^2}{2\gamma}\parallel\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(v)-\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}(0)\parallel^2_{L^2(\Gamma_1)},$$

we obtain the result.

■

Finally, we can reformulate the problem (20) as follows :

For any $\gamma>0$, find $u_\varepsilon^\gamma\in(L^2(\Gamma_0))^2$ such as:

$$J_\varepsilon^\gamma(u^\gamma)=\inf_{v\in(L^2(\Gamma_0))^2}J_\varepsilon^\gamma(v),\tag{22}$$

$$with\quad J_\varepsilon^\gamma(v)=J_\varepsilon(v,0)-J_\varepsilon(0,0)+\frac{\varepsilon^2}{\gamma}\parallel S(v)\parallel^2_{L^2(\Gamma_1)}+\frac{\varepsilon^2}{\gamma}\parallel\frac{\partial S(v)}{\partial\nu}\parallel^2_{L^2(\Gamma_1)}.$$

The problem (22) is a low-regret problem and its solution, if it exists, will be the low-regret control.

Remark 2.3. *Contrary to the linear case, the function J_ε^γ is not convex, so we do not necessarily have the uniqueness of u_ε. Moreover, we are not sure that u_ε converges in O. Thus, we use the adapted penalization method defined by J. L Lions in (J. L. Lions, 1968) for the search for low-regret control.*

2.3 Adapted Low-Regret Control

In this part, we are interested in finding a solution of the following minimization problem:

$$\inf_{v\in(L^2(\Gamma_0))^2}J_{\varepsilon a}^\gamma(v),\tag{23}$$

where

$$J_{\varepsilon a}^\gamma(v)=J_\varepsilon(v,0)-J_\varepsilon(0,0)+\parallel v_0-\tilde{u_0}\parallel^2_{L^2(\Gamma_1)}+\parallel v_1-\tilde{u_1}\parallel^2_{L^2(\Gamma_1)}+\frac{\varepsilon^2}{\gamma}\parallel S(v)\parallel^2_{L^2(\Gamma_1)}+\frac{\varepsilon^2}{\gamma}\parallel\frac{\partial S(v)}{\partial\nu}\parallel^2_{L^2(\Gamma_1)},\tag{24}$$

with $\tilde{u}=(\tilde{u_0},\tilde{u_1})\in(L^2(\Gamma_0))^2$ is a no-regret control and the control u_ε^γ solution of (23) will be the adapted low-regret control .

2.4 Existence of Adapted Low-Regret Control

The following proposition shows the existence of an adapted low-regret control.

Proposition 2.3. *There is at least a low-regret adapted control $u_\varepsilon^\gamma\in(L^2(\Gamma_0))^2$ solution of (23).*

Proof :

From the definition of $J_{\varepsilon a}^\gamma$, we have:

$$J^{\gamma}_{\varepsilon a}(v) \geq -J_{\varepsilon}(0,0), \forall v \in (L^2(\Gamma_0))^2,$$

thus :

$$-J_{\varepsilon}(0,0) \leq J_{\varepsilon}(v,0) - J_{\varepsilon}(0,0) + \frac{\varepsilon^2}{\gamma} \parallel S(v) \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(v)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)} + \parallel v_0 - \tilde{u_0} \parallel^2_{L^2(\Gamma_1)} + \parallel v_1 - \tilde{u_1} \parallel^2_{L^2(\Gamma_1)} .$$

Let's define $J^{\gamma}_{\varepsilon a}$ by:

$$J^{\gamma}_{\varepsilon a} : L^2(\Gamma_0) \longrightarrow \mathbb{R},$$
$$(v_0, v_1) \longmapsto J^{\gamma}_{\varepsilon a}(v_0, v_1).$$

We denote by:

$$A = \{v \in (L^2(\Gamma_0))^2, J^{\gamma}_{\varepsilon a}(v) \geq -J(0,0)\}.$$

We have :

$$J^{\gamma}_{\varepsilon a}(0) = \frac{\varepsilon^2}{\gamma} \parallel S(0) \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(0)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)} \geq -J(0,0).$$

We assume $A \neq \emptyset$. Therefore:

$$d^{\gamma}_{\varepsilon} = \inf_{v \in (L^2(\Gamma_0))^2} J^{\gamma}_{\varepsilon a}(v)$$

exists.

Let $v_n = v_n(\varepsilon, \gamma)$ be a minimizing sequence such as,

$$d^{\gamma}_{\varepsilon} = \lim_{n \longrightarrow \infty} J^{\gamma}_{\varepsilon a}(v_n), \tag{25}$$

then, $-J(0,0) \leq J^{\gamma}_{\varepsilon a}(v_n) \leq d^{\gamma}_{\varepsilon} + 1$.

As $J^{\gamma}_{\varepsilon a}(v_n) = J_{\varepsilon}(v_n,0) - J_{\varepsilon}(0,0) + \parallel v_{0n} - \tilde{u_0} \parallel^2_{L^2(\Gamma_1)} + \parallel v_{1n} - \tilde{u_1} \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel S(v_n) \parallel^2_{L^2(\Gamma_1)} +$

$$+ \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(v_n)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)},$$

$$= \parallel z_{\varepsilon}(v_n,0) - z_d \parallel^2_{L^2(\Omega)} + N_0 \|v_{0n}\|^2_{L^2(\Gamma_0)} + N_1 \|v_{1n}\|^2_{L^2(\Gamma_0)} - \parallel z_{\varepsilon}(0,0) - z_d \parallel^2_{L^2(\Omega)} +$$

$$+ \parallel v_{0n} - \tilde{u_0} \parallel^2_{L^2(\Gamma_1)} + \parallel v_{1n} - \tilde{u_1} \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel S(v_n) \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(v_n)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)} .$$

$J^{\gamma}_{\varepsilon a}(v_n) \leq d^{\gamma}_{\varepsilon} + 1 \Rightarrow \parallel z_{\varepsilon}(v_n,0) - z_d \parallel^2_{L^2(\Omega)} + N_0 \|v_{0n}\|^2_{L^2(\Gamma_0)} + N_1 \|v_{1n}\|^2_{L^2(\Gamma_0)} - \parallel z_{\varepsilon}(0,0) - z_d \parallel^2_{L^2(\Omega)} + \parallel v_{0n} - \tilde{u_0} \parallel^2_{L^2(\Gamma_1)} + \parallel v_{1n} - \tilde{u_1} \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel S(v_n) \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(v_n)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)} \leq d^{\gamma}_{\varepsilon} + 1.$

Therefore, it exists a constant c^{γ}_{ε} independent of n such as: $\begin{cases} N_0 \|v_{0n}\|^2_{L^2(\Gamma_0)} \leq c^{\gamma}_{\varepsilon}, \\ N_1 \|v_{1n}\|^2_{L^2(\Gamma_0)} \leq c^{\gamma}_{\varepsilon}, \end{cases} \Rightarrow \begin{cases} \|v_{0n}\|_{L^2(\Gamma_0)} \leq \sqrt{\frac{c^{\gamma}_{\varepsilon}}{N_0}}, \\ \|v_{1n}\|_{L^2(\Gamma_0)} \leq \sqrt{\frac{c^{\gamma}_{\varepsilon}}{N_1}}, \end{cases}$

and so we can extract sequences v_n such as $v_n(\varepsilon, \gamma) \rightharpoonup u^{\gamma}_{\varepsilon}$ in $(L^2(\Gamma_0))^2$.

■

Give some preliminary results that will be used for the characterization of the control :

$z_{\varepsilon}(v^{\gamma}, g)$ is differentiable on $(L^2(\Gamma_0))^2$:

$z_{\varepsilon}(v^{\gamma} + w, g) - z_{\varepsilon}(v^{\gamma}, g) = w_0 . \frac{\partial z^{\gamma}_{\varepsilon}}{\partial v_0}(v, g) + w_1 \frac{\partial z_{\varepsilon}}{\partial v_1}(v^{\gamma}, g) + \parallel (w_0, w_1) \parallel_{(L^2(\Gamma_0))^2} \epsilon(w_0, w_1),$

Proposition 2.4. *$\zeta_\varepsilon^\gamma := \zeta_\varepsilon(v,0)$ is solution of systems:*

$$\begin{cases} -\triangle^2 \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} - 3\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right)\left(z_\varepsilon^\gamma\right)^2 - \varepsilon \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} = -\dfrac{\partial z_\varepsilon}{\partial v_0}(u_\varepsilon^\gamma, 0) & in \quad \Omega, \\ \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = 0; \quad \dfrac{\partial}{\partial \nu}\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) + \triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = 0 & on \quad \Gamma_0, \\ \varepsilon\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = 0; \quad \varepsilon\dfrac{\partial}{\partial \nu}\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) + \triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = 0 & on \quad \Gamma_1. \end{cases} \tag{26}$$

$$\begin{cases} -\triangle^2 \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1} - 3\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right)\left(z_\varepsilon^\gamma\right)^2 - \varepsilon \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1} = -\dfrac{\partial z_\varepsilon}{\partial v_1}(u_\varepsilon^\gamma, 0) & in \quad \Omega, \\ \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1} - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right) = 0; \quad \dfrac{\partial}{\partial \nu}\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right) + \triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right) = 0 & on \quad \Gamma_0, \\ \varepsilon\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1} - \dfrac{\partial}{\partial \nu}\triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right) = 0; \quad \varepsilon\dfrac{\partial}{\partial \nu}\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right) + \triangle\left(\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right) = 0 & on \quad \Gamma_1. \end{cases} \tag{27}$$

Proof :

⊗ Let us show that ζ_ε^γ is a solution of (26):

$$\bullet - \triangle^2\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) - 3\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) z_\varepsilon^2 - \varepsilon\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} = -\triangle^2\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right) -$$

$$-3\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right)\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right),$$

$$= \lim_{\lambda\longrightarrow 0}\left(\frac{-\triangle^2\zeta_\varepsilon(\lambda v_0^\gamma, 0) - 3\zeta_\varepsilon(\lambda v_0^\gamma, 0)\,(z_\varepsilon)^2 - \varepsilon\zeta_\varepsilon(\lambda v_0^\gamma, 0)}{\lambda}\right) - \lim_{\lambda\longrightarrow 0}\left(\frac{-\triangle^2\zeta_\varepsilon(0,0) - 3\zeta_\varepsilon(0,0)\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\zeta_\varepsilon(0,0)}{\lambda}\right),$$

$$= \lim_{\lambda\longrightarrow 0}\left(\frac{z_\varepsilon(\lambda v_0^\gamma, 0) - z_\varepsilon(0,0)}{\lambda}\right) = -\frac{\partial z_\varepsilon}{\partial v_0}(u_\varepsilon^\gamma, 0),$$

thus,

$$-\triangle^2\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) - 3\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right)\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = -\frac{\partial z_\varepsilon}{\partial v_0}(u_\varepsilon^\gamma, 0) \quad in \quad \Omega.$$

•*For boundary conditions* Γ_0 :

$$\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} - \frac{\partial}{\partial \nu}\triangle\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = \left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right) - \frac{\partial}{\partial \nu}\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right),$$

$$= \left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \dfrac{\partial}{\partial \nu}\triangle\zeta_\varepsilon(\lambda v_0^\gamma,)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(0,0) - \dfrac{\partial}{\partial \nu}\triangle\zeta_\varepsilon(0,0)}{\lambda}\right) = 0,$$

$$\frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) + \triangle\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = \frac{\partial}{\partial \nu}\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right) + \triangle\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right),$$

$$= \left(\lim_{\lambda\longrightarrow 0}\frac{\dfrac{\partial \zeta_\varepsilon}{\partial \nu}(\lambda v_0^\gamma, 0) - \triangle\zeta_\varepsilon(\lambda v_0^\gamma, 0)}{\lambda}\right) - \left(\lim_{\lambda\longrightarrow 0}\frac{\dfrac{\partial \zeta_\varepsilon}{\partial \nu}(0,0) - \triangle\zeta_\varepsilon(0,0)}{\lambda}\right) = 0.$$

•*For boundary conditions* Γ_1 :

$$\varepsilon\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} - \frac{\partial}{\partial \nu}\triangle\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right) = \varepsilon\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right) - \frac{\partial}{\partial \nu}\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda}\right),$$

$$= \left(\lim_{\lambda \longrightarrow 0} \frac{\varepsilon \zeta_\varepsilon(\lambda v_0^\gamma, 0) - \frac{\partial}{\partial \nu} \triangle \zeta_\varepsilon(\lambda v_0^\gamma, 0)}{\lambda} \right) - \left(\lim_{\lambda \longrightarrow 0} \frac{\varepsilon \zeta_\varepsilon(0,0) - \frac{\partial}{\partial \nu} \triangle \zeta_\varepsilon(0,0)}{\lambda} \right) = 0,$$

$$\varepsilon \frac{\partial}{\partial \nu} \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} \right) + \triangle \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0} \right) = \varepsilon \frac{\partial}{\partial \nu} \left(\lim_{\lambda \longrightarrow 0} \frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda} \right) + \triangle \left(\lim_{\lambda \longrightarrow 0} \frac{\zeta_\varepsilon(\lambda v_0^\gamma, 0) - \zeta_\varepsilon(0,0)}{\lambda} \right),$$

$$= \left(\lim_{\lambda \longrightarrow 0} \frac{\varepsilon \frac{\partial \zeta_\varepsilon}{\partial \nu}(\lambda v_0^\gamma, 0) - \triangle \zeta_\varepsilon(\lambda v_0^\gamma, 0)}{\lambda} \right) - \left(\lim_{\lambda \longrightarrow 0} \frac{\varepsilon \frac{\partial \zeta_\varepsilon}{\partial \nu}(0,0) - \triangle \zeta_\varepsilon(0,0)}{\lambda} \right) = 0.$$

⊗ Using similar reasoning as before, we show that ζ_ε^γ is solution of (27).

■

Proposition 2.5. $z_\varepsilon^\gamma := z_\varepsilon(v^\gamma, 0)$ *is solution of systems:*

$$\begin{cases} -\triangle^2 \frac{\partial z_\varepsilon^\gamma}{\partial v_0} - 3 \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) \left(z_\varepsilon^\gamma \right)^2 - \varepsilon \frac{\partial z_\varepsilon^\gamma}{\partial v_0} = 0 & in \quad \Omega, \\ \frac{\partial z_\varepsilon^\gamma}{\partial v_0} - \frac{\partial}{\partial \nu} \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) = v_0; \quad \frac{\partial}{\partial \nu} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) + \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) = 0 & on \quad \Gamma_0, \\ \varepsilon \frac{\partial z_\varepsilon^\gamma}{\partial v_0} - \frac{\partial}{\partial \nu} \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) = 0; \quad \varepsilon \frac{\partial}{\partial \nu} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) + \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) = 0 & on \quad \Gamma_1, \end{cases} \tag{28}$$

$$\begin{cases} -\triangle^2 \frac{\partial z_\varepsilon^\gamma}{\partial v_1} - 3 \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_1} \right) \left(z_\varepsilon^\gamma \right)^2 - \varepsilon \frac{\partial z_\varepsilon^\gamma}{\partial v_1} = 0 & in \quad \Omega, \\ \frac{\partial z_\varepsilon^\gamma}{\partial v_1} - \frac{\partial}{\partial \nu} \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_1} \right) = 0; \quad \frac{\partial}{\partial \nu} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_1} \right) + \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_1} \right) = v_1 & on \quad \Gamma_0, \\ \varepsilon \frac{\partial z_\varepsilon^\gamma}{\partial v_1} - \frac{\partial}{\partial \nu} \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_1} \right) = 0; \quad \varepsilon \frac{\partial}{\partial \nu} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_1} \right) + \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_1} \right) = 0 & on \quad \Gamma_1. \end{cases} \tag{29}$$

Proof :
⊗ Let us show that ζ_ε^γ is a solution of (28):

$$\bullet - \triangle^2 \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) - 3 \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) \left(z_\varepsilon^\gamma \right)^2 - \varepsilon \frac{\partial z_\varepsilon^\gamma}{\partial v_0} = \frac{\partial}{\partial v_0} \left(-\triangle^2 z_\varepsilon^\gamma - \left(z_\varepsilon^\gamma \right)^3 - \varepsilon z_\varepsilon^\gamma \right) = 0,$$

because $-\triangle^2 z_\varepsilon^\gamma - \left(z_\varepsilon^\gamma \right)^3 - \varepsilon z_\varepsilon^\gamma = 0 \quad$ in $\quad \Omega$, thus,

$$-\triangle^2 \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) - 3 \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) \left(z_\varepsilon^\gamma \right)^2 - \varepsilon \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) = 0 \quad in \quad \Omega.$$

•*For boundary conditions* Γ_0 :

$$\frac{\partial z_\varepsilon^\gamma}{\partial v_0} - \frac{\partial}{\partial \nu} \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) = \left(\lim_{\lambda \longrightarrow 0} \frac{z_\varepsilon(\lambda v_0^\gamma, 0) - z_\varepsilon(0,0)}{\lambda} \right) - \frac{\partial}{\partial \nu} \triangle \left(\lim_{\lambda \longrightarrow 0} \frac{z_\varepsilon(\lambda v_0^\gamma, 0) - z_\varepsilon(0,0)}{\lambda} \right),$$

$$= \left(\lim_{\lambda \longrightarrow 0} \frac{z_\varepsilon(\lambda v_0^\gamma, 0) - \frac{\partial}{\partial \nu} \triangle z_\varepsilon(\lambda v_0^\gamma, 0)}{\lambda} \right) - \left(\lim_{\lambda \longrightarrow 0} \frac{z_\varepsilon(0,0) - \frac{\partial}{\partial \nu} \triangle z_\varepsilon(0,0)}{\lambda} \right) = v_0,$$

$$\frac{\partial}{\partial \nu} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) + \triangle \left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0} \right) = \frac{\partial}{\partial \nu} \left(\lim_{\lambda \longrightarrow 0} \frac{z_\varepsilon(\lambda v_0^\gamma, 0) - z_\varepsilon(0,0)}{\lambda} \right) + \triangle \left(\lim_{\lambda \longrightarrow 0} \frac{z_\varepsilon(\lambda v_0^\gamma, 0) - z_\varepsilon(0,0)}{\lambda} \right),$$

$$= \left(\lim_{\lambda \longrightarrow 0} \frac{\frac{\partial z_\varepsilon}{\partial \nu}(\lambda v_0^\gamma, 0) - \triangle z_\varepsilon(\lambda v_0^\gamma, 0)}{\lambda} \right) - \left(\lim_{\lambda \longrightarrow 0} \frac{\frac{\partial z_\varepsilon}{\partial \nu}(0,0) - \triangle z_\varepsilon(0,0)}{\lambda} \right) = 0.$$

$\bullet For\ \ boundary\ \ conditions\ \ \Gamma_1$:

$$\varepsilon\frac{\partial z_\varepsilon^\gamma}{\partial v_0}-\frac{\partial}{\partial\nu}\triangle\left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0}\right)=\varepsilon\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(\lambda v_0^\gamma,0)-z_\varepsilon(0,0)}{\lambda}\right)-\frac{\partial}{\partial\nu}\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(\lambda v_0^\gamma,0)-z_\varepsilon(0,0)}{\lambda}\right),$$

$$=\left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon z_\varepsilon(\lambda v_0^\gamma,0)-\dfrac{\partial}{\partial\nu}\triangle z_\varepsilon(\lambda v_0^\gamma,0)}{\lambda}\right)-\left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon z_\varepsilon(0,0)-\dfrac{\partial}{\partial\nu}\triangle z_\varepsilon(0,0)}{\lambda}\right)=0,$$

$$\varepsilon\frac{\partial}{\partial\nu}\left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0}\right)+\triangle\left(\frac{\partial z_\varepsilon^\gamma}{\partial v_0}\right)=\varepsilon\frac{\partial}{\partial\nu}\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(\lambda v_0^\gamma,0)-z_\varepsilon(0,0)}{\lambda}\right)+\triangle\left(\lim_{\lambda\longrightarrow 0}\frac{z_\varepsilon(\lambda v_0^\gamma,0)-z_\varepsilon(0,0)}{\lambda}\right),$$

$$=\left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon\dfrac{\partial z_\varepsilon}{\partial\nu}(\lambda v_0^\gamma,0)-\triangle z_\varepsilon(\lambda v_0^\gamma,0)}{\lambda}\right)-\left(\lim_{\lambda\longrightarrow 0}\frac{\varepsilon\dfrac{\partial z_\varepsilon}{\partial\nu}(0,0)-\triangle z_\varepsilon(0,0)}{\lambda}\right)=0.$$

⊗ Using similar reasoning as before, we show that z_ε^γ is solution of (29).

■

2.5 Characterization of Adapted Low-Regret Control

Proposition 2.6. *The adapted low-regret control $u_\varepsilon^\gamma=(u_{0\varepsilon}^\gamma,u_{1\varepsilon}^\gamma)$ solution to (23) is characterized by a unique solution $\{\zeta_\varepsilon^\gamma,z_\varepsilon^\gamma,\beta_\varepsilon^\gamma,\phi_\varepsilon^\gamma\}$, of the system:*

$$\begin{cases}-\triangle^2\zeta_\varepsilon^\gamma-3\zeta_\varepsilon^\gamma\left(z_\varepsilon^\gamma\right)^2-\varepsilon\zeta_\varepsilon^\gamma=-\left(z_\varepsilon^\gamma-z_d\right) & in\quad \Omega,\\ \zeta_\varepsilon^\gamma-\dfrac{\partial\triangle\zeta_\varepsilon^\gamma}{\partial\nu}=0;\quad \dfrac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}+\triangle\zeta_\varepsilon^\gamma=0 & on\quad \Gamma_0,\\ \varepsilon\zeta_\varepsilon^\gamma-\dfrac{\partial\triangle\zeta_\varepsilon^\gamma}{\partial\nu}=0;\quad \varepsilon\dfrac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}+\triangle\zeta_\varepsilon^\gamma=0 & on\quad \Gamma_1,\end{cases}$$

$$\begin{cases}-\triangle^2 z_\varepsilon^\gamma-\left(z_\varepsilon^\gamma\right)^3-\varepsilon z_\varepsilon^\gamma=0 & in\quad \Omega,\\ z_\varepsilon^\gamma-\dfrac{\partial\triangle z_\varepsilon^\gamma}{\partial\nu}=u_{0\varepsilon}^\gamma;\quad \dfrac{\partial z_\varepsilon^\gamma}{\partial\nu}+\triangle z_\varepsilon^\gamma=u_{1\varepsilon}^\gamma & on\quad \Gamma_0,\\ \varepsilon z_\varepsilon^\gamma-\dfrac{\partial\triangle z_\varepsilon^\gamma}{\partial\nu}=0;\quad \varepsilon\dfrac{\partial z_\varepsilon}{\partial\nu}+\triangle z_\varepsilon^\gamma=0 & on\quad \Gamma_1,\end{cases}$$

$$\begin{cases}-\triangle^2\beta_\varepsilon^\gamma-3\beta_\varepsilon^\gamma\left(z_\varepsilon^\gamma\right)^2-\varepsilon\beta_\varepsilon^\gamma=0 & in\quad \Omega,\\ \beta_\varepsilon^\gamma-\dfrac{\partial\triangle\beta_\varepsilon^\gamma}{\partial\nu}=0;\quad \dfrac{\partial\beta_\varepsilon^\gamma}{\partial\nu}+\triangle\beta_\varepsilon^\gamma=0 & on\quad \Gamma_0,\\ \varepsilon\beta_\varepsilon^\gamma-\dfrac{\partial\triangle\beta_\varepsilon^\gamma}{\partial\nu}=-\dfrac{\varepsilon^2}{\gamma}(\zeta_\varepsilon(u_\varepsilon^\gamma)-\zeta_\varepsilon(0));\ \varepsilon\dfrac{\partial\beta_\varepsilon^\gamma}{\partial\nu}+\triangle\beta_\varepsilon^\gamma=-\dfrac{\varepsilon^2}{\gamma}\left(\dfrac{\partial(\zeta_\varepsilon(u_\varepsilon^\gamma)-\zeta_\varepsilon(0))}{\partial\nu}\right) & on\quad \Gamma_1,\end{cases}$$

$$\begin{cases}-\triangle^2\phi_\varepsilon^\gamma-3\left(z_\varepsilon^\gamma\right)^2\phi_\varepsilon^\gamma-\varepsilon\phi_\varepsilon^\gamma=z_\varepsilon^\gamma-z_d-\beta_\varepsilon^\gamma & in\quad \Omega,\\ \phi_\varepsilon^\gamma-\dfrac{\partial\triangle\phi_\varepsilon^\gamma}{\partial\nu}=0;\quad \dfrac{\partial\phi_\varepsilon^\gamma}{\partial\nu}+\triangle\phi_\varepsilon^\gamma=0 & on\quad \Gamma_0,\\ \varepsilon\phi_\varepsilon^\gamma-\dfrac{\partial\triangle\phi_\varepsilon^\gamma}{\partial\nu}=0;\quad \varepsilon\dfrac{\partial\phi_\varepsilon^\gamma}{\partial\nu}+\triangle\phi_\varepsilon^\gamma=0 & on\quad \Gamma_1,\end{cases}$$

$\phi_\varepsilon^\gamma+N_0u_{0\varepsilon}^\gamma+N_1u_{1\varepsilon}^\gamma=\tilde{u_0}-u_{0\varepsilon}^\gamma+\tilde{u_1}-u_{1\varepsilon}^\gamma \quad in \quad L^2(\Gamma_0)$,
where $z_\varepsilon^\gamma:=z_\varepsilon(u_\varepsilon^\gamma,0)$.

Proof :

Let u_ε^γ be the solution of (11)-(22) on $(L^2(\Gamma_0))^2$. The Euler-Lagrange necessary condition gives for every $w=(w_0,w_1)\in(L^2(\Gamma_0))^2$:

$$\langle\frac{\partial z_\varepsilon^\gamma}{\partial v_0},z_\varepsilon^\gamma-z_d\rangle_\Omega+\langle\frac{\partial z_\varepsilon^\gamma}{\partial v_1},z_\varepsilon^\gamma-z_d\rangle_\Omega+\langle N_0u_{0\varepsilon}^\gamma+u_{0\varepsilon}^\gamma-\tilde{u_0},w_0\rangle_{\Gamma_0}+\langle N_1u_{1\varepsilon}^\gamma+u_{1\varepsilon}^\gamma-\tilde{u_1},w_1\rangle_{\Gamma_0}+\frac{\varepsilon^2}{\gamma}\langle\frac{\partial\zeta_\varepsilon}{\partial v_0}(w_0),S(u_{0\varepsilon}^\gamma)\rangle_{\Gamma_1}++\frac{\varepsilon^2}{\gamma}\langle\frac{\partial\zeta_\varepsilon}{\partial v_1}(w_1),S(u_{1\varepsilon}^\gamma)\rangle_{\Gamma_1}+$$
$$\frac{\varepsilon^2}{\gamma}\langle\frac{\partial}{\partial\nu}\left(\frac{\partial\zeta_\varepsilon}{\partial v_0}\right)(w_0),\frac{\partial S(u_{0\varepsilon}^\gamma)}{\partial\nu}\rangle_{\Gamma_1}+\frac{\varepsilon^2}{\gamma}\langle\frac{\partial}{\partial\nu}\left(\frac{\partial\zeta_\varepsilon}{\partial v_1}\right)(w_1),\frac{\partial S(u_{1\varepsilon}^\gamma)}{\partial\nu}\rangle_{\Gamma_1}=0.$$

Indeed, $\quad J_{\varepsilon a}^\gamma(u_{0\varepsilon}^\gamma+\lambda w_0,u_{1\varepsilon}^\gamma+\lambda w_1)-J_{\varepsilon a}^\gamma(u_{0\varepsilon}^\gamma,u_{1\varepsilon}^\gamma)=J_\varepsilon(u_{0\varepsilon}^\gamma+\lambda w_0,0)+J_\varepsilon(u_{1\varepsilon}^\gamma+\lambda w_1,0)-J_\varepsilon(u_{0\varepsilon}^\gamma,0)-J_\varepsilon(u_{1\varepsilon}^\gamma,0)+$

$+ \parallel u^{\gamma}_{0\varepsilon} + \lambda w_0 - \tilde{u_0} \parallel^2_{L^2(\Gamma_1)} + \parallel u^{\gamma}_{1\varepsilon} + \lambda w_1 - \tilde{u_1} \parallel^2_{L^2(\Gamma_1)} - \parallel u^{\gamma}_{0\varepsilon} - \tilde{u_0} \parallel^2_{L^2(\Gamma_1)} - \parallel u^{\gamma}_{1\varepsilon} - \tilde{u_1} \parallel^2_{L^2(\Gamma_1)} +$
$+ \frac{\varepsilon^2}{\gamma} \parallel S(u^{\gamma}_{0\varepsilon} + \lambda w_0) \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel S(u^{\gamma}_{1\varepsilon} + \lambda w_1) \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(u^{\gamma}_{0\varepsilon} + \lambda w_0)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(u^{\gamma}_{1\varepsilon} + \lambda w_1)}{\partial \nu} \parallel^2_{L^2(\Gamma_1)} - \frac{\varepsilon^2}{\gamma} \parallel$
$S(u^{\gamma}_{0\varepsilon}) \parallel^2_{L^2(\Gamma_1)} - \frac{\varepsilon^2}{\gamma} \parallel S(u^{\gamma}_{1\varepsilon}) \parallel^2_{L^2(\Gamma_1)} - \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(u^{\gamma}_{0\varepsilon})}{\partial \nu} \parallel^2_{L^2(\Gamma_1)} - \frac{\varepsilon^2}{\gamma} \parallel \frac{\partial S(u^{\gamma}_{1\varepsilon})}{\partial \nu} \parallel^2_{L^2(\Gamma_1)},$

$$J^{\gamma}_{\varepsilon a}(u^{\gamma}_{0\varepsilon} + \lambda w_0, u^{\gamma}_{1\varepsilon} + \lambda w_1) - J^{\gamma}_{\varepsilon a}(u^{\gamma}_{0\varepsilon}, u^{\gamma}_{1\varepsilon}) = \lambda^2 \parallel \frac{z^{\gamma}_{\varepsilon}(u^{\gamma}_{0\varepsilon} + \lambda w_0, 0) - z_{\varepsilon}(u^{\gamma}_{0\varepsilon}, 0)}{\lambda} \parallel^2_{L^2(\Omega)} +$$

$$+\lambda^2 \parallel \frac{z_{\varepsilon}(u^{\gamma}_{1\varepsilon} + \lambda w_1, 0) - z_{\varepsilon}(u^{\gamma}_{1\varepsilon}, 0)}{\lambda} \parallel^2_{L^2(\Omega)} + \lambda^2(N_0 + 1) \parallel w_0 \parallel^2_{L^2(\Gamma_0)} + \lambda^2(N_1 + 1) \parallel w_1 \parallel^2_{L^2(\Gamma_0)} +$$

$$+\frac{(\varepsilon\lambda)^2}{\gamma}\left(\parallel \frac{S(u^{\gamma}_{0\varepsilon} + \lambda w_0) - S(u^{\gamma}_{0\varepsilon})}{\lambda} \parallel^2_{L^2(\Omega)} + \parallel \frac{S(u^{\gamma}_{1\varepsilon} + \lambda w_1) - S(u^{\gamma}_{1\varepsilon})}{\lambda} \parallel^2_{L^2(\Omega)}\right) +$$

$$+\frac{(\varepsilon\lambda)^2}{\gamma}\left(\parallel \frac{\frac{\partial S(u^{\gamma}_{0\varepsilon} + \lambda w_0)}{\partial \nu} - \frac{\partial S(u^{\gamma}_{0\varepsilon})}{\partial \nu}}{\lambda} \parallel^2_{L^2(\Omega)} + \parallel \frac{\frac{\partial S(u^{\gamma}_{1\varepsilon} + \lambda w_1)}{\partial \nu} - \frac{\partial S(u^{\gamma}_{1\varepsilon})}{\partial \nu}}{\lambda} \parallel^2_{L^2(\Omega)}\right) +$$

$+2\langle z^{\gamma}_{\varepsilon}(u^{\gamma}_{0\varepsilon} + \lambda w_0, 0) - z_{\varepsilon}(u^{\gamma}_{0\varepsilon}, 0), z^{\gamma}_{\varepsilon} - z_d\rangle_{\Omega} + 2\langle z_{\varepsilon}(u^{\gamma}_{1\varepsilon} + \lambda w_1, 0) - z_{\varepsilon}(u^{\gamma}_{1\varepsilon}, 0), z^{\gamma}_{\varepsilon} - z_d\rangle_{\Omega} +$
$+ 2\lambda\langle N_0 u^{\gamma}_{0\varepsilon} + u^{\gamma}_{0\varepsilon} - \tilde{u_0}, w_0\rangle_{\Gamma_0} + 2\lambda\langle N_1 u^{\gamma}_{1\varepsilon} + u^{\gamma}_{1\varepsilon} - \tilde{u_1}, w_1\rangle_{\Gamma_0} + \frac{2\varepsilon^2}{\gamma}\langle S(u^{\gamma}_{0\varepsilon} + \lambda w_0, 0) - S(u^{\gamma}_{0\varepsilon}, 0), S(u^{\gamma}_{0\varepsilon})\rangle_{\Gamma_1} + \frac{2\varepsilon^2}{\gamma}\langle S(u^{\gamma}_{1\varepsilon} +$
$\lambda w_1, 0) - S(u^{\gamma}_{1\varepsilon}, 0), S(u^{\gamma}_{1\varepsilon})\rangle_{\Gamma_1} + + \frac{2\varepsilon^2}{\gamma}\langle \frac{\partial S(u^{\gamma}_{0\varepsilon} + \lambda w_0)}{\partial \nu} - \frac{\partial S(u^{\gamma}_{0\varepsilon})}{\partial \nu}, \frac{\partial S(u^{\gamma}_{0\varepsilon})}{\partial \nu}\rangle_{\Gamma_1} +$

$$+\frac{2\varepsilon^2}{\gamma}\langle \frac{\partial S(u^{\gamma}_{1\varepsilon} + \lambda w_1)}{\partial \nu} - \frac{\partial S(u^{\gamma}_{1\varepsilon})}{\partial \nu}, \frac{\partial S(u^{\gamma}_{1\varepsilon})}{\partial \nu}\rangle_{\Gamma_1}.$$

Dividing by λ and applying the limit , we obtain:

$$\lim_{\lambda \longrightarrow 0} \frac{J^{\gamma}_{\varepsilon a}(u^{\gamma}_{0\varepsilon} + \lambda w_0, u^{\gamma}_{1\varepsilon} + \lambda w_1) - J^{\gamma}_{\varepsilon a}(u^{\gamma}_{0\varepsilon}, u^{\gamma}_{1\varepsilon})}{\lambda} = 2\langle \frac{\partial z^{\gamma}_{\varepsilon}}{\partial v_0}, z^{\gamma}_{\varepsilon} - z_d\rangle_{\Omega} + 2\langle \frac{\partial z^{\gamma}_{\varepsilon}}{\partial v_1}, z^{\gamma}_{\varepsilon} - z_d\rangle_{\Omega} +$$

$$+2\langle N_0 u^{\gamma}_{0\varepsilon} + u^{\gamma}_{0\varepsilon} - \tilde{u_0}, w_0\rangle_{\Gamma_0} + 2\langle N_1 u^{\gamma}_{1\varepsilon} + u^{\gamma}_{1\varepsilon} - \tilde{u_1}, w_1\rangle_{\Gamma_0} + \frac{2\varepsilon^2}{\gamma}\langle \frac{\partial S^{\gamma}_{\varepsilon}}{\partial v_0}, S^{\gamma}_{\varepsilon}\rangle_{\Gamma_1} + \frac{2\varepsilon^2}{\gamma}\langle \frac{\partial S^{\gamma}_{\varepsilon}}{\partial v_1}, S^{\gamma}_{\varepsilon}\rangle_{\Gamma_1} +$$

$$+\frac{2\varepsilon^2}{\gamma}\langle \frac{\partial}{\partial \nu}\left(\frac{\partial S^{\gamma}_{\varepsilon}}{\partial v_0}\right), \frac{\partial S^{\gamma}_{\varepsilon}}{\partial \nu}\rangle_{\Gamma_1} + \frac{2\varepsilon^2}{\gamma}\langle \frac{\partial}{\partial \nu}\left(\frac{\partial S^{\gamma}_{\varepsilon}}{\partial v_1}\right), \frac{\partial S^{\gamma}_{\varepsilon}}{\partial \nu}\rangle_{\Gamma_1}$$

,

because $\lim_{\lambda \longrightarrow 0} \frac{z_{\varepsilon}(u^{\gamma}_{0\varepsilon} + \lambda w_0, 0) - z_{\varepsilon}(u^{\gamma}_{0\varepsilon}, 0)}{\lambda} = \frac{\partial z^{\gamma}_{\varepsilon}}{\partial v_0}$ et $\lim_{\lambda \longrightarrow 0} \frac{S(u^{\gamma}_{0\varepsilon} + \lambda w_0) - S(u^{\gamma}_{0\varepsilon})}{\lambda} = \frac{\partial S^{\gamma}_{\varepsilon}}{\partial v_0}$.
The terms $\parallel \frac{\partial z^{\gamma}_{\varepsilon}}{\partial v_0} \parallel^2_{L^2(\Omega)}$, $\parallel \frac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial v_0} \parallel^2_{L^2(\Omega)}$ are bounded and tend to 0 when $\lambda \to 0$.
As $S(u^{\gamma}_{0\varepsilon} + \lambda w_0) - S(u^{\gamma}_{0\varepsilon}) = \zeta(u^{\gamma}_{0\varepsilon} + \lambda w_0) - \zeta(u^{\gamma}_{0\varepsilon})$, then:

$$\lim_{\lambda \longrightarrow 0} \frac{J^{\gamma}_{\varepsilon a}(u^{\gamma}_{0\varepsilon} + \lambda w_0, u^{\gamma}_{1\varepsilon} + \lambda w_1) - J^{\gamma}_{\varepsilon a}(u^{\gamma}_{0\varepsilon}, u^{\gamma}_{1\varepsilon})}{\lambda} = 2\langle \frac{\partial z^{\gamma}_{\varepsilon}}{\partial v_0}, z^{\gamma}_{\varepsilon} - z_d\rangle_{\Omega} + 2\langle \frac{\partial z^{\gamma}_{\varepsilon}}{\partial v_1}, z^{\gamma}_{\varepsilon} - z_d\rangle_{\Omega} +$$

$$+2\langle N_0 u^{\gamma}_{0\varepsilon} + u^{\gamma}_{0\varepsilon} - \tilde{u_0}, w_0\rangle_{\Gamma_0} + 2\langle N_1 u^{\gamma}_{1\varepsilon} + u^{\gamma}_{1\varepsilon} - \tilde{u_1}, w_1\rangle_{\Gamma_0} + \frac{2\varepsilon^2}{\gamma}\langle \frac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial v_0}, S^{\gamma}_{\varepsilon}\rangle_{\Gamma_1} + \frac{2\varepsilon^2}{\gamma}\langle \frac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial v_1}, S^{\gamma}_{\varepsilon}\rangle_{\Gamma_1} +$$

$$+\frac{2\varepsilon^2}{\gamma}\langle \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial v_0}\right), \frac{\partial S^{\gamma}_{\varepsilon}}{\partial \nu}\rangle_{\Gamma_1} + \frac{2\varepsilon^2}{\gamma}\langle \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial v_1}\right), \frac{\partial S^{\gamma}_{\varepsilon}}{\partial \nu}\rangle_{\Gamma_1}$$

.

The Euler-Lagrange condition becomes:

$$\langle\frac{\partial z_\varepsilon^\gamma}{\partial v_0}, z_\varepsilon^\gamma - z_d\rangle_\Omega + \langle\frac{\partial z_\varepsilon^\gamma}{\partial v_1}, z_\varepsilon^\gamma - z_d\rangle_\Omega + \langle N_0 u_{0\varepsilon}^\gamma + u_{0\varepsilon}^\gamma - \tilde{u_0}, w_0\rangle_{\Gamma_0} + \langle N_1 u_{1\varepsilon}^\gamma + u_{1\varepsilon}^\gamma - \tilde{u_1}, w_1\rangle_{\Gamma_0} +$$
$$+\frac{\varepsilon^2}{\gamma}\langle\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}, S_\varepsilon^\gamma\rangle_{\Gamma_1} + \frac{\varepsilon^2}{\gamma}\langle\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}, S_\varepsilon^\gamma\rangle_{\Gamma_1} + \frac{\varepsilon^2}{\gamma}\langle\frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_0}\right), \frac{\partial S_\varepsilon^\gamma}{\partial \nu}\rangle_{\Gamma_1} + \frac{\varepsilon^2}{\gamma}\langle\frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v_1}\right), \frac{\partial S_\varepsilon^\gamma}{\partial \nu}\rangle_{\Gamma_1} = 0.$$

Consider β_ε^γ solution of the following system:

$$\begin{cases} -\triangle^2\beta_\varepsilon^\gamma - 3\beta_\varepsilon^\gamma\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\beta_\varepsilon^\gamma = 0 & in \quad \Omega, \\ \beta_\varepsilon^\gamma - \frac{\partial\triangle\beta_\varepsilon^\gamma}{\partial \nu} = 0; \quad \frac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \triangle\beta_\varepsilon^\gamma = 0 & on \quad \Gamma_0, \\ \varepsilon\beta_\varepsilon^\gamma - \frac{\partial\triangle\beta_\varepsilon^\gamma}{\partial \nu} = -\frac{\varepsilon^2}{\gamma}(\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0)); \ \varepsilon\frac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \triangle\beta_\varepsilon^\gamma = -\frac{\varepsilon^2}{\gamma}\left(\frac{\partial(\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0))}{\partial \nu}\right) & on \quad \Gamma_1. \end{cases} \tag{30}$$

By multiplying the first equation of (30) by $\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)$, we have:

$$\langle -\triangle^2\beta_\varepsilon^\gamma - 3\beta_\varepsilon^\gamma\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = 0,$$
$$\Rightarrow \langle -\triangle^2\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega - \langle 3\beta_\varepsilon^\gamma\left(z_\varepsilon^\gamma\right)^2, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega - \langle \varepsilon\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = 0,$$

and applying Green formulation to the first member, we have:

$$\langle\triangle^2\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)(w)\rangle_\Omega = \langle\frac{\partial(\triangle\beta_\varepsilon^\gamma(\nu))}{\partial \nu}, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Gamma - \langle\triangle\beta_\varepsilon^\gamma(\nu), \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_\Gamma + \langle\frac{\partial \beta_\varepsilon^\gamma(\nu)}{\partial \nu}, \triangle\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Gamma -$$
$$-\langle\beta_\varepsilon^\gamma(\nu), \frac{\partial}{\partial \nu}\left(\triangle\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_\Gamma + \langle\beta_\varepsilon^\gamma(\nu), \triangle^2\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$
$$= \langle\frac{\partial(\triangle\beta_\varepsilon^\gamma(\nu))}{\partial \nu}, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_{\Gamma_1} - \langle\triangle\beta_\varepsilon^\gamma(\nu), \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_{\Gamma_1} - \langle\frac{\partial \beta_\varepsilon^\gamma(\nu)}{\partial \nu}, \varepsilon\frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_{\Gamma_1} -$$
$$-\langle\beta_\varepsilon^\gamma(\nu), \varepsilon\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_{\Gamma_1} + \langle\beta_\varepsilon^\gamma(\nu), \triangle^2\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$
$$= \langle\frac{\partial(\triangle\beta_\varepsilon^\gamma(\nu))}{\partial \nu} - \varepsilon\beta_\varepsilon^\gamma(\nu), \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_{\Gamma_1} - \langle\triangle\beta_\varepsilon^\gamma(\nu) + \varepsilon\frac{\partial \beta_\varepsilon^\gamma(\nu)}{\partial \nu}, \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_{\Gamma_1} +$$
$$+\langle\beta_\varepsilon^\gamma(\nu), \triangle^2\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$

$$= \frac{\varepsilon^2}{\gamma}\langle\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0), \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_{\Gamma_1} + \frac{\varepsilon^2}{\gamma}\langle\left(\frac{\partial(\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0))}{\partial \nu}\right), \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_{\Gamma_1} + \langle\beta_\varepsilon^\gamma(\nu), \triangle^2\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$

$$\langle -\triangle^2\beta_\varepsilon^\gamma - 3\beta_\varepsilon^\gamma\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = -\frac{\varepsilon^2}{\gamma}\langle\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0), \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_{\Gamma_1} - \frac{\varepsilon^2}{\gamma}\langle\frac{\partial(\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0))}{\partial \nu}, \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_{\Gamma_1} -$$
$$-\langle\beta_\varepsilon^\gamma(\nu), \triangle^2\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega - 3\langle\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\left(z_\varepsilon^\gamma\right)^2\rangle_\Omega - \langle\varepsilon\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$
$$\Rightarrow \langle -\triangle^2\beta_\varepsilon^\gamma - 3\beta_\varepsilon^\gamma\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\beta_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = -\frac{\varepsilon^2}{\gamma}\langle\left(\frac{\partial(\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0))}{\partial \nu}\right), \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_{\Gamma_1} -$$
$$-\frac{\varepsilon^2}{\gamma}\langle\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0), \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_{\Gamma_1} + \langle -\triangle^2\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0) - 3\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\left(z_\varepsilon^\gamma\right)^2 - \varepsilon\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0), \beta_\varepsilon^\gamma(\nu)\rangle_\Omega,$$

we obtain:

$$\frac{\varepsilon^2}{\gamma}\langle S_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\rangle_{\Gamma_1} + \frac{\varepsilon^2}{\gamma}\langle\frac{\partial S_\varepsilon^\gamma}{\partial \nu}, \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_0}(w,0)\right)\rangle_{\Gamma_1} + \langle\frac{\partial z_\varepsilon^\gamma}{\partial v_0}, \beta_\varepsilon^\gamma(\nu)\rangle_\Omega = 0.$$

By again multiplying the first equation of (30) by $\dfrac{\partial \zeta_\varepsilon}{\partial v_1}(w,0)$ and using a similar reasoning as before, we get:

$$\frac{\varepsilon^2}{\gamma}\langle S_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v_1}\rangle_{\Gamma_1} + \frac{\varepsilon^2}{\gamma}\langle \frac{\partial S_\varepsilon^\gamma}{\partial \nu}, \frac{\partial}{\partial \nu}\left(\frac{\partial \zeta_\varepsilon}{\partial v_1}(w,0)\right)\rangle_{\Gamma_1} + \langle \frac{\partial z_\varepsilon^\gamma}{\partial v_1}, \beta_\varepsilon^\gamma(v)\rangle_\Omega = 0.$$

The Euler-Lagrange condition becomes:

$$\langle \frac{\partial z_\varepsilon^\gamma}{\partial v_0}, z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma\rangle_\Omega + \langle \frac{\partial z_\varepsilon^\gamma}{\partial v_1}, z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma\rangle_\Omega + \langle N_0 u_{0\varepsilon}^\gamma + u_{0\varepsilon}^\gamma - \tilde{u_0}, w_0\rangle_{\Gamma_0} + \langle N_1 u_{1\varepsilon}^\gamma + u_{1\varepsilon}^\gamma - \tilde{u_1}, w_1\rangle_{\Gamma_0} = 0.$$

Let us now introduce the adjoint ϕ_ε^γ defined by:

$$\begin{cases} -\triangle^2\phi_\varepsilon^\gamma - 3\left(z_\varepsilon^\gamma\right)^2\phi_\varepsilon^\gamma - \varepsilon\phi_\varepsilon^\gamma = z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma & in \quad \Omega, \\ \phi_\varepsilon^\gamma - \dfrac{\partial \triangle \phi_\varepsilon^\gamma}{\partial \nu} = 0; \quad \dfrac{\partial \phi_\varepsilon^\gamma}{\partial \nu} + \triangle\phi_\varepsilon^\gamma = 0 & on \quad \Gamma_0, \\ \varepsilon\phi_\varepsilon^\gamma - \dfrac{\partial \triangle \phi_\varepsilon^\gamma}{\partial \nu} = 0; \quad \varepsilon\dfrac{\partial \phi_\varepsilon^\gamma}{\partial \nu} + \triangle\phi_\varepsilon^\gamma = 0 & on \quad \Gamma_1. \end{cases}$$

• By multiplying the first equation of ϕ_ε^γ by $\dfrac{\partial z_\varepsilon}{\partial v_0}(w,0)$, we obtain :

$$\langle -\triangle^2\phi_\varepsilon^\gamma - 3\left(z_\varepsilon^\gamma\right)^2\phi_\varepsilon^\gamma - \varepsilon\phi_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = \langle z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$

and by using again similar reasoning as before, we get:

$$\langle \triangle^2\phi_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = \langle \frac{\partial(\triangle\phi_\varepsilon^\gamma(v))}{\partial \nu}, \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Gamma - \langle \triangle\phi_\varepsilon^\gamma(v), \frac{\partial}{\partial \nu}\left(\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\right)\rangle_\Gamma +$$

$$+\langle \frac{\partial \phi_\varepsilon^\gamma(v)}{\partial \nu}, \triangle\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Gamma - \langle \phi_\varepsilon^\gamma(v), \frac{\partial}{\partial \nu}\left(\triangle\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\right)\rangle_\Gamma + \langle \phi_\varepsilon^\gamma(v), \triangle^2\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$

$$= \langle -\phi_\varepsilon^\gamma(v), w_0\rangle_{\Gamma_0} + \langle \phi_\varepsilon^\gamma(v), \triangle^2\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$

$$\Rightarrow \langle -\triangle^2\phi_\varepsilon^\gamma - 3\left(z_\varepsilon^\gamma\right)^2\phi_\varepsilon^\gamma - \varepsilon\phi_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = \langle \phi_\varepsilon^\gamma(v), w_0\rangle_{\Gamma_0} -$$

$$-\langle \phi_\varepsilon^\gamma(v), \triangle^2\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega - 3\langle \phi_\varepsilon^\gamma, \left(z_\varepsilon^\gamma\right)^2\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega - \langle \varepsilon\phi_\varepsilon^\gamma(v), \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$

$$\Rightarrow \langle -\triangle^2\phi_\varepsilon^\gamma - 3\left(z_\varepsilon^\gamma\right)^2\phi_\varepsilon^\gamma - \varepsilon\phi_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega = \langle \phi_\varepsilon^\gamma(v), w_0\rangle_{\Gamma_0} +$$

$$+\langle \phi_\varepsilon^\gamma(v), -\triangle^2\frac{\partial z_\varepsilon}{\partial v_0}(w,0) - 3\left(z_\varepsilon^\gamma\right)^2\frac{\partial z_\varepsilon}{\partial v_0}(w,0) - \varepsilon\frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega,$$

$$\Rightarrow \langle \phi_\varepsilon^\gamma(v), w_0\rangle_{\Gamma_0} = \langle z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v_0}(w,0)\rangle_\Omega.$$

By again multiplying the first equation of ϕ_ε^γ by $\dfrac{\partial z_\varepsilon}{\partial v_1}(w,0)$, we have :

$$\langle \phi_\varepsilon^\gamma(v), w_1\rangle_{\Gamma_0} = \langle z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v_1}(w,0)\rangle_\Omega.$$

From the Euler-Lagrange condition, we finally get:

$\langle \phi_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + u_{0\varepsilon}^\gamma - \tilde{u_0} + N_1 u_{0\varepsilon}^\gamma + u_{1\varepsilon}^\gamma - \tilde{u_1}, w\rangle_{\Gamma_0} = 0, \forall w \in (L^2(\Gamma_0))^2,$
the adapted low-regret control u_ε^γ is characterized by:

$$\phi_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + N_1 u_{1\varepsilon}^\gamma = \tilde{u_0} + \tilde{u_1} - u_{0\varepsilon}^\gamma - u_{1\varepsilon}^\gamma.$$

■

2.6 Singular Optimality System (SOS)

In this section, we give the SOS for low-regret control for the problem (1)

Lemma 2.4. *It exists constant $C > 0$ such as:*

$$\begin{cases} \| u^{\gamma}_{0\varepsilon} \|_{L^2(\Gamma_0)} \leq C, \\ \| u^{\gamma}_{1\varepsilon} \|_{L^2(\Gamma_0)} \leq C, \\ \| z^{\gamma}_{\varepsilon} \|_{L^2(\Omega)} \leq C, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \| \zeta^{\gamma}_{\varepsilon} \|_{L^2(\Gamma_1)} \leq C, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \| \dfrac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial \nu} \|_{L^2(\Gamma_1)} \leq C, \end{cases} \tag{31}$$

Proof :

u^{γ}_{ε} is a solution of (25), therefore:

$$J^{\gamma}_{a\varepsilon}(u^{\gamma}_{\varepsilon}) \leq J^{\gamma}_{a\varepsilon}(v), \forall \quad v \in (L^2(\Gamma_0))^2.$$

In the particular case where $v = 0$, we obtain:

$$\| z^{\gamma}_{\varepsilon} - z_d \|^2_{L^2(\Omega)} + N_0 \|u^{\gamma}_{0\varepsilon}\|^2_{L^2(\Gamma_0)} + N_1 \|u^{\gamma}_{1\varepsilon}\|^2_{L^2(\Gamma_0)} + \| u^{\gamma}_{0\varepsilon} - \tilde{u} \|^2_{L^2(\Gamma_1)} +$$

$$+ \frac{\varepsilon^2}{\gamma} \| \zeta^{\gamma}_{\varepsilon} \|^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma} \| \frac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial \nu} \|^2_{L^2(\Gamma_1)} \leq \| z_d \|^2_{L^2(\Gamma_1)} = c;$$

hence the result.

■

Theorem 2.1. *The low-regret control u^{γ} for problem (1) is characterized by $\{\zeta^{\gamma}, z^{\gamma}, \beta^{\gamma}, \phi^{\gamma}\}$:*

$$\begin{cases} -\triangle \zeta^{\gamma} - 3\zeta^{\gamma} (z^{\gamma})^2 = -(z^{\gamma} - z_d) \quad in \quad \Omega, \zeta^{\gamma} = 0; \quad \dfrac{\partial \zeta^{\gamma}}{\partial \nu} = 0 \quad on \quad \Gamma_0, \end{cases}$$

$$\begin{cases} -\triangle z^{\gamma} - (z^{\gamma})^3 = 0 & in \quad \Omega, \\ z^{\gamma} = u^{\gamma}_0; \quad \dfrac{\partial z^{\gamma}}{\partial \nu} = u^{\gamma}_1 & on \quad \Gamma_0, \end{cases}$$

$$\begin{cases} -\triangle \beta^{\gamma} - 3\beta^{\gamma} (z^{\gamma})^2 = 0 & in \quad \Omega, \\ \beta^{\gamma} = 0; \quad \dfrac{\partial \beta^{\gamma}}{\partial \nu} = 0 & on \quad \Gamma_0, \end{cases}$$

$$\begin{cases} -\triangle \phi^{\gamma} - 3\phi^{\gamma} (z^{\gamma})^2 = z^{\gamma} - z_d - \beta^{\gamma} & in \quad \Omega, \\ \phi^{\gamma} = 0; \quad \dfrac{\partial \phi^{\gamma}}{\partial \nu} = 0 & on \quad \Gamma_0, \end{cases}$$

$$\phi^{\gamma} + N_0 u^{\gamma}_0 + N_1 u^{\gamma}_1 = \tilde{u}_0 - u^{\gamma}_0 + \tilde{u}_1 - u^{\gamma}_1 \quad in \quad L^2(\Gamma_0). \tag{32}$$

Proof :

From the Proposition 2.6, we deduce that z^{γ}_{ε} is solution of the system:

$$\begin{cases} -\triangle^2 z^{\gamma}_{\varepsilon} - \left(z^{\gamma}_{\varepsilon}\right)^3 - \varepsilon z^{\gamma}_{\varepsilon} = 0 & in \quad \Omega, \\ z^{\gamma}_{\varepsilon} - \dfrac{\partial \triangle z^{\gamma}_{\varepsilon}}{\partial \nu} = u^{\gamma}_{0\varepsilon}; \quad \dfrac{\partial z^{\gamma}_{\varepsilon}}{\partial \nu} + \triangle z^{\gamma}_{\varepsilon} = u^{\gamma}_{1\varepsilon} & on \quad \Gamma_0, \\ \varepsilon z^{\gamma}_{\varepsilon} - \dfrac{\partial \triangle z^{\gamma}_{\varepsilon}}{\partial \nu} = 0; \quad \varepsilon \dfrac{\partial z_{\varepsilon}}{\partial \nu} + \triangle z^{\gamma}_{\varepsilon} = 0 & on \quad \Gamma_1. \end{cases} \tag{33}$$

By changing variable, $\eta^{\gamma}_{\varepsilon} = \triangle z^{\gamma}_{\varepsilon}$, thus the system (33) becomes:

$$\begin{cases} -\triangle \eta^{\gamma}_{\varepsilon} - \left(z^{\gamma}_{\varepsilon}\right)^3 - \varepsilon z^{\gamma}_{\varepsilon} = 0 & in \quad \Omega, \\ z^{\gamma}_{\varepsilon} - \dfrac{\partial \eta^{\gamma}_{\varepsilon}}{\partial \nu} = u^{\gamma}_{0\varepsilon}; \quad \dfrac{\partial z^{\gamma}_{\varepsilon}}{\partial \nu} + \eta^{\gamma}_{\varepsilon} = u^{\gamma}_{1\varepsilon} & on \quad \Gamma_0, \\ \varepsilon z^{\gamma}_{\varepsilon} - \dfrac{\partial \eta^{\gamma}_{\varepsilon}}{\partial \nu} = 0; \quad \varepsilon \dfrac{\partial z_{\varepsilon}}{\partial \nu} + \eta^{\gamma}_{\varepsilon} = 0 & on \quad \Gamma_1. \end{cases} \tag{34}$$

From the Lemma 2.3, $\| z_\varepsilon^\gamma \|^2_{L^2(\Omega)} \leq C$, therefore z_ε^γ converges weakly in $L^2(\Omega)$ and tends to z^γ when $\varepsilon \longrightarrow 0$.

We deduce from the first equation in (34) that: $\| \triangle\eta_\varepsilon^\gamma + \left(z_\varepsilon^\gamma\right)^3 \|_{L^2(\Omega)} = \| \varepsilon z_\varepsilon^\gamma \|_{L^2(\Omega)} \leq \varepsilon C \longrightarrow 0$.
Therefore,

$$\begin{cases} \triangle\eta_\varepsilon^\gamma \longrightarrow -\left(z_\varepsilon^\gamma\right)^3 & in \quad L^2(\Omega), \\ \dfrac{\partial\eta_\varepsilon^\gamma}{\partial\nu} \longrightarrow 0 & on \quad L^2(\Gamma_1), \\ \eta_\varepsilon^\gamma \longrightarrow 0 & on \quad L^2(\Gamma_1). \end{cases} \tag{35}$$

When $\varepsilon \longrightarrow 0$. We recap that:

$$\begin{cases} -\triangle\eta^\gamma - (z^\gamma)^3 = 0 & in \quad \Omega, \\ \dfrac{\partial\eta^\gamma}{\partial\nu} = 0 & on \quad \Gamma_1, \\ \eta^\gamma = 0 & on \quad \Gamma_1. \end{cases} \tag{36}$$

By using the unique continuation theorem of Mizohata (S. Mizohata, 1958), we deduce from (36) that we also have:

$$\frac{\partial\eta^\gamma}{\partial\nu} = \eta^\gamma = 0 \qquad on \quad \Gamma_0. \tag{37}$$

On the other hand, Lemma 2.3 also gives: $\| u_{0\varepsilon}^\gamma \|_{L^2(\Gamma_0)} \leq C$ and $\| u_{1\varepsilon}^\gamma \|_{L^2(\Gamma_0)} \leq C$, therefore,

$$(u_{0\varepsilon}^\gamma, u_{1\varepsilon}^\gamma) \rightharpoonup (u_0^\gamma, u_1^\gamma) \qquad in \qquad L^2(\Gamma_0) \times L^2(\Gamma_0). \tag{38}$$

From (34)-(35)-(38) we obtain:

$$\begin{cases} -\triangle z^\gamma - (z^\gamma)^3 = 0 & in \quad \Omega, \\ z^\gamma = u_0^\gamma; \quad \dfrac{\partial z^\gamma}{\partial\nu} = u_1^\gamma & on \quad \Gamma_0. \end{cases} \tag{39}$$

Again, we use the estimate of Lemma 2.3, we deduce the following limits:
$\dfrac{\varepsilon}{\sqrt{\gamma}}\zeta_\varepsilon^\gamma \rightharpoonup \lambda_0^\gamma$ weakly in $L^2(\Gamma_1)$ and $\dfrac{\varepsilon}{\sqrt{\gamma}}\dfrac{\partial\zeta_\varepsilon^\gamma}{\partial\nu} \rightharpoonup \lambda_1^\gamma$ weakly in $L^2(\Gamma_1)$.
Thus $\dfrac{\varepsilon^2}{\gamma}\zeta_\varepsilon^\gamma \rightharpoonup 0$ and $\dfrac{\varepsilon^2}{\gamma}\dfrac{\partial\zeta_\varepsilon^\gamma}{\partial\nu} \rightharpoonup 0$ when $\varepsilon \longrightarrow 0$.
We obtain:

$$\Big\{ -\triangle\zeta^\gamma - 3\zeta^\gamma (z^\gamma)^2 = -(z^\gamma - z_d) \quad in \quad \Omega, \zeta^\gamma = 0; \quad \frac{\partial\zeta^\gamma}{\partial\nu} = 0 \qquad on \quad \Gamma_0, \tag{40}$$

and $\phi_\varepsilon^\gamma = -N_0 u_{0\varepsilon}^\gamma - N_1 u_{1\varepsilon}^\gamma + \tilde{u}_0 - u_{0\varepsilon}^\gamma + \tilde{u}_1 - u_{1\varepsilon}^\gamma \qquad in \qquad L^2(\Gamma_0)$.
Finally, from (23) and (38), we have:

$$\phi_\varepsilon^\gamma \rightharpoonup \phi^\gamma = -N_0 u_0^\gamma - N_1 u_1^\gamma + \tilde{u}_0 - u_0^\gamma + \tilde{u}_1 - u_1^\gamma \qquad in \qquad L^2(\Gamma_0). \tag{41}$$

■

2.7 Characterization of the No-Regret Control

Now, we give optimality System for no-regret control.

Theorem 2.2. *The no-regret control $\tilde{u} = (\tilde{u}_0, \tilde{u}_1)$ for problem (1) is characterized by the unique $\{\zeta, z, \beta, \phi\}$, solution to:*

$$\begin{cases} -\triangle\zeta - 3z^2\zeta = -(z - z_d) & in \quad \Omega, \\ \zeta = 0; \quad \dfrac{\partial\zeta}{\partial\nu} = 0 & on \quad \Gamma_0, \end{cases}$$

$$\begin{cases} -\triangle z - z^3 = 0 & in \quad \Omega, \\ z = \tilde{u}_0; \quad \dfrac{\partial z}{\partial\nu} = \tilde{u}_1 & on \quad \Gamma_0, \end{cases}$$

$$\begin{cases} -\triangle\beta - 3z^2\beta = 0 & in \quad \Omega, \\ \beta = 0; \quad \dfrac{\partial\beta}{\partial\nu} = 0 & on \quad \Gamma_0, \end{cases}$$

$$\begin{cases} -\triangle\phi - 3z^2\phi = z - z_d - \beta & in \quad \Omega, \\ \phi = 0; \quad \dfrac{\partial\phi}{\partial\nu} = 0 & on \quad \Gamma_0, \end{cases}$$

$$\phi + N_0\tilde{u}_0 + N_1\tilde{u}_1 = 0 \qquad in \qquad L^2(\Gamma_0).$$

Proof :
From Theorem 2.1 and we go to the limit in $\gamma \longrightarrow 0$, we obtain:

$$\begin{cases} \zeta^\gamma \rightharpoonup \zeta = 0, \\ \beta^\gamma \rightharpoonup \beta = 0, \\ \phi^\gamma \rightharpoonup \phi = 0 \end{cases} \tag{42}$$

$$on \quad \Gamma_0,$$

well; From (32), we have:

$$(u_0^\gamma, u_1^\gamma) \rightharpoonup (\tilde{u}_0, \tilde{u}_1) \qquad in \qquad L^2(\Gamma_0) \times L^2(\Gamma_0), \tag{43}$$

Therefore:

$$\begin{cases} z^\gamma \rightharpoonup z = \tilde{u}_0, \\ \dfrac{\partial z^\gamma}{\partial \nu} \rightharpoonup \dfrac{\partial z}{\partial \nu} = \tilde{u}_1 \end{cases} \tag{44}$$

$$on \quad \Gamma_0,$$

From what precedes, it exists a unique $\tilde{u}$ characterized by $\{\zeta, z, \beta, \phi\}$ solution of the system (1).

■

3. Conclusion

In this work, we have examined a ill-posed problem with incomplete data using the regularization method and the adapted low-regret control method. This method allowed us to generate the missing information on Γ_1 without which the control of the system was delicate and the adapted low-regret control method allowed us to obtain the characterization of the control of problem (1). By using the low-regret method and by moving to the limit on $\gamma \longrightarrow 0$, we obtain the no-regret control.

References

Barry, M., Nakoulima, O., & Ndiaye, G. B. (2013). Cauchy system for parabolic operator. *International Journal of Evolution Equations, 8*(4), 277–290.

Barry, M., & Ndiaye, G. B. (2014). Cauchy system for an hyperbolic operator. *Journal of Nonlinear Evolution Equations and Applications, 4*, 37–52.

Berhail, A., & Omrane, A. (2015). Optimal control of the ill-posed Cauchy elliptic problem. *International Journal of Differential Equations, Article ID 468918, 9 pages*.

Kenne, C., Eugering, G. L., & Mophou, G. (2020). Optimal control of a population dynamics model with missing birth rate. *SIAM Journal on Control and Optimization, 58*(3), 1289–1313.

Kernevez, J. P. (1980). *Enzyme mathematics, North-Holland Publishing Company.*

Lions, J. L. (1983). *Contrôle des systmes distribus singuliers, Gauthiers-Villard, BORDAS, Paris.*

Lions, J. L. (1968). *Contrôle optimal de systmes gouverns par des quations aux drives partielles. DUNOD, Gauthiers-Villard, Paris.*

Lions, J. L. (1986). Contrôle de Pareto de systmes distribus. Le cas stationnaire. Comptes Rendus de l'Acadmie des Sciences - Series I. *Mathematics, 302*(6), 223-227.

Lions, J. L. (1986). Contrôle de Pareto de systmes distribus, le cas d'volution. *C. R. Acad. Sci.Paris Ser. I., 302*, 413-417.

Lions, J. L. (1992). Contrôle moindres regrets des systmes distribus. Comptes Rendus de l'Acadmie des Sciences - Series I. *Mathematics, 315*, 1253C1257.

LOUISON, L. (2015). *Analyse mathmatique et contr?le optimal pour les quations d'advection-diffusion: Application au problme de transfert de nutriments pour les plantes en agrocologie, 2 octobre.*

Mizohata, S. (1958). Unicit du prolongement des solutions des quations elliptiques du quatrime ordre. *Proceedings of the Japan Academy, 34*, 687–692.

Mophou, G., Foko Tiomela, R. G., & Seibou, A. (2020). *Optimal control of averaged state of a parabolic equation with missing boundary condition, International Journal of Control, 93*10, 2358–2369.

Nakoulima, O. (1994). Contrôle de systmes mal poss de type elliptique. *Journal de Mathmatiques Pures et Appliques, 73*(9), 441–453.

Nakoulima, O., Omrane, A., & Velin, J. (2000). Perturbations moindres regrets dans les systmes distribus donnes manquantes. *Compte Rendus de l'Acadmie des Sciences: Series I: Mathematics, 330*(9), 801–806. https://dx.doi.org/10.1016/s0764-4442(00)00263-9

Nakoulima, O., Omrane, A., & Velin, J. (2002). No-regret control for nonlinear distributed systems with incomplete data. *J. Maths. Pures Appl, 81*. https://dx.doi.org/10.1016/S0021-7824(02)01268-0

Nakoulima, O., Omrane, A., & Velin, J. (2003). On the Pareto control and no-regret control for distributed systems with incomplete data. *SIAM Journal on Control and Optimization, 42*(4), 1167-1184. https://dx.doi.org/10.1137/S0363012900380188

Nakoulima, O., Omrane, A., & Velin, J. (2003). Low-regret perturbations in distributed systems with incomplete data. *SIAM Journal on Control and Optimization, 42*(4), 1167-1184.

Nakoulima, O., Omrane, A., & Dorville, R. (2004). Low-regret control of singular distributed systems: The ill-posed backwards heat problem. *Applied Mathematics Letters, 17*, 549-552.

Nakoulima, O., & Mophou, G. M. (2009). Control of Cauchy system for an elliptic operator. *Acta Mathematica Sinica, 25*(11), 1819–1834.

Smoler, J. (1983). *Shock waves, and reaction-diffusion equation, Springer Verlag, Berlin-Heidelberg, New York.*

Sougalo, S., & Nakoulima, O. (1998). *Contrôle optimal pour le problme de Cauchy pour un oprateur Elliptique, Prpublication du Dpartement de Mathmatiques et Informatique DMI, Universit des Antilles et de la Guyane, Guadeloupe, France.*

Tindano, T., Tao, S., & Sawadogo, S. (2022). Optimal control for evolution problem of incomplete data. *Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680, 1*, 1-19.

Velin, J. (2004). No-regret distributed control of system governed by quasilinear elliptic equations with incomplete data: the degenerate case. *Journal de Mathmatiques Pures et Appliques, 83*(4), 503-539.