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Triangles, Fractals and Spaghetti

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Abstract

The well-known Broken Spaghetti Problem is a geometric problem which can be stated as: *A stick of spaghetti breaks into three parts and all points of the stick have the same probability to be a breaking point. What is the probability that the three sticks, putting together, form a triangle?*

In this note, we describe a hidden geometric pattern behind the symmetric version of this problem, namely a fractal that parametrizes the sample space of this problem. Using that fractal, we address the question about the probability to obtain a δ-equilateral triangle.

Keywords: Fractal, Triangle, Probability

1. Introduction

The Broken Spaghetti Problem, also called the broken stick problem, is an old mathematical problem. It goes back to British mathematicians and it has interested mathematicians like E. Lemoine [7], H. Poincaré [8], de Finetti [2, 3]. There are several equivalent ways to formulate this problem but we are interested in the version stated above. The solution to the problem is

$$\widetilde{\mathbb{P}} = \frac{1}{4};$$

and it was already known to British mathematicians. This problem has been generalized to higher dimensional objects, see for instance [1, 4, 6]. In [6], the authors addressed a variant of the Broken Spaghetti Problem by changing triangles to other geometric objects. But, let us stay on the original problem. As mentioned in [5], E. Lemoine was the first to publish an article on this problem and he used an exhaustion process to come out with the answer (see [7]). There is also a geometric approach given by H. Poincaré in *Calcul des Probabilités* [8] where the space of all possibilities is adequately described: an equilateral triangle. Then the probability turns to be a relative area. The geometric approach was known before Poincaré but what he did better is the proof of why the fact that the broken points are equally likely translates to a uniform distribution on the sample space. The latter allows one to compute the probability as the relative area and it was assume to be obvious by other authors. In [5], G. S. Goodman gave another proof of why the samples are uniformly distributed using a beautiful argument from elementary geometry. Since Lemoine method uses uniform distribution on a discretized version of the problem and a limit process, his result combine with Poincaré's one could be interpreted like a convergence of a discrete uniform distribution to a continuous one.

Goodman also raised the question about the choice on how to sample this problem and how it affects the outcome of the probability. In general, the problem is mostly sample as followed (see Figure 1):

- l_1, l_2 and l_3 are the length of the three sticks;
- l_1 l_2 and l_3 are ordered in such a way that l_1 is the length of the stick on the left, l_2 is the length of the middle stick and l_3 is the length for the one on the right.

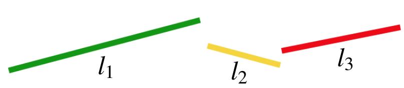

Figure 1. Sampling of the (non symmetric) spaghetti problem

Using this way, the sample $l_1 = 3/4, l_2 = 1/8, l_3 = 1/8$ is considered to be different to $l_1 = 1/8, l_2 = 3/4, l_3 = 1/8$. Although, one can consider a symmetric way to sample this problem and in this case, the order on which the three sticks appears does not matter. Let us call this problem the *Symmetric Broken Spaghetti Problem*.

In these notes, we describe the geometric shape of the sample space for the Symmetric Broken Spaghetti Problem. It happens that it is a fractal and the probability for the symmetric version is:

$$\mathbb{P} = \frac{1}{8}.$$

The interpretation of the fractal enable to give the probability to obtain a δ-equilateral triangle.

Nowadays, the Broken Spaghetti Problem is also used to introduce the notion of probability to high school students. The authors have presented this problem during the BRIS-NLAGA 2022 held in Ziguinchor/Senegal. The goal was to introduce the notion of probability to students and the notion of fractal as well.

As one can see, the symmetric version of the Broken Spaghetti Problem does note have the same probability like the original problem. This brings to mind the Bertrand paradox in which a problem have different probability depending on the way the sampling is made. Nonetheless, our goal was not to insist on the paradoxical behavior of the Broken Spaghetti Problem but rather to attach a fractal to this problem and thereby to show to students how rich this simple problem is. We end this introduction with a quote from G. S. Goodman [5] on this problem:

..."Because of this, the problem of the Broken Stick, often snubbed as a mere mathematical diversion by those who forget that probability theory had its origins in mathematical diversions, deserves to occupy a more dignified place in the hierarchy of mathematical though."

2. Geometric Approach Reviewed

In this section, we set up some materials and we recall the geometric proof of the Broken Spaghetti Problem. Without lost of generality we can assume that the stick has length 1, and it breaks randomly at two points. Let l_1, l_2 and l_3 be the (ordered) list of lengths of the three sticks. Therefore, the following equation holds:

$$l_1 + l_2 + l_3 = 1 \qquad (1).$$

So, a sample here is a vector $v := (l_1, l_2, l_3) \in \mathbb{R}^3$ such that (1) is satisfied. It follows that the set of all possibilities

$$\widetilde{\mathcal{E}} = \{(l_1, l_2, l_3) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*, l_1 + l_2 + l_3 = 1\},$$

is an equilateral triangle: the two dimensional simplex in $\mathbb{R}^3$. There is an another way to represent $\widetilde{\mathcal{E}}$ just by drawing it in $\mathbb{R}^2$. In this case, $\widetilde{\mathcal{E}}$ is an equilateral triangle in $\mathbb{R}^2$ with side-length equal to $\frac{2\sqrt{3}}{3}$ and the coordinates of a point M in $\widetilde{\mathcal{E}}$ are given by the distance between M and each of the three sides (Figure 2).

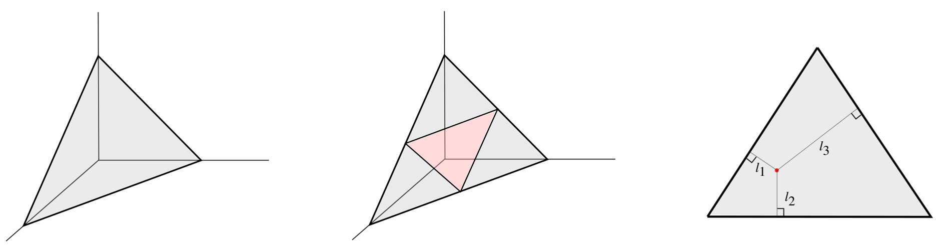

Figure 2. The sample space and its representation in $\mathbb{R}^2$

Moreover l_1, l_2 and l_3 in $\widetilde{\mathcal{E}}$ are sides of a triangle if and only if they satisfy the triangle inequalities:

$$l_2 + l_3 \leq l_1, \quad l_1 + l_3 \leq l_2, \quad l_1 + l_2 \leq l_3 \qquad (2).$$

Conditions (1) and (2) are equivalent to the following:

$$l_1 \leq \frac{1}{2}, \quad l_2 \leq \frac{1}{2}, \quad l_3 \leq \frac{1}{2}, \quad l_1 + l_2 + l_3 = 1.$$

Then, the sample that give a positive answer to the problem are:

$$\widetilde{\mathcal{T}} = \{(l_1, l_2, l_3) \in \widetilde{\mathcal{E}}; l_1 \leq \frac{1}{2},\ \ l_2 \leq \frac{1}{2},\ \ l_3 \leq \frac{1}{2}\}.$$

The set $\widetilde{\mathcal{T}}$ is also an equilateral triangle: the one that joins the middle points of the side of $\widetilde{\mathcal{E}}$.

By Poincaré [8], the fact that the sticks broke randomly at two points translates to a uniform distribution $\widetilde{\mathcal{E}}$. So, the probability $\widetilde{\mathbb{P}}$ is the amount of $\widetilde{\mathcal{T}}$ we have in $\widetilde{\mathcal{E}}$ (relative area):

$$\widetilde{\mathbb{P}} = \frac{\text{Area}(\widetilde{\mathcal{T}})}{\text{Area}(\widetilde{\mathcal{E}})} = \frac{1}{4}.$$

Now, let us turn to the Symmetric Broken Spaghetti Problem. For that, we consider samples (l_1, l_2, l_3) up to the action of the symmetric group $\mathcal{S}_3$:

$$\sigma.(l_1, l_2, l_3) = (l_{\sigma(1)}, l_{\sigma(2)}, l_{\sigma(3)});$$

where $\sigma \in \mathcal{S}_3$.

Therefore, the sample space $\mathcal{E}$ of the Symmetric Broken Spaghetti Problem is:

$$\mathcal{E} = \widetilde{\mathcal{E}}/\mathcal{S}_3.$$

3. The Geometry of the Sample Space

In this section, we describe the sample space $\mathcal{E}$. Since $\mathcal{S}_3$ is generated by (12), (13) and (23), the action of $\mathcal{S}_3$ on $\widetilde{\mathcal{E}}$ allows one to consider the points up to symmetries along the three bisectors of $\widetilde{\mathcal{E}}$. Using this, we describe $\mathcal{E}$ inductively.

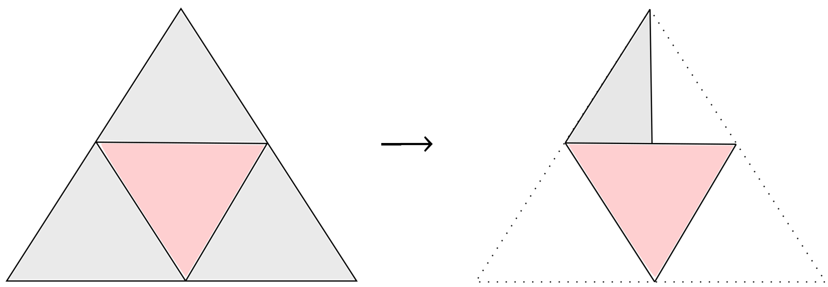

Figure 3. Sample space after one step

Step 1: Let A_1 be the triangle joining the middle points of the side of $\widetilde{\mathcal{E}}$. Then, $\widetilde{\mathcal{E}} - A_1$ is the union of three triangles T_1, T_2 and T_3 as depicted in Figure 3-left. The triangle T_i is the set of points $(l_1, l_2, l_3) \in \widetilde{\mathcal{E}}$ such that $l_i \geq 1/2$. So a point in T_i is equivalent to a point in T_j; that is we can delete two of the three triangles (let us say T_1 and T_2) from $\widetilde{\mathcal{E}}$. In T_3, a point (l_1, l_2, l_3) is equivalent to (l_2, l_1, l_3). So, we can delete half the triangle T_3. At this step, we obtain a space $\widetilde{\mathcal{E}}_1$ made with two triangles one of which is free from relations (see Figure 3-right).

Step 2: From $\widetilde{\mathcal{E}}_1$, let A_2 be the triangle joining the middle points of the sides of A_1. Then, A_1 splits into four triangles A_2, T_1, T_2 and T_3. Again by applying symmetries, we have $T_1 \sim T_2 \sim T_3$. So, one can remove two of the three triangles and also half of the last one to obtained $\widetilde{\mathcal{E}}_2$ (see Figure 4).

Step n: At this step, we divide A_{n-1} into four triangles by drawing A_n: the triangle that connects the middle points of the sides of A_{n-1}. Thefore, $A_{n-1} = A_n \cup T_1 \cup T_2 \cup T_3$ and $T_1 \sim T_2 \sim T_3$. We construct $\widetilde{\mathcal{E}}_n$ by deleting T_2 and T_3 and by removing half of T_1.

The Sample space is then given by:

$$\mathcal{E} := \varinjlim_{n} \widetilde{\mathcal{E}}_n.$$

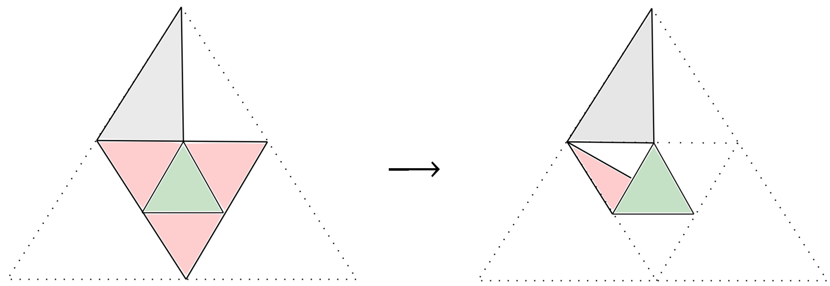

Figure 4. Sample space after two steps

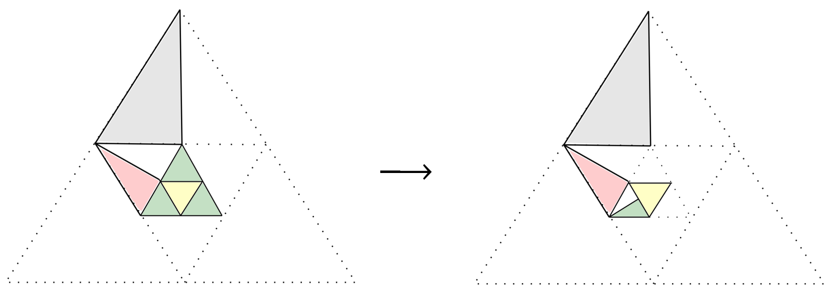

Figure 5. Sample space after three steps

Depending on the pieces one delete during the induction, we obtain different shapes of the sample space. In Figure 6, we draw two examples of the sample space.

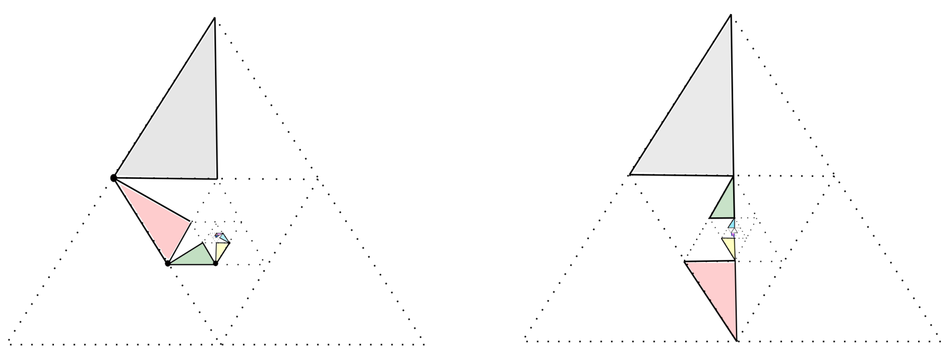

Figure 6. Sample space of the Symmetric Broken Spaghetti Problem represented in two different ways

4. Interpretation of the New Sample Space

Since the sample space changed, a natural question is to know if the probability of the Symmetric Broken Spaghetti Problem changed as well. Actually, it changes as we will see right away. The sample space $\mathcal{E}$ is the union of infinitely many triangles $(T_i)_{i=1,\ldots,\infty}$. Then,

$$\text{Area}(\mathcal{E}) = \sum_{i=1}^{\infty} \text{Area}(T_i).$$

The first triangle is one-eight of $\widetilde{\mathcal{E}}$, and each triangle T_i is also one-eight of T_{i-1}. Therefore, we have $\text{Area}(T_n) = (\frac{1}{8})^n \text{Area}(\widetilde{\mathcal{E}})$. It follows that:

$$\text{Area}(\mathcal{E}) = (\frac{1}{8} + \frac{1}{8^2} + \cdots + \frac{1}{8^n} + \ldots)\text{Area}(\widetilde{\mathcal{E}}) = \frac{1}{7}\text{Area}(\widetilde{\mathcal{E}})$$

The area of the set $\mathcal{T}$ of samples that satisfies the three triangles inequalities is:

$$\text{Area}(\mathcal{T}) = (\frac{1}{8^2} + \cdots + \frac{1}{8^n} + \dots)\text{Area}(\widetilde{\mathcal{E}}) = \frac{1}{7\times 8}\text{Area}(\widetilde{\mathcal{E}})$$

The probability to obtain a triangle for the Symmetric Broken Spaghetti Problem is now:

$$\mathbb{P} = \frac{1}{8}.$$

As one can see, $\mathcal{E}$ is a sequence of triangular pieces converging to the point

$$G := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

This agree with what one expected to have that is the probability to obtain an equilateral triangle is 0. Let us give the interpretation of triangular pieces in $\mathcal{E}$. A triangle τ given by (l_1, l_2, l_3) is *δ-equilateral* (respectively *(δ, δ')-equilateral*) if $\max\{|l_1 - l_2|, |l_1 - l_3|, |l_2 - l_3|\} \leq \delta$ (respectively $\delta \leq \max\{|l_1 - l_2|, |l_1 - l_3|, |l_2 - l_3|\} \leq \delta'$).

For each triangle T_i, set:

$$\delta_i := \sup\{\max\{|l_1 - l_2|, |l_1 - l_3|, |l_2 - l_3|\}, (l_1, l_2, l_3) \in T_i\}.$$

Then, each triangular piece T_i $(i \geq 2)$ in $\mathcal{E}$ represents the sample of points that give a (δ_{i+1}, δ_i)-equilateral triangle and the truncated sequence starting from a piece T_i represents the sample of points that give a δ_i-equilateral triangle.

So,

$$\mathbb{P}_{i+1,i} = \frac{1}{7\times 8^i}, \quad \mathbb{P}_i = \frac{1}{8^{i-1}};$$

where $\mathbb{P}_{i+1,i}$ and $\mathbb{P}_i$ are the probabilities to obtained a (δ_{i+1}, δ_i)-equilateral triangle and a δ_i-equilateral triangle, respectively.

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