Journal of Mathematics Research; Vol. 14, No. 6; December 2022
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

On the Collatz Conjecture

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Received: October 20, 2022 Accepted: November 23, 2022 Online Published: November 28, 2022

doi:10.5539/jmr.v14n6p28 URL: https://doi.org/10.5539/jmr.v14n6p28

Abstract

The Collatz conjecture (or Syracuse conjecture) states: all Syracuse sequences converge to 1. We present a Syracuse sequence, and we prove that the conjecture is true, first by using the fact that all convergent integer sequences are eventually constant. We then prove wrong 2 hypotheses: the case where the sequence tends to infinity, and the case where the sequence has no limit and is eventually periodic. We conclude by elimination, afterward.

Keywords: Onion-factorization, Collatz conjecture, Syracuse sequence, onion factors

1. Introduction

Let's consider natural numbers, which set is $\mathbb{N}$. We define a sequence of numbers built this way: given any non-zero natural number c, the next term is $c/2$ if c is even, $(3c+1)/2$ if c is odd; repeatedly, we can build as many terms as we want. As a result, we have a sequence of odd and even numbers, with c as the initial term. We call it a Syracuse sequence.

Equivalently, one can introduce another sequence of odd numbers only, with c as the initial term if c is odd. If c is even, the initial term is obtained after dividing c by 2 enough times to obtain an odd number, say r times and the initial term is $c' = \frac{c}{2^r}$, r non-zero natural number. If we consider a Syracuse sequence with c' as the initial term, the next term is $c'' = \frac{3c'+1}{2^s}$; after c'' we have $c''' = \frac{3c''+1}{2^t}$, . . . , the **exponents** s and t being the necessary times we divide by 2 to obtain odd numbers c'' and c''', respectively. Generally, any Syracuse sequence $(U_n)_{n\in\mathbb{N}}$ has its 2 consecutive odd terms linked by the following expression:

$$U_{n+1} = \frac{3U_n+1}{2^{m_n}}\ ;\ \ m_n \text{ non} - \text{zero natural number.} \tag{1}$$

It's tempting to seek a general term of the sequence using (1). But we quickly notice in (1) that the exponent m_n is not predetermined, and its value depends on $3U_n+1$, more specifically U_n.
We have:

$$U_1 = \frac{3U_0+1}{2^{m_0}}, U_2 = \frac{3U_1+1}{2^{m_1}}, \dots, U_{n+1} = \frac{3U_n+1}{2^{m_n}}; m_i \in \mathbb{N}, i = 1,2,\dots,n. Also: U_2 = \frac{3^2U_0+3^1+2^{m_0}}{2^{m_0}2^{m_1}} \tag{2}$$

By using mathematical induction, one can prove:

$$U_n = \frac{3^nU_0+3^{n-1}+\sum_{k=2}^{n}\prod_{i=0}^{k-2}3^{n-k}\,2^{m_i}}{\prod_{i=0}^{n-1}2^{m_i}}, \forall n \geq 2 \tag{3}$$

The relation (3) gives the general term of the Syracuse sequence, given U_0 the initial term. We notice in fact that the values of $m_i, i = 0,1,\dots,n-1$ are unknown. The only way to know them is to calculate one by one all the terms, from the second to the nth term, using the relation (1). This means, at infinity, the progression of the sequence is still a mystery.

Since Lothar Collatz, the originator of the Collatz conjecture aroused in mathematicians interest in this problem, numerous researchers have been publishing papers on it. Some of them are gathered in a paper discussing the conjecture and related problems (Jeffey C. Lagarias, 2010). This is due to the fact that, many connexions to this topic were found in other branches of mathematics, like for instance ergodic theory and Markov chains, in a survey paper by K. R. Matthews (Jeffey C. Lagarias, 2010, p. 79).

Despite such abundance of publication, the conjecture has not been resolved yet. Recently, a significant progress was made using probabilistic arguments (Terrence Tao, 2019).

The approach developped in this paper is new, and a deep study of syracuse sequences will be necessary so that, the tools proposed match the needs for the resolution of the conjecture.

Since each m_i can be known only between 2 consecutive odd terms; and based upon the particular property of a Syracuse sequence that, any odd term multiple of 3 cannot have a previous term, first we will establish all possible expressions of any previous term and its next one. Then, we will use the concept of onion-factorization, to test the consistency of the expression of the initial term, when we extend the sequence infinitely.

TERMINOLOGY

- In the following, we deal with only the odd terms of the Syracuse sequence. And we choose to refer to the odd terms by “terms” rather than “odd terms”;

- We mean by previous terms in a Syracuse sequence, all the terms with the same rank n behind the term with rank n+1; by next term, we mean the term with rank n+1 after the term with rank n. n is an arbitrary natural number;

- We mean by same form-previous terms, all previous terms with the same algebraic form 6m+1, 6m+3 or 6m+5, m natural number;

- ”Initial term” refers to the first term of a Syracuse sequence, while “initial previous term” refers to the first term of a sequence of previous terms.

2. Syracuse Sequences

2.1 Previous Terms

2.1.1 The General Expression of Previous Terms

$(U_n)_{n\in\mathbb{N}}$ is a Syracuse sequence. By using the relation (1), we can write

$$U_n = \frac{2^{m_n}U_{n+1} - 1}{3} \tag{4}$$

By choosing any term U_{n+1}, expression (4) helps find its previous one in the Syracuse sequence. Since U_n is an integer, 3 has to divide the numerator $2^{m_n}U_{n+1} - 1$. But, if any term has a next one in the Syracuse sequence, do all terms necessarily have previous ones? We quickly notice that if $2^{m_n}U_{n+1}$ is a multiple of 3, that is, U_{n+1} multiple of 3, then 3 cannot divide $2^{m_n}U_{n+1} - 1$, since it is a subtraction between a multiple of 3 and a non-multiple of 3. This leads us to a property verified by all Syracuse sequences:

Property. *Given any random term $U_n, n \geq 1$, in a Syracuse sequence, it doesn't exist a previous term U_{n-1} if and only if U_n is a multiple of 3.* (5)

Now, let's select some examples to see how the relation (4) works. Let 7; 7 is not a multiple of 3, and by trying which m_n suits, we find

$$\frac{2^2 \times \mathbf{7} - 1}{3} = 9$$

But, since we have tried at random, is there any other m_n that can work too? How many?

We find for $m_n = 4$:

$$\frac{2^4 \times \mathbf{7} - 1}{3} = 37$$

If we continue, we figure out different values of U_{n-1} for $m_n = 6,8,10,\ldots$, all being previous terms of $U_n = 7$ in the Syracuse sequence.

In comparison, to determine the next term to 7, we have

$$\frac{3 \times 7 + 1}{2^2} = 11,$$

and $m_n = 2$ is the only choice here; 11 is then the unique term that follows 7 in the Syracuse sequence.

Let any U_{i+1} in the Syracuse sequence with rank i+1. We note $(U_{i,p})_{p\in\mathbb{N}}$ the monotonically increasing sequence of its previous terms: i is there to signify that behind U_{i+1} in the Syracuse sequence, all the previous terms have the same rank i, and p is to put order among them. Let $U_{i,0}$ the initial previous term, and $U_{i,p}$ the (p+1)th; we have the following expressions:

$$U_{i,p} = \frac{2^{b_p}U_{i+1} - 1}{3}, \qquad U_{i,0} = \frac{2^{b}U_{i+1} - 1}{3} \quad , \tag{6}$$

b fixed; and for $p \geq 1,\ b_p > b > 0$.

Since U_{i+1} has previous terms, it can't be a multiple of 3.

We have the difference:

$$\mathbb{N} \ni (U_{i,p} - U_{i,0}) = \frac{2^{b_p}U_{i+1} - 1}{3} - \frac{2^{b}U_{i+1} - 1}{3} = \frac{U_{i+1}(2^{b_p} - 2^{b})}{3}$$

Then, 3 necessarily divides $2^{b_p} - 2^b = 2^b\left(2^{(b_p-b)} - 1\right)$, hence it should exist a natural number d, non-zero, such that $2^{(b_p-b)} - 1 = 4^d - 1$.

We then have $b_p - b = 2d, b_p = b + 2d$, and the (p+1)th previous term of U_{i+1} becomes:

$$U_{i,p} = \frac{2^b 4^d U_{i+1} - 1}{3},$$

with d an arbitrary non-zero positive integer.

Also, we can write $U_{i,0} = \frac{2^b 4^0 U_{i+1} - 1}{3}$ according to (6), and d can take any non-zero number in the set $\mathbb{N}$. Then the general term of $(U_{i,p})_{p\in\mathbb{N}}$, sequence of previous terms of U_{i+1} is:

$$U_{i,p} = \frac{2^b 4^p U_{i+1} - 1}{3}, \forall p \in \mathbb{N}, \text{ b fixed.} \tag{7}$$

Every U_{i+1} is either 3m, 3m+1, or 3m+2, m natural number. Since we consider the terms of a Syracuse sequence to be odds, we have 3(2a+1) for the multiple of 3, 3(2a)+1, and 3(2a+1)+2, a natural number. U_{i+1} is not a multiple of 3, then it could only be either 6a+1 or 6a+5.

<u>VALUE OF b</u>

b is the exponent to determine the initial previous term of a monotonically increasing sequence. It is then the minimum positive integer needed to obtain $U_{i,0}$. Then we can try in the expression of $U_{i,0}$ the lower possible value of b: 1, then add 1 if the test fails ($U_{i,0}$ is not a natural number), and try the new value of b, till the test is valid ($U_{i,0}$ is a natural number).

- Case 1: $U_{i+1} = 6a + 1$

We use

$$U_{i,0} = \frac{2^b 4^0(6a + 1) - 1}{3}$$

b=1, $\frac{2(6a+1)-1}{3} = \frac{12a+1}{3} = 4a + \frac{1}{3} \notin \mathbb{N}$; b=2, $\frac{2^2(6a+1)-1}{3} = \frac{24a+3}{3} = 8a + 1 \in \mathbb{N}$. The expression (7) becomes:

$$U_{i,p} = \frac{2^2 4^p(6a + 1) - 1}{3} = \frac{4^{p+1}(6a + 1) - 1}{3}, \forall p \in \mathbb{N}. \tag{8}$$

- Case 2: $U_{i+1} = 6a + 5$

The initial previous term here is

$$U_{i,0} = \frac{2^b 4^0(6a + 5) - 1}{3}$$

b=1, $\frac{2(6a+5)-1}{3} = \frac{12a+9}{3} = 4a + 3 \in \mathbb{N}$. Hence

$$U_{i,p} = \frac{2 \times 4^p(6a+5) - 1}{3}, \forall p \in \mathbb{N}\,. \tag{9}$$

2.1.2 Connexions Among Previous Terms

Now, for a given term 6a+1 or 6a+5, we can thanks to (8) and (9), easily list without omission all the p+1 first previous terms. Furthermore, we know that $U_{i,p}$ can be expressed as either 6m+1, 6m+3 or 6m+5, m natural number.

Let's take two previous terms, both in the same algebraic form: $6m_1 + 1$ and $6m_2 + 1$, both previous to 6a+5. Assume $6m_2 + 1 > 6m_1 + 1$. We then have:

$$\frac{2 \times 4^{p_1}(6a+5) - 1}{3} = 6m_1 + 1;\ \frac{2 \times 4^{p_2}(6a+5) - 1}{3} = 6m_2 + 1,$$

and the difference:

$$6m_2 + 1 - (6m_1 + 1) = 6(m_2 - m_1) = \frac{2(4^{p_2} - 4^{p_1})(6a+5)}{3} \tag{10}$$

We have $4^{p_2} - 4^{p_1} = 4^{p_1}(4^s - 1)$ with $s + p_1 = p_2$, s non-zero natural number. The product of the means equals the product of the extremes, and (10) becomes:

$$9(m_2 - m_1) = 4^{p_1}(4^s - 1)(6a+5), \tag{11}$$

and $4^s - 1 = 3\sum_{k=0}^{s-1} 4^k$.

In order to make the second member $4^{p_1}(4^s - 1)(6a+5)$ a multiple of 9, so to match the first member $9(m_2 - m_1)$, s must take the value that makes $\sum_{k=0}^{s-1} 4^k$ a multiple of 3. For instance, s = 3.

We notice that such a condition on s is the same if we replace 6a+5 by 6a+1, even if in addition we choose the algebraic form 6m+5 or 6m+3 for the two previous terms. In fact, (11) in this case becomes

$$9(m_2 - m_1) = 2 \times 4^{p_1}(4^s - 1)(6a+1),$$

and we still have the condition mentioned above on s. Also, nothing changes about s no matter the value of p_1. Finally, whatever the two previous terms are, it only requires them to have the same algebraic form, for the condition on s to be applied.

CONDITION ON s

The number s is such that $\sum_{k=0}^{s-1} 4^k$ is a multiple of 3.

- For $s = 3n$, n non-zero natural number, let's prove that $S_n = \sum_{k=0}^{3n-1} 4^k$ is always a multiple of 3.

n=1, $S_1 = 21$ is a multiple of 3. Let's suppose S_n is a multiple of 3, and let's prove it at order n+1.

We have

$$S_{n+1} = \sum_{k=0}^{3(n+1)-1} 4^k = \sum_{k=0}^{3n+2} 4^k = S_n + \sum_{k=3n}^{3n+2} 4^k = S_n + 4^{3n}(1 + 4 + 4^2) = S_n + 21 \times 4^{3n}$$

which is a multiple of 3. In conclusion, S_n is a multiple of 3.

- For cases s=3n+1 and s=3n+2, we respectively have

$$\sum_{k=0}^{3n} 4^k = \sum_{k=0}^{3n-1} 4^k + 4^{3n} = S_n + 4^{3n},$$

and

$$\sum_{k=0}^{3n+1} 4^k = \sum_{k=0}^{3n-1} 4^k + 4^{3n} + 4^{3n+1} = S_n + 4^{3n} + 4^{3n+1}\,.$$

.

We know that S_n is a multiple of 3, while $4^{3n} + 4^{3n+1} = 4^{3n} \times 5$ and 4^{3n} aren't. Then, $\sum_{k=0}^{3n} 4^k$ and $\sum_{k=0}^{3n+1} 4^k$ are both not multiples of 3.

Finally, s=3n, n non-zero natural number, is the only possibility that validates the condition. This means, between two consecutive same form-previous terms, the gap is s = 3.

2.1.3 Previous Terms of $U_{i+1} = 6a + 1$

The general term is

$$U_{i,p} = \frac{4^{p+1}(6a+1) - 1}{3}, \forall p \in \mathbb{N}$$

Case 1: a=3k

The initial previous term is

$$\frac{4^{0+1}(6(3k)+1) - 1}{3} = 6(4k) + 1$$

the second previous term is

$$\frac{4^{1+1}(6(3k)+1) - 1}{3} = 6(16k) + 5$$

and the third

$$\frac{4^{2+1}(6(3k)+1) - 1}{3} = 6(64k+3) + 3.$$

Since s=3 between two consecutive same form-previous terms, we have in general:

$$\begin{aligned} V_n &= 6j + 1 = \frac{4^{3n+1}(6(3k)+1) - 1}{3} \\ W_n &= 6j + 5 = \frac{4^{3n+2}(6(3k)+1) - 1}{3} \quad \forall n \in \mathbb{N}\,. \\ X_n &= 6j + 3 = \frac{4^{3n+3}(6(3k)+1) - 1}{3} \end{aligned}$$

V_n, W_n, X_n are general terms of monotonically increasing sequences $(V_n)_{n\in\mathbb{N}}, (W_n)_{n\in\mathbb{N}}, (X_n)_{n\in\mathbb{N}}$, respectively. Notice that the three sequences are sub-sequences of $\left(U_{i,p}\right)_{p\in\mathbb{N}}$.

$$V_n = 6k \times 4^{3n+1} + \frac{4^{3n+1} - 1}{3} = 6k \times 4^{3n+1} + \sum_{p=0}^{3n} 4^p = 6k \times 4^{3n+1} + \sum_{p=1}^{3n} 4^p + 1;$$

$$\sum_{p=1}^{3n} 4^p = 4 + 4^2 + \ldots + 4^{3n}\,;\ n \neq 0$$

and

$$\begin{gathered} 4 + 4^2 + 4^3 = 21 \times 4 \\ 4^4 + 4^5 + 4^6 = 21 \times 4^4 \\ \vdots \\ 4^{3n-2} + 4^{3n-1} + 4^{3n} = 21 \times 4^{3n-2} \end{gathered}$$

Then by adding members of the same side, we get:

$$\sum_{p=1}^{3n} 4^p = 21 \times 4 + 21 \times 4^4 + \cdots + 21 \times 4^{3n-2} = 21\left(4 + \cdots + 4^{3n-2}\right) = 21 \times 4\left(1 + 4^3 + \cdots 4^{3n-3}\right)$$

$$= 6 \times 14 \sum_{p=0}^{n-1} 4^{3p} \ ;$$

Finally, we have:

$$\boldsymbol{V_n = 6\left[4^{3n+1}k + 14\sum_{p=0}^{n-1} 4^{3p}\right] + 1 \ \forall\, n \geq 1;\ V_0 = 6(4k) + 1}\ ;$$

$$W_n = 6k \times 4^{3n+2} + \frac{4^{3n+2} - 1}{3} = 6k \times 4^{3n+2} + 1 + 4 + 4^2 + \ldots + 4^{3n+1} = 6k \times 4^{3n+2} + 5 + 4^2 + \ldots + 4^{3n+1}$$

We have:

$$\begin{gathered} 4^2 + 4^3 + 4^4 = 4^2 \times 21 \\ 4^5 + 4^6 + 4^7 = 4^5 \times 21 \\ \vdots \\ 4^{3n-1} + 4^{3n} + 4^{3n+1} = 4^{3n-1} \times 21 \end{gathered}$$

Adding same side members and reporting it into the general term, it follows:

$$W_n = 6k \times 4^{3n+2} + 5 + 4^2 \times 21 + \ldots + 4^{3n-1} \times 21 = 6k \times 4^{3n+2} + 5 + 21 \times 4 \times 4(1 + 4^3 + \ldots + 4^{3n-3}),$$

then

$$\boldsymbol{W_n = 6\left[4^{3n+2}k + 14\sum_{p=0}^{n-1} 4^{3p+1}\right] + 5 \ \forall\, n \geq 1;\ W_0 = 6(16k) + 5}\ ;$$

$$X_n = 6k \times 4^{3n+3} + \frac{4^{3n+3} - 1}{3} = 6k \times 4^{3n+3} + \sum_{p=0}^{3n+2} 4^p \ ; \ \sum_{p=0}^{3n+2} 4^p = 1 + 4 + \ldots + 4^{3n+2}\ ;$$

$$\begin{gathered} 1 + 4 + 4^2 = 21 \\ 4^3 + 4^4 + 4^5 = 4^3 \times 21 \\ \vdots \\ 4^{3n} + 4^{3n+1} + 4^{3n+2} = 4^{3n} \times 21\ ; \end{gathered}$$

This implies:

$$X_n = 6k \times 4^{3n+3} + 21\left(1 + 4^3 + \ldots + 4^{3n}\right) = 6k \times 4^{3n+3} + 21\sum_{p=0}^{n} 4^{3p}$$

$$21\sum_{p=0}^{n} 4^{3p} = 6 \times 3\sum_{p=0}^{n} 4^{3p} + 3\sum_{p=1}^{n} 4^{3p} + 3 = 6 \times 3\sum_{p=0}^{n} 4^{3p} + 3 \times 4\sum_{p=1}^{n} 4^{3p-1} + 3$$

Then,

$$\boldsymbol{X_n = 6\left[4^{3n+3}k + 3\sum_{p=0}^{n} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1}\right] + 3 \ \forall\, n \geq 1;\ X_0 = 6(64k + 3) + 3}\ ;$$

Case 2: a=3k+1

The initial previous term is

$$\frac{4^{0+1}(6(3k+1)+1) - 1}{3} = 6(4k+1) + 3$$

then

$$V_n = 6j + 3 = \frac{4^{3n+1}(6(3k+1)+1)-1}{3} \;;$$

$$V_n \;=\; 6k \times 4^{3n+1} + \frac{7 \times 4^{3n+1} - 1}{3} = 6k \times 4^{3n+1} + \frac{6 \times 4^{3n+1}}{3} + \frac{4^{3n+1} - 1}{3} = 6k \times 4^{3n+1} + 2 \times 4^{3n+1} + \sum_{p=1}^{3n} 4^p + 1$$

$$We\ already\ know \sum_{p=1}^{3n} 4^p = 21 \sum_{p=0}^{n-1} 4^{3p+1} = 6 \times 3 \sum_{p=0}^{n-1} 4^{3p+1} + 3 \sum_{p=0}^{n-1} 4^{3p+1}$$

$$\Rightarrow V_n = 6k \times 4^{3n+1} - 4^{3n+1} + 3 \sum_{p=0}^{n} 4^{3p+1} + 6 \times 3 \sum_{p=0}^{n-1} 4^{3p+1} + 1;\ 1 - 4^{3n+1} = -3 \sum_{p=0}^{3n} 4^p = -3 - 3 \sum_{p=1}^{3n} 4^p$$

$$\Rightarrow V_n = 6k \times 4^{3n+1} + 6 \times 2 \sum_{p=0}^{n} 4^{3p} + 6 \times 3 \sum_{p=0}^{n-1} 4^{3p+1} - 6 + 3 - 3 \sum_{p=1}^{3n} 4^p$$

Finally,

$$\boldsymbol{V_n \;=\; 6\left[4^{3n+1}k + 3 \sum_{p=0}^{n-1} 4^{3p+1} + 2 \sum_{p=0}^{n} 4^{3p} - 2 \sum_{p=1}^{3n} 4^{p-1} - 1\right] + 3, \forall\, n \geq 1;\ V_0 = 6(4k+1) + 3}\ ;$$

$$W_0 = \frac{4^{1+1}(6(3k+1)+1)-1}{3} = 6(16k+6) + 1$$

then:

$$W_n = 6j + 1 = \frac{4^{3n+2}(18k+7)-1}{3} = 6k \times 4^{3n+2} + 2 \times 4^{3n+2} + \sum_{p=0}^{3n+1} 4^p$$

$$\sum_{p=0}^{3n+1} 4^p = 1 + 4 + \ldots + 4^{3n+1} = 5 + 21 \times 4^2 \sum_{p=0}^{n-1} 4^{3p} = 5 + 21 \times 4^2 \sum_{p=1}^{n} 4^{3p-3} = 5 + 4^2 \times 6 \times 3 \sum_{p=1}^{n} 4^{3p-3} + 3 \sum_{p=1}^{n} 4^{3p-1}$$

$$2 \times 4^{3n+2} = 3 \times 4^{3(n+1)-1} - 4^{3n+2}; 5 - 4^{3n+2} = 1 + 4\left(1 - 4^{3n+1}\right) = 1 - 4 \times 3 \sum_{p=0}^{3n} 4^p$$

$$\Rightarrow W_n = 6k \times 4^{3n+2} + 1 - 6 \times 2 \sum_{p=0}^{3n} 4^p + 6 \times 3 \sum_{p=1}^{n} 4^{3p-1} + 6 \times 2 \sum_{p=1}^{n+1} 4^{3p-2}$$

In conclusion,

$$\boldsymbol{W_n \;=\; 6\left[4^{3n+2}k + 3 \sum_{p=1}^{n} 4^{3p-1} + 2 \sum_{p=1}^{n+1} 4^{3p-2} - 2 \sum_{p=0}^{3n} 4^{p}\right] + 1, \forall\, n \geq 1;\ W_0 = 6(16k+6) + 1}\ ;$$

$$X_0 = \frac{4^{2+1}(18k+7)-1}{3} = 6(64k+24) + 5$$

and it follows:

$$X_n = \frac{4^{3n+3}(18k+7)-1}{3} = 6k \times 4^{3n+3} + 2 \times 4^{3n+3} + \sum_{p=0}^{3n+2} 4^p$$

$$\sum_{p=0}^{3n+2} 4^p = 21\sum_{p=0}^{n} 4^{3p} = 21\sum_{p=1}^{n} 4^{3p} + 4^2 + 5; \sum_{p=1}^{n} 4^{3p} = 6 \times 3\sum_{p=1}^{n} 4^{3p} + 3\sum_{p=1}^{n} 4^{3p}\,; 2 \times 4^{3n+3} = 3 \times 4^{3(n+1)} - 4^{3n+3}$$

$$\Rightarrow X_n = 6k \times 4^{3n+3} + 4^2 + 5 + 6 \times 3\sum_{p=1}^{n} 4^{3p} + 3\sum_{p=1}^{n+1} 4^{3p} - 4^{3n+3}$$

$$4^2 - 4^{3n+3} = 4^2\left(1 - 4^{3n+1}\right) = -4^2 \times 3\sum_{p=0}^{3n} 4^p$$

$$\Rightarrow X_n = 6k \times 4^{3n+3} + 5 + 6 \times 3\sum_{p=1}^{n} 4^{3p} + 3 \times 4\sum_{p=1}^{n+1} 4^{3p-1} - 6 \times 8\sum_{p=0}^{3n} 4^p$$

In conclusion,

$$\boldsymbol{X_n = 6\left[4^{3n+3}k + 3\sum_{p=1}^{n} 4^{3p} + 2\sum_{p=1}^{n+1} 4^{3p-1} - 2\sum_{p=0}^{3n} 4^{p+1}\right] + 5, \forall\, n \geq 1;\ X_0 = 6(64k+24)+5}\,;$$

Case 3: a=3k+2

The initial previous term is

$$\frac{4^{0+1}(6(3k+2)+1)-1}{3} = 6(4k+2)+5$$

then:

$$V_n = 6j + 5 = \frac{4^{3n+1}(6(3k+2)+1)-1}{3} = 6k \times 4^{3n+1} + 4 \times 4^{3n+1} + \sum_{p=0}^{3n} 4^p$$

$$\sum_{p=0}^{3n} 4^p = 1 + \sum_{p=1}^{3n} 4^p = 1 + 21\sum_{p=0}^{n-1} 4^{3p+1} = 1 + 6 \times 3\sum_{p=0}^{n-1} 4^{3p+1} + 3\sum_{p=0}^{n-1} 4^{3p+1}\,; 4 \times 4^{3n+1} = 3 \times 4^{3n+1} + 4^{3n+1}$$

$$\Rightarrow V_n = 6k \times 4^{3n+1} + 6 \times 3\sum_{p=0}^{n-1} 4^{3p+1} + 3\sum_{p=0}^{n} 4^{3p+1} + 4^{3n+1} + 1; 4^{3n+1} + 1 = 4^{3n+1} - 1 + 2 = 5 + 3\sum_{p=1}^{3n} 4^p$$

$$\boldsymbol{\Rightarrow V_n = 6\left[4^{3n+1}k + 3\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=0}^{n} 4^{3p} + 2\sum_{p=1}^{3n} 4^{p-1}\right] + 5 \forall\, n \geq 1;\ V_0 = 6(4k+2)+5}\,;$$

$$\frac{4^{1+1}(6(3k+2)+1)-1}{3} = 6(16k+11)+3$$

then it follows:

$$W_n = 6j + 3 = \frac{4^{3n+2}k(18k + 13) - 1}{3} = 6k \times 4^{3n+2} + 4 \times 4^{3n+2} + \sum_{p=0}^{3n+1} 4^p$$

$$\sum_{p=0}^{3n+1} 4^p = 1 + 4 + \sum_{p=2}^{3n+1} 4^p = 5 + 21\sum_{p=1}^{n} 4^{3p-1} = 5 + 6 \times 3\sum_{p=1}^{n} 4^{3p-1} + 3\sum_{p=1}^{n} 4^{3p-1}$$

$$4 \times 4^{3n+2} + 5 = 6 + 3 \times 4^{3n+2} + 4^{3n+2} - 1 = 6 + 3 \times 4^{3n+2} + 3\sum_{p=0}^{3n+1} 4^p = 6 + 3 \times 4^{3(n+1)-1} + 3\sum_{p=1}^{3n+1} 4^p + 3$$

$$\Rightarrow \boldsymbol{W_n = 6\left[4^{3n+2}k + 3\sum_{p=1}^{n} 4^{3p-1} + 2\sum_{p=1}^{n+1} 4^{3p-2} + 2\sum_{p=1}^{3n+1} 4^{p-1} + 1\right] + 3 \;\forall\, n \geq 1;\; W_0 = 6(16k + 11) + 3}\,;$$

$$\frac{4^{2+1}(6(3k + 2) + 1) - 1}{3} = 6(64k + 46) + 1$$

and we have:

$$X_n = 6j + 1 = \frac{4^{3n+3}(18k + 13) - 1}{3} = 6k \times 4^{3n+3} + 4 \times 4^{3n+3} + \sum_{p=0}^{3n+2} 4^p\,;$$

$$\sum_{p=0}^{3n+2} 4^p = 21\sum_{p=0}^{n} 4^{3p} = 6 \times 3\sum_{p=1}^{n} 4^{3p} + 3\sum_{p=1}^{n} 4^{3p} + 1 + 4 + 4^2$$

$$4 \times 4^{3n+3} = 3 \times 4^{3(n+1)} + 4^{3n+3}$$

$$1 + 4 + 4^2 + 4^{3n+3} = 1 + 4 + 4^2(4^{3n+1} - 1 + 2) = 1 + 4 + 4^2 \times 3\sum_{p=0}^{3n} 4^p + 2 \times 4^2$$

$$= 1 + 6^2 + 6 \times 2\sum_{p=0}^{3n} 4^{p+1}$$

$$\Rightarrow \boldsymbol{X_n = 6\left[4^{3n+3}k + 3\sum_{p=1}^{n} 4^{3p} + 2\sum_{p=1}^{n+1} 4^{3p-1} + 2\sum_{p=0}^{3n} 4^{p+1} + 6\right] + 1 \;\forall\, n \geq 1;\; X_0 = 6(64k + 46) + 1}\,.$$

2.1.4 Previous Terms of $U_{i+1} = 6a + 5$

The general term is

$$U_{i,p} = \frac{2 \times 4^p(6a + 5) - 1}{3}, \forall p \in \mathbb{N}\,.$$

Case 1: a=3k

The initial previous term is

$$\frac{2 \times 4^0(6(3k) + 5) - 1}{3} = 6(2k) + 3\,.$$

So:

$$V_n = 6j+3 = \frac{2\times 4^{3n}(6(3k)+5)-1}{3} = 12k\times 4^{3n} + \frac{9\times 4^{3n}+4^{3n}-1}{3} = 12k\times 4^{3n} + 3\times 4^{3n} + \sum_{p=0}^{3n-1} 4^p$$

$$\sum_{p=0}^{3n-1} 4^p = 21\sum_{p=0}^{n-1} 4^{3p} = 6\times 3\sum_{p=0}^{n-1} 4^{3p} + 3\sum_{p=1}^{n-1} 4^{3p} + 3$$

$$\Rightarrow \boldsymbol{V_n = 6\left[2\times 4^{3n}k + 3\sum_{p=0}^{n-1} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1}\right] + 3\ \forall\, n \geq 1;\ V_0 = 6(2k)+3}\ ;$$

The second previous term is

$$\frac{2\times 4^1(18k+5)-1}{3} = 6(8k+2)+1$$

then

$$W_n = 6j+1 = \frac{2\times 4^{3n+1}(18k+5)-1}{3} = 12k\times 4^{3n+1} + 3\times 4^{3n+1} + \sum_{p=0}^{3n} 4^p\ ;$$

$$\sum_{p=0}^{3n} 4^p = 1 + \sum_{p=1}^{3n} 4^p = 1 + 21\sum_{p=0}^{n-1} 4^{3p+1} = 16\times 3\sum_{p=0}^{n-1} 4^{3p+1} + 3\sum_{p=0}^{n-1} 4^{3p+1}$$

$$\Rightarrow W_n = 12k\times 4^{3n+1} + 1 + 6\times 3\sum_{p=0}^{n-1} 4^{3p+1} + 3\times 4\sum_{p=0}^{n-1} 4^{3p} + 3\times 4\times 4^{3n}$$

Finally,

$$\boldsymbol{W_n = 6\left[2\times 4^{3n+1}k + 3\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=0}^{n} 4^{3p}\right] + 1\ \forall\, n \geq 1;\ W_0 = 6(8k+2)+1}\ ;$$

The third previous term is

$$\frac{2\times 4^2(18k+5)-1}{3} = 6(32k+8)+5$$

and it implies

$$X_n = 6j+5 = \frac{4^{3n+2}(18k+5)-1}{3} = 12k\times 4^{3n+2} + 3\times 4^{3n+2} + \sum_{p=0}^{3n+1} 4^p$$

$$\sum_{p=0}^{3n+1} 4^p = 1 + 4 + \sum_{p=2}^{3n+1} 4^p = 5 + 21\sum_{p=0}^{n-1} 4^{3p+2}$$

$$\Rightarrow X_n = 12k\times 4^{3n+2} + 5 + 6\times 3\sum_{p=0}^{n-1} 4^{3p+2} + 3\times 4\sum_{p=0}^{n} 4^{3p+1}$$

$$\Rightarrow \boldsymbol{X_n = 6\left[2\times 4^{3n+2}k + 3\sum_{p=0}^{n-1} 4^{3p+2} + 2\sum_{p=0}^{n} 4^{3p+1}\right] + 5\ \forall\, n \geq 1;\ X_0 = 6(32k+8)+5}\ ;$$

Case 2: a=3k+1

The initial previous term is

$$\frac{2\times 4^0(6(3k+1)+5)-1}{3} = 6(2k+1)+1$$

then

$$V_n = \frac{2\times 4^{3n}(6(3k+1)+5)-1}{3} = 12k\times 4^{3n} + 7\times 4^{3n} + \sum_{p=0}^{3n-1} 4^p \text{ , } n \geq 1$$

$$\sum_{p=0}^{3n-1} 4^p = 21\sum_{p=0}^{n-1} 4^{3p} \text{ ; } 7\times 4^{3n} = 4\times 4^{3n} + 3\times 4^{3n}$$

$$\Rightarrow V_n = 12k\times 4^{3n} + 6\times 3\sum_{p=0}^{n-1} 4^{3p} + 3\sum_{p=1}^{n} 4^{3p} + 4\times 4^{3n} + 3;$$

$$4\times 4^{3n} + 3 = 4^{3n+1} - 1 + 4 = 4 + 3\sum_{p=0}^{3n} 4^p = 4 + 3 + 3\sum_{p=1}^{3n} 4^p = 6 + 1 + 3\times 4\sum_{p=1}^{3n} 4^{p-1}$$

$$\Rightarrow \boldsymbol{V_n = 6\left[2\times 4^{3n}k + 3\sum_{p=0}^{n-1} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1} + 2\sum_{p=1}^{3n} 4^{p-1} + 1\right] + 1\ \forall\, n \geq 1;\ V_0 = 6(2k+1)+1}\ ;$$

The second previous term:

$$\frac{2\times 4^1(18k+11)-1}{3} = 6(8k+4)+5$$

then

$$W_n = \frac{2\times 4^{3n+1}(18k+11)-1}{3} = 12k\times 4^{3n+1} + 7\times 4^{3n+1} + \sum_{p=0}^{3n} 4^p$$

$$= 12k\times 4^{3n+1} + 7\times 4^{3n+1} + 1 + 21\sum_{p=0}^{n-1} 4^{3p+1} \quad ;$$

$$7\times 4^{3n+1} = 4^{3n+2} + 3\times 4^{3n+1}$$

$$\Rightarrow W_n = 12k\times 4^{3n+1} + 4^{3n+2} + 1 + 6\times 3\sum_{p=0}^{n-1} 4^{3p+1} + 3\sum_{p=1}^{n} 4^{3p+1} + 12\ ;$$

$$4^{3n+2} + 1 + 12 = 4^{3n+2} - 1 + 9 + 5 = 3\sum_{p=1}^{3n+1} 4^p + 12 + 5$$

$$\Rightarrow \boldsymbol{W_n = 6\left[2\times 4^{3n+1}k + 3\times\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=1}^{n} 4^{3p} + 2\sum_{p=1}^{3n+1} 4^{p-1} + 2\right] + 5\ \forall\, n \geq 1; W_0 = 6(8k+4)+5}\ ;$$

The third previous term is

$$\frac{2\times 4^2(18k+11)-1}{3} = 6(32k+19)+3$$

then

$$X_n = \frac{2\times 4^{3n+2}(18k+11)-1}{3} = 12k\times 4^{3n+2} + 7\times 4^{3n+2} + \sum_{p=0}^{3n+1} 4^p$$

$$\sum_{p=0}^{3n+1} 4^p = 5 + 21\sum_{p=0}^{n-1} 4^{3p+2} = ,7 \times 4^{3n+2} = 4^{3n+3} + 3 \times 4^{3n+2}$$

$$\Rightarrow X_n = 12k \times 4^{3n+2} + 4^{3n+3} - 1 + 6 + 6 \times 3\sum_{p=0}^{n-1} 4^{3p+2} + 3\sum_{p=0}^{n} 4^{3p+2}$$

$$X_n = 12k \times 4^{3n+2} + 6 \times 3\sum_{p=0}^{n-1} 4^{3p+2} + 6 \times 2\sum_{p=0}^{n} 4^{3p+1} + 3\sum_{p=1}^{3n+2} 4^p + 3 + 6$$

In the end,

$$\boldsymbol{X_n = 6\left[2 \times 4^{3n+2}k + 3\sum_{p=0}^{n-1} 4^{3p+2} + 2\sum_{p=0}^{n} 4^{3p+1} + 2\sum_{p=1}^{3n+2} 4^{p-1} + 1\right] + 3\ \forall\, n \geq 1;\ X_0 = 6(32k + 19) + 3}\,;$$

Case 3: a=3k+2

The initial previous term is

$$\frac{2 \times 4^0(6(3k+2)+5) - 1}{3} = 6(2k+1) + 5\,;$$

We then have

$$V_n = 6j + 5 = \frac{2 \times 4^{3n}(6(3k+2)+5) - 1}{3} = 12k \times 4^{3n} + 11 \times 4^{3n} + \sum_{p=0}^{3n-1} 4^p\,;\ \sum_{p=0}^{3n-1} 4^p = 21\sum_{p=0}^{n-1} 4^{3p}$$

$$V_n = 12k \times 4^{3n} + 8 \times 4^{3n} + 6 \times 3\sum_{p=0}^{n-1} 4^{3p} + 3\sum_{p=0}^{n} 4^{3p} = 12k \times 4^{3n} + 2\left(4^{3n+1} - 1\right) + 5 + 6 \times 3\sum_{p=0}^{n-1} 4^{3p} + 3\sum_{p=1}^{n} 4^{3p}$$

$$\boldsymbol{\Rightarrow V_n = 6\left[2 \times 4^{3n}k + 3\sum_{p=0}^{n-1} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1} + \sum_{p=0}^{3n} 4^p\right] + 5\ \forall\, n \geq 1;\ V_0 = 6(2k+1) + 5}\,;$$

The second previous term is

$$\frac{2 \times 4^1(18k+17) - 1}{3} = 6(8k+7) + 3$$

then

$$W_n = 6j + 3 = \frac{2 \times 4^{3n+1}(18k+17) - 1}{3} = 12k \times 4^{3n+1} + 11 \times 4^{3n+1} + \sum_{p=0}^{3n} 4^p$$

$$\sum_{p=0}^{3n} 4^p = 1 + 21\sum_{p=0}^{n-1} 4^{3p+1}$$

$$\Rightarrow W_n = 12k \times 4^{3n+1} + 2 \times 4^{3n+2} + 3 - 2 + 6 \times 3\sum_{p=0}^{n-1} 4^{3p+1} + 3\sum_{p=0}^{n} 4^{3p+1}$$

Finally

$$\boldsymbol{W_n = 6\left[2 \times 4^{3n+1}k + 3\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=0}^{n} 4^{3p} + \sum_{p=0}^{3n+1} 4^p\right] + 3\ \forall\, n \geq 1;\ W_0 = 6(8k+7) + 3}\,;$$

The third previous term is

$$\frac{2 \times 4^2(18k+17) - 1}{3} = 6(32k+30) + 1\,;$$

It follows

$$X_n = 6j + 1 = 12k \times 4^{3n+2} + 11 \times 4^{3n+2} + \sum_{p=0}^{3n+1} 4^p = 12k \times 4^{3n+2} + 11 \times 4^{3n+2} + 5 + 21 \sum_{p=0}^{n-1} 4^{3p+2}$$

$$X_n = 12k \times 4^{3n+2} + 2 \times 4^{3n+3} - 2 + 1 + 6 + 6 \times 3 \sum_{p=0}^{n-1} 4^{3p+2} + 3 \sum_{p=0}^{n} 4^{3p+2}$$

In conclusion,

$$\boldsymbol{X_n = 6\left[2 \times 4^{3n+2}k + 3\sum_{p=0}^{n-1} 4^{3p+2} + 2\sum_{p=0}^{n} 4^{3p+1} + \sum_{p=0}^{3n+2} 4^p + 1\right] + 1 \ \forall\, n \geq 1;\ X_0 = 6(32k + 30) + 1\ .}$$

Let's summarize:

$$\boldsymbol{PREVIOUS\ TERMS\ OF\ U_{i+1} = 6a + 1}:$$

$$a = 3\boldsymbol{k} \begin{cases} V_n = 6\left[4^{3n+1}\boldsymbol{k} + 14\sum_{p=0}^{n-1} 4^{3p}\right] + 1 \ \forall\, n \geq 1; V_0 = 6(4\boldsymbol{k}) + 1 \\ W_n = 6\left[4^{3n+2}\boldsymbol{k} + 14\sum_{p=0}^{n-1} 4^{3p+1}\right] + 5 \ \forall\, n \geq 1; W_0 = 6(16\boldsymbol{k}) + 5 \\ X_n = 6\left[4^{3n+3}\boldsymbol{k} + 3\sum_{p=0}^{n} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1}\right] + 3 \ \forall\, n \geq 1; X_0 = 6(64\boldsymbol{k} + 3) + 3 \end{cases}$$

$$a = 3\boldsymbol{k} + 1 \begin{cases} V_n = 6\left[4^{3n+1}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=0}^{n} 4^{3p} - 2\sum_{p=1}^{3n} 4^{p-1} - 1\right] + 3 \ \forall\, n \geq 1; V_0 = 6(4\boldsymbol{k} + 1) + 3 \\ W_n = 6\left[4^{3n+2}\boldsymbol{k} + 3\sum_{p=1}^{n} 4^{3p-1} + 2\sum_{p=1}^{n+1} 4^{3p-2} - 2\sum_{p=0}^{3n} 4^{p}\right] + 1 \ \forall\, n \geq 1; W_0 = 6(16\boldsymbol{k} + 6) + 1 \\ X_n = 6\left[4^{3n+3}\boldsymbol{k} + 3\sum_{p=1}^{n} 4^{3p} + 2\sum_{p=1}^{n+1} 4^{3p-1} - 2\sum_{p=0}^{3n} 4^{p+1}\right] + 5 \ \forall\, n \geq 1; X_0 = 6(64\boldsymbol{k} + 24) + 5 \end{cases}$$

$$a = 3\boldsymbol{k} + 2 \begin{cases} V_n = 6\left[4^{3n+1}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=0}^{n} 4^{3p} + 2\sum_{p=1}^{3n} 4^{p-1}\right] + 5 \ \forall\, n \geq 1; V_0 = 6(4\boldsymbol{k} + 2) + 5 \\ W_n = 6\left[4^{3n+2}\boldsymbol{k} + 3\sum_{p=1}^{n} 4^{3p-1} + 2\sum_{p=1}^{n+1} 4^{3p-2} + 2\sum_{p=1}^{3n+1} 4^{p-1} + 1\right] + 3 \ \forall\, n \geq 1; W_0 = 6(16\boldsymbol{k} + 11) + 3 \\ X_n = 6\left[4^{3n+3}\boldsymbol{k} + 3\sum_{p=1}^{n} 4^{3p} + 2\sum_{p=1}^{n+1} 4^{3p-1} + 2\sum_{p=0}^{3n} 4^{p+1} + 6\right] + 1 \ \forall\, n \geq 1; X_0 = 6(64\boldsymbol{k} + 46) + 1 \end{cases}$$

$\boldsymbol{PREVIOUS\ TERMS\ OF\ U_{i+1} = 6a + 5}$:

$$a = 3\boldsymbol{k}\begin{cases} V_n = 6\left[2 \times 4^{3n}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1}\right] + 3\ \forall\, n \geq 1; V_0 = 6(2\boldsymbol{k}) + 3 \\ W_n = 6\left[2 \times 4^{3n+1}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=0}^{n} 4^{3p}\right] + 1\ \forall\, n \geq 1; W_0 = 6(8\boldsymbol{k} + 2) + 1 \\ X_n = 6\left[2 \times 4^{3n+2}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p+2} + 2\sum_{p=0}^{n} 4^{3p+1}\right] + 5\ \forall\, n \geq 1; X_0 = 6(32\boldsymbol{k} + 8) + 5 \end{cases}$$

$$a = 3\boldsymbol{k} + 1\begin{cases} V_n = 6\left[2 \times 4^{3n}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1} + 2\sum_{p=1}^{3n} 4^{p-1} + 1\right] + 1\ \forall\, n \geq 1; V_0 = 6(2\boldsymbol{k} + 1) + 1 \\ W_n = 6\left[2 \times 4^{3n+1}\boldsymbol{k} + 3 \times \sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=1}^{n} 4^{3p} + 2\sum_{p=1}^{3n+1} 4^{p-1} + 2\right] + 5\ \forall\, n \geq 1; W_0 = 6(8\boldsymbol{k} + 4) + 5 \\ X_n = 6\left[2 \times 4^{3n+2}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p+2} + 2\sum_{p=0}^{n} 4^{3p+1} + 2\sum_{p=1}^{3n+2} 4^{p-1} + 1\right] + 3\ \forall\, n \geq 1; X_0 = 6(32\boldsymbol{k} + 19) + 3 \end{cases}$$

$$a = 3\boldsymbol{k} + 2\begin{cases} V_n = 6\left[2 \times 4^{3n}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p} + 2\sum_{p=1}^{n} 4^{3p-1} + \sum_{p=0}^{3n} 4^{p}\right] + 5\ \forall\, n \geq 1; V_0 = 6(2\boldsymbol{k} + 1) + 5 \\ W_n = 6\left[2 \times 4^{3n+1}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p+1} + 2\sum_{p=0}^{n} 4^{3p} + \sum_{p=0}^{3n+1} 4^{p}\right] + 3\ \forall\, n \geq 1; W_0 = 6(8\boldsymbol{k} + 7) + 3 \\ X_n = 6\left[2 \times 4^{3n+2}\boldsymbol{k} + 3\sum_{p=0}^{n-1} 4^{3p+2} + 2\sum_{p=0}^{n} 4^{3p+1} + \sum_{p=0}^{3n+2} 4^{p} + 1\right] + 1\ \forall\, n \geq 1; X_0 = 6(32\boldsymbol{k} + 30) + 1 \end{cases}$$

The relations above, which we call "general relations" tell, for any random term U_{i+1} in a Syracuse sequence, the expressions of all its previous terms with rank i, for all possible values of number a: 3k, 3k+1 or 3k+2, k natural number. In reverse, any random term U_i is necessarily among V_n, W_n or X_n, so to determine its next term U_{i+1} using the same relations. In fact, let any odd natural number q; it can be written either 6m+1, 6m+3 or 6m+5, m natural number. Its next term in a Syracuse sequence is necessarily written $6a + 1$ or $6a + 5$, a natural number. Since the previous terms are formulated exhaustively, q is either V_n, W_n or $X_n, n \in \mathbb{N}$.

2.2 Onion-Factorization

2.2.1 Definition

Let C_1 a natural number.

We call onion-factorization of C_1 in $\mathbb{Q}$, set of rational numbers, a development of C_1 into an arbitrary given $n \in \mathbb{N}$ onion factors $C_2, \ldots, C_{n+1}$, all natural numbers, n non-zero, such that $C_1 = A_1C_2 + B_1, C_2 = A_2C_3 + B_2, \ldots, C_n = A_nC_{n+1} + B_n, A_i \in \mathbb{Q}, B_i \in \mathbb{Q}, i = 1,2, \ldots, n; C_i \in \mathbb{N}, i = 1, \ldots, n+1$ and:

$$C_1 = A_1(A_2(\ldots(A_n(C_{n+1}) + B_n)\ldots) + B_2) + B_1$$

About consecutive onion factors C_i and C_{i+1}, we can look at them as respectively the (integer) dividend and the (integer) quotient of a division in which, the divisor A_i and the remainder B_i are rationals, $i = 1, \ldots, n$.

<u>Example:</u>

We look for an onion-factorization in $\mathbb{Q}$ of 33.

We have

$$33 = 6 \times 5 + 3;\ 5 = 4 \times 1 + 1;\ 1 = \frac{4}{3} \times 1 - \frac{1}{3};\ 1 = \frac{2}{3} \times 1 + \frac{1}{3}$$

The first onion factor is 33, the second 5, the third 1, the fourth 1 and the fifth 1. We put them together and it follows:

$$33 = 6\left(4\left(\frac{4}{3}\left(\frac{2}{3}(1) + \frac{1}{3}\right) - \frac{1}{3}\right) + 1\right) + 3\,;$$

We also have

$$33 = 2 \times 16 + 1;\ 16 = 2 \times 8;\ 8 = 2 \times 4;\ 4 = 2 \times 2;\ 2 = 2 \times 1;\ 1 = 2 \times 0 + 1$$

The onion factors here are in order 33, 16, 8, 4, 2, 1 and 0. Then, another onion-factorization of 33 is:

$$33 = 2(2(2(2(2(2(0) + 1))))) + 1\,.$$

Remark: An onion-factorization in $\mathbb{N}$ is an onion-factorization in $\mathbb{Q}$.

2.2.2 Theorem

Let's consider U_0 the initial term of a Syracuse sequence $(U_n)_{n\in\mathbb{N}}$. We define U_0 as a previous term. Then it exists an onion-factorization in $\mathbb{Q}$ of the initial term such that, the more we extend the sequence by calculating new terms, the more onion factors we can append. Each of the appended onion factors is a value of k, found in the expression of any previous term and its next one.

Proof:

The general relations imply that, for any term $6j + 1, 6j + 3, 6j + 5$ in a Syracuse sequence, j is always equal to $Ak + B$, A non-zero and B natural numbers.

Since U_0 is a previous term, the initial term is either V_n, W_n or $X_n, n \in \mathbb{N}$. The 2nd term is naturally a next term, then is either $6a + 1$ or $6a + 5$, $a \in \mathbb{N}$, but is also a previous term to the third one. Except the initial term, every term in the Syracuse sequence is both a previous and a next term.

$\forall\, n \geq 1, U_n$ is not expressed the same whether we consider it as a previous term or a next term. In fact, for any $6a + 1$ or $6a + 5$ as a next term, a is either $3k$, $3k + 1$ or $3k + 2$; but as a previous term, a is expressed differently, and its expression tells straight on its parity. For instance we have for a odd: $6(2k + 1) + 1$ and $6(2k + 1) + 5$.

Let the initial term $U_0 = 6(A_0k_0 + B_0) + 3$ (one can replace 3 by 1 or 5, it changes nothing on what follows). As the general relations show it, like U_0, the expression of the next term will contain k_0. If $U_1 = 6j_1 + 1$ or $6j_1 + 5$, we have $j_1 = A_1k_0 + B_1$. With U_1 as a previous term, j_1 becomes $A_2k_1 + B_2$, that is, $A_1k_0 + B_1 = A_2k_1 + B_2$. We then have $k_0 = \frac{A_2}{A_1}k_1 + \frac{B_2 - B_1}{A_1}$ and the initial term becomes

$U_0 = 6\left(A_0\left(\frac{A_2}{A_1}(k_1) + \frac{B_2 - B_1}{A_1}\right) + B_0\right) + 3$. If we move to the third term U_2, as a next term its expression contains k_1, and if $U_2 = 6j_2 + 1$ or $6j_2 + 5$, we have $j_2 = A_3k_1 + B_3$; as a previous term, j_2 turns $A_4k_2 + B_4$ and $A_3k_1 + B_3 = A_4k_2 + B_4$. It follows $k_1 = \frac{A_4}{A_3}k_2 + \frac{B_4 - B_3}{A_3}$ and the initial term becomes

$$U_0 = 6\left(A_0\left(\frac{A_2}{A_1}\left(\frac{A_4}{A_3}(k_2) + \frac{B_4 - B_3}{A_3}\right) + \frac{B_2 - B_1}{A_1}\right) + B_0\right) + 3\,.$$

In general, let's consider terms from U_0 to $U_n, n \in \mathbb{N},\ n \geq 3$. We represent the move from a previous term to the next term by a right arrow. For a purpose of simplification, we just represent each term $6j + 1, 6j + 3$ or $6j + 5$ by $j = Ak + B$, and it goes:

$$A_0k_0 + B_0 \to (A_1k_0 + B_1 = A_2k_1 + B_2) \to (A_3k_1 + B_3 = A_4k_2 + B_2) \to A_5k_2 + B_5 \ \ldots \\ (A_{2n-1}k_{n-1} + B_{2n-1} = A_{2n}k_n + B_{2n}) \qquad (12)$$

$A_i \in \mathbb{N}, B_i \in \mathbb{N}, i = 0, \ldots, 2n; k_i \in \mathbb{N}, i = 0, \ldots, n; n \in \mathbb{N}, n \geq 3.$

This representation displays $n + 1$ terms in a Syracuse sequence. By exploiting any equality in brackets, linking k_i and $k_{i+1}, i = 0,1, \ldots, n - 1$ in (12), we have:

$$k_0 = \frac{A_2}{A_1}k_1 + \frac{(B_2 - B_1)}{A_1}; k_1 = \frac{A_4}{A_3}k_2 + \frac{(B_4 - B_3)}{A_3}; \dots ; k_{n-1} = \frac{A_{2n}}{A_{2n-1}}k_n + \frac{(B_{2n} - B_{2n-1})}{A_{2n-1}}. \quad (\mathbf{13})$$

From (13), we can link directly $k_0$and $k_n$with the following:

$$k_0 = \frac{A_2}{A_1}\left(\frac{A_4}{A_3}\left(\cdots\left(\frac{A_{2n}}{A_{2n-1}}(k_n) + \frac{B_{2n} - B_{2n-1}}{A_{2n-1}}\right)\cdots\right) + \frac{B_4 - B_3}{A_3}\right) + \frac{B_2 - B_1}{A_1}.$$

Finally,

$$U_0 = 6\left(A_0\left(\frac{A_2}{A_1}\left(\frac{A_4}{A_3}\left(\cdots\left(\frac{A_{2n}}{A_{2n-1}}(k_n) + \frac{B_{2n} - B_{2n-1}}{A_{2n-1}}\right)\cdots\right) + \frac{B_4 - B_3}{A_3}\right) + \frac{B_2 - B_1}{A_1}\right) + B_0\right) + 3.$$

$A_i, i\ even$ corresponds to previous terms, then it is a power of 2; $A_i, iodd$ corresponds to next terms, then equals 3. So $\frac{A_i(ieven)}{A_i(iodd)}$ is rational.U_0 is an onion-factorization in $\mathbb{Q}$.

2.2.3 Corollary

Let a convergent Syracuse sequence $(U_n)_{n\in\mathbb{N}}$, with $U_0 = 6(A_0k_0 + B_0) + 3; A_0, B_0, k_0 \in \mathbb{N}$. Then the sequence converges to 1, and the initial term has an onion-factorization in $\mathbb{Q}$ such that, while we extend the sequence infinitely, we can append only a finite number of onion factors, the last onion factor corresponding to the limit of the sequence. The initial term is as follows:

$$U_0 = 6\left(A_0\left(\frac{A_2}{A_1}\left(\frac{A_4}{A_3}\left(\cdots\left(\frac{A_{2n}}{A_{2n-1}}(\cdots(0)\cdots) + \frac{B_{2n} - B_{2n-1}}{A_{2n-1}}\right)\cdots\right) + \frac{B_4 - B_3}{A_3}\right) + \frac{B_2 - B_1}{A_1}\right) + B_0\right) + 3;$$

$(n \geq 3), A_i\ non-zero, A_i \in \mathbb{N}, B_i \in \mathbb{N}, i \in \mathbb{N}.$

<u>Proof:</u>

We got a convergent Syracuse sequence. Since we are dealing with a sequence of positive integers, it necessarily is eventually constant. This means, it exists a rank $p \in \mathbb{N}$ from which $U_p = U_{p+1} = \cdots = constant$. A closer look at the general relations, and we notice that this is possible only as soon as the previous term is $6(4k) + 1$ and its next term $6(3k) + 1$, with $k = 0$. Then the sequence converges to constant = 1.

Consequently, and since we can build an onion-factorization of the initial term, with as many onion factors as we want, we have from the rank p on, an infinite number of onion factors, all equal to 0 while we extend the sequence infinitely. Next term $6\big(3(0)\big) + 1$ is equal to previous term $6\big(4(0)\big) + 1$, then

$3 \times 0 = 4 \times 0 \Leftrightarrow 0 = \frac{4}{3}(0)$. It follows:

$$0 = \frac{4}{3}\left(\frac{4}{3}\left(\frac{4}{3}\left(\underbrace{\cdots}_{infinitely}\frac{4}{3}(0)\underbrace{\cdots}_{infinitely}\right)\right)\right)$$

This implies that, in the onion-factorization of U_0, we can have only a finite number of onion factors, which equal 0 and are the last onion factors, including the 0 we express above as an onion-factorization, and of course a finite number of non-zero onion factors.

Thanks to the theorem, The onion-factorization of U_0 while we extend the sequence infinitely, can be expressed as follows:

$$U_0 = 6\left(A_0\left(\frac{A_2}{A_1}\left(\frac{A_4}{A_3}(\cdots(0)\cdots) + \frac{B_4 - B_3}{A_3}\right) + \frac{B_2 - B_1}{A_1}\right) + B_0\right) + 3.$$

<u>Remark:</u> The onion-factorization of 0 mentioned above implies that, the calculus is made from the last onion factor to the first one. Doing so is right since each onion factor k_i is known if and only if we know k_{i+1}, $i \in \mathbb{N}$. Operating in the opposite sense leads to an indeterminate result $(\infty \times 0)$.

3. Results

3.1 Hypothesis: The Syracuse Sequence Has No Limit and Is Eventually Periodic

Let $(U_n)_{n\in\mathbb{N}}, U_0$ its initial term. Let $N \geq 2$ the number of terms in the cycle. Since the sequence is eventually periodic, it exists a previous term $U_{n_0} = 6j_{n_0} + 1$ or $U_{n_0} = 6j_{n_0} + 5$ or $U_{n_0} = 6j_{n_0} + 3, j_{n_0} = A_{n_0}k_{n_0} + B_{n_0}, n_0 \in \mathbb{N}$, from which N terms are repeated infinitely, one after the other. In the onion-factorization of U_{n_0}, let the onion factors follow the progression of the sequence as we did in the corollary, that is, when we extend the sequence infinitely. The following representation is the same one we use in the proof of the theorem, and we have, when we make in the cycle a single turn from U_{n_0} to U_{n_0}:

$$\boldsymbol{A_{n_0}k_{n_0}} + \boldsymbol{B_{n_0}} \rightarrow \left(A_{n_0+1}k_{n_0} + B_{n_0+1} = A_{n_0+2}k_{n_0+1} + B_{n_0+2}\right) \rightarrow A_{n_0+3}k_{n_0+1} + B_{n_0+3} \ \cdots$$
$$\left(A_{n_0+2N-3}k_{n_0+N-2} + B_{n_0+2N-3} = A_{n_0+2N-2}k_{n_0+N-1} + B_{n_0+2N-2}\right) \rightarrow \left(A_{n_0+2N-1}k_{n_0+N-1} + B_{n_0+2N-1} = \boldsymbol{A_{n_0}k_{n_0}} + \boldsymbol{B_{n_0}}\right)$$

The onion-factorization in $\mathbb{Q}$ of j_{n_0}, while we extend the sequence infinitely, goes:

$$j_{n_0} = A_{n_0}\left(\frac{A_{n_0+2}}{A_{n_0+1}}\left(\cdots\left(\frac{A_{n_0}}{A_{n_0+2N-1}}\left(\frac{A_{n_0+2}}{A_{n_0+1}}\left(\underbrace{\cdots}_{infinitely}(?)\underbrace{\cdots}_{infinitely}\right) + \frac{B_{n_0+2} - B_{n_0+1}}{A_{n_0+1}}\right) + \frac{B_{n_0} - B_{n_0+2N-1}}{A_{n_0+2N-1}}\right)\cdots\right) + \frac{B_{n_0+2} - B_{n_0+1}}{A_{n_0+1}}\right) + B_{n_0}.$$

The question mark is there for the sequence has no limit. According to the definition on onion-factorization, we must know the value of the onion factor at the limit of the sequence, on which all the remaining onion factors depend. Then, j_{n_0} couldn't be calculated, the same case with U_{n_0}.

In conclusion, U_0 can't be determined, which invalidates the existence of the sequence.

3.2 Hypothesis: The Syracuse Sequence Diverges and Tends to Infinity

Let$(U_n)_{n\in\mathbb{N}}$, $U_0 = 6j_0 + 3, j_0 \in \mathbb{N}$ its initial term, considered a previous term as well. By application of the theorem, we can build an onion-factorization of $j_0$with an infinite number of onion factors, while we extend the sequence infinitely. The hypothesis implies that $lim_{n\to\infty} U_n = \infty$, and U_n is getting infinitely large if and only if, in the expression $6(Ak + B) + 1$ or $6(Ak + B) + 5$, k is getting infinitely large. In fact, A and B may get very bigger, but are always quantified values. Onion factors being values of k, it follows:

$$j_0 = A_0\left(\frac{A_2}{A_1}\left(\cdots\left(\frac{A_{2n}}{A_{2n-1}}\left(\underbrace{\cdots}_{infinitely}(\infty)\underbrace{\cdots}_{infinitely}\right) + \frac{B_{2n} - B_{2n-1}}{A_{2n-1}}\right)\cdots\right) + \frac{B_2 - B_1}{A_1}\right) + B_0;$$

$A_i \in \mathbb{N}, A_i \ non-zero\,; B_i \in \mathbb{N}, i \in \mathbb{N}.$

Such an expression of j_0 is quite unusual, and one can hardly understand what it may implicate. But, unlike the expression of the onion-factorization of j_{n_0}, there is no "question mark" here, and the onion factor corresponding to the limit of the Syracuse sequence, is infinitely large. Before rushing into any conclusion, let's see how infinite can be dealt with in an onion-factorization.

For this, we need to take a look at the following:

$$1 = \frac{1}{3}\times 3 = \frac{1}{3}\left(\frac{1}{3}\times 9\right) = \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\times 27\right)\right) = \cdots = \frac{1}{3}\left(\frac{1}{3}\left(\cdots\left(\frac{1}{3}\left(\underbrace{\cdots}_{infinitely}(\infty)\underbrace{\cdots}_{infinitely}\right)\right)\cdots\right)\right)$$

This is strange, because the last expression alone doesn't seem to equal 1. But since we know how it is obtained, we can write the following:

$$\frac{1}{3}\left(\frac{1}{3}\left(\cdots\left(\frac{1}{3}\left(\underbrace{\cdots}_{infinitely}(\infty)\underbrace{\cdots}_{infinitely}\right)\right)\cdots\right)\right) = \frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}\times\cdots = 1$$

Notice that in this onion-factorization, $B_{2n-1} - B_{2n} = 0, n\, non-zero \in \mathbb{N}$.

The equality (12) shows that we can calculate such an onion-factorization with an infinite number of onion factors, once it can be written trough a classical operation on which calculus can be made.

Generally, any onion-factorization with an infinite number of onion factors is calculable under the condition that, it exists an integer M such that $\forall\, n \geq M, B_{2n-1} - B_{2n} = 0$.

The onion-factorization of j_0 is obtained, with onion factors obeying the general relations. We then need to check, in this case where the Syracuse sequence diverges, whether those general relations allow to have the condition of calculability.

Since the sequence diverges, for all integer N, it exists integers $n_1, n_2;\ n_2 > n_1 \geq N,$ such that $U_{n_2} > U_{n_1}$.

We make the hypothesis that, in the progression of the sequence, it exists an integer M such that

$\forall n \geq M, B_{2n-1} - B_{2n} = 0$. To illustrate this, let's represent a move from a previous term to a next term by a right arrow, as we did in (12). When turning a next term into a previous term, the hypothesis must be locally respected.

Starting with one of the general relations that permit $U_{n_2} > U_{n_1}$, it follows:

$$6(2k_0+1)+5 \rightarrow (6(3k_0+\mathbf{2})+5 = 6(4k_1+\mathbf{2})+5) \rightarrow (6(3k_1+\mathbf{2})+1 = 6(8k_2+\mathbf{2})+1) \rightarrow (6(3k_2+\mathbf{0})+5 = 6(16k_3+\mathbf{0})+5) \rightarrow (6(3k_3)+1 = 6(4k_4)+1) \rightarrow (6(3k_4)+1 = 6(4k_5)+1) \cdots$$

This progression is undoubtedly decreasing to 1. Absurd. Then the condition of calculability is not fulfilled.

Besides, we know that as an onion-factorization, j_0 is inconsistent. In fact, by definition, infinite is not a quantifiable value. This means that, we can't estimate the values of onion factors when the terms of the sequence are getting infinitely large. There is consequently an impossibility to know any onion factor with a finite rank.

To conclude, j_0 and U_0 are impossible to calculate, and this makes the hypothesis impossible to realize.

Only the hypothesis of a convergent Syracuse sequence remains at the table.

4. Conclusion

Almost all the numerical details in the general relations have not been used in the proof. That is exactly where the conjecture is too complex to address, and they have been skipped by choice, while we have built the proof on the notion of onion-factorization. Meanwhile, those details may tell us more about interesting properties of Syracuse sequences.

References

Jeffrey, C. L. (2010). *The ultimate challenge: The 3x+1 Problem*. American Mathematical Society.

Terrence, T. (2019). *Almost all orbits of the Collatz map attain almost bounded values.* https://doi.org/10.48550/arXiv.1909.03562

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