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Analytical Solution of Some Systems of Nonlinear Fractional Differential Equations by the SBA Method

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Abstract

We suggest a new approach to the Some Blaise Abbo (SBA) method for solving systems of nonlinear fractional partial differential equations and we have tested it with two examples.

Keywords: system of equations, nonlinear fractional partial differential equations, SBA method, fractional integral, Mittag-Leffler function

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1. Introduction

In general, there are several numerical methods for solving systems of nonlinear fractional partial differential equations. Among these different methods we will suggest a new approach of the SBA method for solving these systems.

2. Definitions and Basic Properties

2.1 Gamma Function

2.1.1 Definition

One of the basic functions of fractional calculus is Euler's Gamma function $\Gamma(\alpha)$. It is defined by the following integral (Harrat A., 2018; Nebie, A.W., 2022; Khalouta, A.; Tellab, B., 2018; Ouedraogo, S., Abbo, B., Yaro, R., & Pare, Y.,2020; Sahadevan, R., & Bakkyaraj,T., 2015).

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt \tag{1}$$

where α is any complex number such $Re(\alpha) > 0$. The Gamma function Γ is decreasing on $[0, 1]$

2.1.2 Properties

An important property of the Gamma function $\Gamma(\alpha)$ is the following recurrence relation (Nebie, A.W., 2022; Khalouta, A.; Tellab, B., 2018).

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \ \ with \ \ \alpha > 0 \tag{2}$$

2.2 Mittag-Leffler Function

For $z \in \mathbb{C}$, the Mittag-Leffler function $E_\alpha(z)$ is defined as follows (Harrat A., 2018; Nebie, A.W., 2022; Khalouta, A.; Tellab, B., 2018).

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \ \ with \ \ \alpha > 0 \tag{3}$$

particular case

$$E_1(z) = e^z$$

This function can be generalized for two positive parameters α and β as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \tag{4}$$

3. Fractional Integral

A primitive of a continuous function on $[a; b]$ is given by the expression (Harrat A., 2018; Nebie, A.W., 2022; Khalouta, A.; Tellab, B., 2018).

$$(I_0 h)(t) = \int_0^t h(x)dx \tag{5}$$

For a primitive of order 2, we have:

$$(I_0^2 h)(t) = \int_0^t \left(\int_0^x h(s)ds\right)dx = \int_0^t (t-x)h(x)dx \tag{6}$$

If $h(t) = C$ with a constant C, then we have:

$$I_a^\alpha(C) = \frac{Ct^\alpha}{\Gamma(\alpha+1)} \tag{7}$$

4. Convergence and Uniqueness

Consider the general form of the following fractional order partial differential equation:

$$(P) : \begin{cases} \dfrac{\partial^\alpha u_i(x_1,...,x_n,t)}{\partial t^\alpha} = R(u_i(x_1,...,x_n,t)) + N(u_i(x_1,...,x_n,t)) \\ u_i(x_1,...,x_n,0) = f_i(x_1,...,x_n) \end{cases}, \quad \forall\, i = 1,...,n \tag{8}$$

with $0 < \alpha \leq 1$.

Put $L_t u_i(x_1,...,x_n,t) = \dfrac{\partial^\alpha u_i(x_1,...,x_n,t)}{\partial t^\alpha}$. Then we have

$$L_t u_i(x_1,...,x_n,t) = R(u_i(x_1,...,x_n,t)) + N(u_i(x_1,...,x_n,t)), \quad \forall\, i = 1,...,n \tag{9}$$

Applying $L_t^{-1}(.) = I_0^\alpha(.)$ the fractional integral to (9), we have:

$$\begin{aligned} u_i(x_1,...,x_n,t) = f_i(x_1,...,x_n) + I_0^\alpha(R(u_i(x_1,...,x_n,t))) + \\ I_0^\alpha(N(u_i(x_1,...,x_n,t))), \quad \forall\, i = 1,...,n \end{aligned} \tag{10}$$

Applying the method of successive approximations to (10), we have

$$\begin{aligned} u_i^k(x_1,...,x_n,t) = f_i(x_1,...,x_n) + I_0^\alpha(R(u_i^k(x_1,...,x_n,t))) + \\ I_0^\alpha(N(u_i^{k-1}(x_1,...,x_n,t))), \quad \forall\, i = 1,...,n;\ k \geq 1 \end{aligned} \tag{11}$$

From (11), we obtain the following SBA algorithm:

$$(SBA) : \begin{cases} u_{i,0}^k(x_1,...,x_n,t) = f_i(x_1,...,x_n) + I_0^\alpha(N(u_i^{k-1}(x_1,...,x_n,t))), \quad \forall\, i = 1,...,n;\ k \geq 1 \\ u_{i,n+1}^k(x_1,...,x_n,t) = I_0^\alpha(R(u_{i,n}^k(x_1,...,x_n,t))), \quad \forall\, i = 1,...,n;\ n > 0 \end{cases} \tag{12}$$

Theorem

Suppose that $\forall k \geq 1$, $N(u_i^{k-1}(x_1,...,x_n,t)) = 0$, $\left|\dfrac{M_i T^\alpha}{\Gamma(\alpha+1)}\right| < 1$, $f_i \in C(\mathbb{R}^n)$, $u_i(x_1,...,x_n,t) \in C(\Omega)$, f_i and u_i are respectively bounded by m_i and M_i such that

$\exists\, m_i = \sup\limits_{(x_1,...,x_n)\in\mathbb{R}^n} |f_i(x_1,...,x_n)|$ and $\exists\ M_i = \sup\limits_{(x_1,...,x_n,t)\in\Omega} |u_i(x_1,...,x_n,t)| > 0$ or $\Omega = \mathbb{R}^n \times [0;T]$; $\forall i = 1,...,n$.

then the SBA algorithm is convergent and the problem (P) admits a unique solution.

Proof: we have the following SBA algorithm:

$$\begin{cases} u_{i,0}^k(x_1,...,x_n,t) = f_i(x_1,...,x_n) + I_0^\alpha(N(u_i^{k-1}(x_1,...,x_n,t))), \quad \forall\, i = 1,...,n;\ k \geq 1 \\ u_{i,n+1}^k(x_1,...,x_n,t) = I_0^\alpha(R(u_{i,n}^k(x_1,...,x_n,t))), \quad \forall\, i = 1,...,n;\ n > 0 \end{cases} \tag{13}$$

or again

$$\begin{cases} u_{i,0}^k(x_1,...,x_n,t) = f_i(x_1,...,x_n), \quad \forall\, i = 1,...,n;\ k \geq 1 \\ u_{i,n+1}^k(x_1,...,x_n,t) = I_0^\alpha(R(u_{i,n}^k(x_1,...,x_n,t))), \quad \forall\, i = 1,...,n;\ n > 0 \end{cases} \tag{14}$$

$$\begin{cases} \left|u_{i,0}^k(x_1,...,x_n,t)\right| &= |f_i(x_1,...,x_n)| \le m_i;\ i=1,...,n;\ k\ge 1 \\ \left|u_{i,1}^k(x_1,...,x_n,t)\right| &= \left|I_0^\alpha(R(u_{i,0}^k(x_1,...,x_n,t)))\right| \le \frac{M_iT^\alpha}{\Gamma(\alpha+1)};\ i=1,...,n;\ k\ge 1 \\ \left|u_{i,2}^k(x_1,...,x_n,t)\right| &= \left|I_0^\alpha(R(u_{i,1}^k(x_1,...,x_n,t)))\right| \le \left(\frac{M_iT^\alpha}{\Gamma(\alpha+1)}\right)^2;\ i=1,...,n;\ k\ge 1 \\ \left|u_{i,3}^k(x_1,...,x_n,t)\right| &= \left|I_0^\alpha(R(u_{i,2}^k(x_1,...,x_n,t)))\right| \le \left(\frac{M_iT^\alpha}{\Gamma(\alpha+1)}\right)^3;\ i=1,...,n;\ k\ge 1 \\ \vdots &= \vdots \\ \left|u_{i,n}^k(x_1,...,x_n,t)\right| &= \left|I_0^\alpha(R(u_{i,n-1}^k(x_1,...,x_n,t)))\right| \le \left(\frac{M_iT^\alpha}{\Gamma(\alpha+1)}\right)^n;\ i=1,...,n;\ k\ge 1;\ \ n>0 \end{cases} \tag{15}$$

Summing member by member of (15) we get:

$$\sum_{n=0}^{+\infty}|u_{i,n}^k(x_1,...,x_n,t)| = m_i + \frac{M_iT^\alpha}{\Gamma(\alpha+1)-M_iT^\alpha};\ i=1,...,n;\ k\ge 1;\ \ n>0$$

hence $\sum_{n=0}^{+\infty}|u_{i,n}^k(x_1,...,x_n,t)|$ is absolutely convergent as a result $\sum_{n=0}^{+\infty}u_{i,n}^k(x_1,...,x_n,t)$ is simply convergent.

Uniqueness of the solution

Let $u_{i,n}^k(x_1,...,x_n,t)$, $v_{i,n}^k(x_1,...,x_n,t)$ be two solutions of (8) with $u_{i,n}^k(x_1,...,x_n,t)$ and for u and v we have the following algorithms:

$$\begin{cases} u_{i,0}^k(x_1,...,x_n,t) = f_i(x_1,...,x_n),\ \ \forall\, i=1,...,n;\ k\ge 1 \\ u_{i,n+1}^k(x_1,...,x_n,t) = I_0^\alpha(R(u_{i,n}^k(x_1,...,x_n,t))),\ \ \forall\, i=1,...,n;\ n>0 \end{cases} \tag{16}$$

and

$$\begin{cases} v_{i,0}^k(x_1,...,x_n,t) = f_i(x_1,...,x_n),\ \ \forall\, i=1,...,n;\ k\ge 1 \\ v_{i,n+1}^k(x_1,...,x_n,t) = I_0^\alpha(R(v_{i,n}^k(x_1,...,x_n,t))),\ \ \forall\, i=1,...,n;\ n>0 \end{cases} \tag{17}$$

by making the difference (16) and (17) we obtain

$$\begin{cases} u_{i,0}^k(x_1,...,x_n,t)-v_{i,0}^k(x_1,...,x_n,t) &= f_i(x_1,...,x_n)-f_i(x_1,...,x_n)=0 \\ & \Rightarrow u_{i,0}^k(x_1,...,x_n,t)=v_{i,0}^k(x_1,...,x_n,t) \\ u_{i,1}^k(x_1,...,x_n,t)-v_{i,1}^k(x_1,...,x_n,t) &= I_0^\alpha(R(u_{i,0}^k(x_1,...,x_n,t)))-I_0^\alpha(R(v_{i,0}^k(x_1,...,x_n,t)))=0 \\ & \Rightarrow u_{i,1}^k(x_1,...,x_n,t)=v_{i,1}^k(x_1,...,x_n,t) \\ u_{i,2}^k(x_1,...,x_n,t)-v_{i,2}^k(x_1,...,x_n,t) &= I_0^\alpha(R(u_{i,1}^k(x_1,...,x_n,t)))-I_0^\alpha(R(v_{i,1}^k(x_1,...,x_n,t)))=0 \\ & \Rightarrow u_{i,2}^k(x_1,...,x_n,t)=v_{i,2}^k(x_1,...,x_n,t) \\ u_{i,3}^k(x_1,...,x_n,t)-v_{i,3}^k(x_1,...,x_n,t) &= I_0^\alpha(R(u_{i,2}^k(x_1,...,x_n,t)))-I_0^\alpha(R(v_{i,2}^k(x_1,...,x_n,t)))=0 \\ & \Rightarrow u_{i,3}^k(x_1,...,x_n,t)=v_{i,3}^k(x_1,...,x_n,t) \\ \vdots &= \vdots \\ u_{i,n}^k(x_1,...,x_n,t)-v_{i,n}^k(x_1,...,x_n,t) &= I_0^\alpha(R(u_{i,n-1}^k(x_1,...,x_n,t)))-I_0^\alpha(R(v_{i,n-1}^k(x_1,...,x_n,t)))=0 \\ & \Rightarrow u_{i,n}^k(x_1,...,x_n,t)=v_{i,n}^k(x_1,...,x_n,t) \end{cases}$$

so $u_{i,n}^k(x_1,...,x_n,t)-v_{i,n}^k(x_1,...,x_n,t)=0 \Rightarrow u_{i,n}^k(x_1,...,x_n,t)=v_{i,n}^k(x_1,...,x_n,t)$; but according to the hypothesis $u_{i,n}^k(x_1,...,x_n,t)=v_{i,n}^k(x_1,...,x_n,t)$; which is contradictory, so the solution of the system is unique.

5. Application

5.1 Example 1

Consider the nonlinear diffusion-reaction PDE system of fractional order in dimension 2 with Cauchy condition.

$$\begin{cases} \frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}+\frac{\partial v(x,y,t)}{\partial x}\times\frac{\partial w(x,y,t)}{\partial y}-\frac{\partial v(x,y,t)}{\partial y}\times\frac{\partial w(x,y,t)}{\partial x}=-u(x,y,t) \\ \frac{\partial^\alpha v(x,y,t)}{\partial t^\alpha}+\frac{\partial w(x,y,t)}{\partial x}\times\frac{\partial u(x,y,t)}{\partial y}+\frac{\partial u(x,y,t)}{\partial x}\times\frac{\partial w(x,y,t)}{\partial y}=v(x,y,t) \\ \frac{\partial^\alpha w(x,y,t)}{\partial t^\alpha}+\frac{\partial u(x,y,t)}{\partial x}\times\frac{\partial v(x,y,t)}{\partial y}+\frac{\partial u(x,y,t)}{\partial y}\times\frac{\partial v(x,y,t)}{\partial x}=w(x,y,t) \\ u(x,y,0)=e^{x+y} \\ v(x,y,0)=e^{x-y} \\ w(x,y,0)=e^{-x+y} \end{cases} \tag{18}$$

with $0 < \alpha \leq 1$

Let's put it this way:

$$\begin{cases} L_t(u(x,y,t)) = \dfrac{\partial^\alpha u(x,y,t)}{\partial t^\alpha};\ \ L_t(v(x,y,t)) = \dfrac{\partial^\alpha v(x,y,t)}{\partial t^\alpha};\ \ L_t(w(x,y,t)) = \dfrac{\partial^\alpha w(x,y,t)}{\partial t^\alpha} \\ R(u(x,y,t)) = u(x,y,t);\ \ R(v(x,y,t)) = v(x,y,t);\ \ R(w(x,y,t)) = w(x,y,t) \\ N_1(v(x,y,t),w(x,y,t)) = \dfrac{\partial v(x,y,t)}{\partial x}\times\dfrac{\partial w(x,y,t)}{\partial y} - \dfrac{\partial v(x,y,t)}{\partial y}\times\dfrac{\partial w(x,y,t)}{\partial x} \\ N_2(u(x,y,t),w(x,y,t)) = \dfrac{\partial w(x,y,t)}{\partial x}\times\dfrac{\partial u(x,y,t)}{\partial y} + \dfrac{\partial u(x,y,t)}{\partial x}\times\dfrac{\partial w(x,y,t)}{\partial y} \\ N_3(u(x,y,t),v(x,y,t)) = \dfrac{\partial u(x,y,t)}{\partial x}\times\dfrac{\partial v(x,y,t)}{\partial y} + \dfrac{\partial u(x,y,t)}{\partial y}\times\dfrac{\partial v(x,y,t)}{\partial x} \end{cases}$$

we have:

$$\begin{cases} L_t(u(x,y,t)) + N_1(v(x,y,t),w(x,y,t)) = -R(u(x,y,t)) \\ L_t(v(x,y,t)) + N_2(u(x,y,t),w(x,y,t)) = R(v(x,y,t)) \\ L_t(w(x,y,t)) + N_3(u(x,y,t),v(x,y,t)) = R(w(x,y,t)) \end{cases} \quad (19)$$

Let's apply $L_t^{-1} = I_0^\alpha(.)$ a $L_t(u(x,y,t))$; $L_t(v(x,y,t))$ and $L_t(w(x,y,t))$ of (19), we get:

$$\begin{cases} u(x,y,t) = e^{x+y} - I_0^\alpha(N_1(v(x,y,t),w(x,y,t))) - I_0^\alpha(R(u(x,y,t))) \\ v(x,y,t) = e^{x-y} - I_0^\alpha(N_2(u(x,y,t),w(x,y,t))) + I_0^\alpha(R(v(x,y,t))) \\ w(x,y,t) = e^{-x+y} - I_0^\alpha(N_3(u(x,y,t),v(x,y,t))) + I_0^\alpha(R(w(x,y,t))) \end{cases} \quad (20)$$

Applying the method of successive approximations to (20), we obtain:

$$\begin{cases} u^k(x,y,t) = e^{x+y} - I_0^\alpha(N_1(v^{k-1}(x,y,t),w^{k-1}(x,y,t))) - I_0^\alpha(R(u^k(x,y,t))), k \geq 1 \\ v^k(x,y,t) = e^{x-y} - I_0^\alpha(N_2(u^{k-1}(x,y,t),w^{k-1}(x,y,t))) + I_0^\alpha(R(v^k(x,y,t))), k \geq 1 \\ w^k(x,y,t) = e^{-x+y} - I_0^\alpha(N_3(u^{k-1}(x,y,t),v^{k-1}(x,y,t))) + I_0^\alpha(R(w^k(x,y,t))), k \geq 1 \end{cases} \quad (21)$$

From (21), we obtain the following SBA algorithm (Bassono, F., 2013; Zongo,G., So, O., & Pare, Y., 2016; Pare, Y., 2010; Pare, Y., Bassono, F. & Some, B., 2012; Pare, Y. (2021).).

$$\begin{cases} \begin{cases} u_0^k(x,y,t) = e^{x+y} - I_0^\alpha(N_1(v^{k-1}(x,y,t),w^{k-1}(x,y,t))),\ \ k \geq 1 \\ u_{n+1}^k(x,y,t) = -I_0^\alpha(R(u_n^k(x,y,t))),\ \ n \geq 0 \end{cases} \\ \begin{cases} v_0^k(x,y,t) = e^{x-y} - I_0^\alpha(N_2(u^{k-1}(x,y,t),w^{k-1}(x,y,t))),\ \ k \geq 1 \\ v_{n+1}^k(x,y,t) = I_0^\alpha(R(v_n^k(x,y,t))),\ \ n \geq 0 \end{cases} \\ \begin{cases} w_0^k(x,y,t) = e^{-x+y} - I_0^\alpha(N_3(u^{k-1}(x,y,t),v^{k-1}(x,y,t))),\ \ k \geq 1 \\ w_{n+1}^k(x,y,t) = I_0^\alpha(R(w_n^k(x,y,t))),\ \ n \geq 0 \end{cases} \end{cases} \quad (22)$$

At step $k = 1$, we apply Picard's principle, we take u^0; v^0 and w^0 such that

$N_1(v^0,w^0) = N_2(u^0,w^0) = N_3(u^0,v^0) = 0$ then $u^0 = v^0 = w^0 = 0$.

The above algorithm becomes:

$$\begin{cases} \begin{cases} u_0^1(x,y,t) = e^{x+y} \\ u_{n+1}^1(x,y,t) = -I_0^\alpha(R(u_n^1(x,y,t))),\ \ n \geq 0 \end{cases} \\ \begin{cases} v_0^1(x,y,t) = e^{x-y} \\ v_{n+1}^1(x,y,t) = I_0^\alpha(R(v_n^1(x,y,t))),\ \ n \geq 0 \end{cases} \\ \begin{cases} w_0^1(x,y,t) = e^{-x+y} \\ w_{n+1}^1(x,y,t) = I_0^\alpha(R(w_n^1(x,y,t))),\ \ n \geq 0 \end{cases} \end{cases} \quad (23)$$

let's calculate $u^1(x,y,t)$, $v^1(x,y,t)$ and $w^1(x,y,t)$

for $n = 0$, we have

$$u_1^1(x,y,t) = -I_0^\alpha(R(u_0^1(x,y,t))) = \frac{-t^\alpha}{\Gamma(\alpha+1)}e^{x+y}$$

$$v_1^1(x,y,t) = I_0^\alpha(R(v_0^1(x,y,t))) = \frac{t^\alpha}{\Gamma(\alpha+1)}e^{x-y}$$

$$w_1^1(x,y,t) = I_0^\alpha(R(w_0^1(x,y,t))) = \frac{t^\alpha}{\Gamma(\alpha+1)}e^{-x+y}$$

for $n = 1$, we have

$$u_2^1(x,y,t) = -I_0^\alpha(R(u_1^1(x,y,t))) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{x+y}$$

$$v_2^1(x,y,t) = I_0^\alpha(R(v_1^1(x,y,t))) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{x-y}$$

$$w_2^1(x,y,t) = I_0^\alpha(R(w_1^1(x,y,t))) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{-x+y}$$

for $n = 2$, we have

$$u_3^1(x,y,t) = -I_0^\alpha(R(u_2^1(x,y,t))) = \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)}e^{x+y}$$

$$v_3^1(x,y,t) = I_0^\alpha(R(v_2^1(x,y,t))) = \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}e^{x-y}$$

$$w_3^1(x,y,t) = I_0^\alpha(R(w_2^1(x,y,t))) = \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}e^{-x+y}$$

recurrently we have:

$$\left\{\begin{array}{lcl} u_0^1(x,y,t) &=& e^{x+y} \\ u_1^1(x,y,t) &=& \frac{-t^\alpha}{\Gamma(\alpha+1)}e^{x+y} \\ u_2^1(x,y,t) &=& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{x+y} \\ u_3^1(x,y,t) &=& \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)}e^{x+y} \\ \vdots &=& \vdots \\ u_n^1(x,y,t) &=& \frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)}e^{x+y} \end{array}\right. ; \left\{\begin{array}{lcl} v_0^1(x,y,t) &=& e^{x-y} \\ v_1^1(x,y,t) &=& \frac{t^\alpha}{\Gamma(\alpha+1)}e^{x-y} \\ v_2^1(x,y,t) &=& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{x-y} \\ v_3^1(x,y,t) &=& \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}e^{x-y} \\ \vdots &=& \vdots \\ v_n^1(x,y,t) &=& \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}e^{x-y} \end{array}\right. ; \left\{\begin{array}{lcl} w_0^1(x,y,t) &=& e^{-x+y} \\ w_1^1(x,y,t) &=& \frac{t^\alpha e^{-x+y}}{\Gamma(\alpha+1)} \\ w_2^1(x,y,t) &=& \frac{t^{2\alpha} e^{-x+y}}{\Gamma(2\alpha+1)} \\ w_3^1(x,y,t) &=& \frac{t^{3\alpha} e^{-x+y}}{\Gamma(3\alpha+1)} \\ \vdots &=& \vdots \\ w_n^1(x,y,t) &=& \frac{t^{n\alpha} e^{-x+y}}{\Gamma(n\alpha+1)} \end{array}\right.$$

The solution at step $k = 1$ is:

$$\left\{\begin{array}{lcl} u^1(x,y,t) &=& e^{x+y}\sum_{n=0}^{+\infty}\frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)} = e^{x+y}.E_\alpha(-t^\alpha) \\ v^1(x,y,t) &=& e^{x-y}\sum_{n=0}^{+\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = e^{x-y}.E_\alpha(t^\alpha) \\ w^1(x,y,t) &=& e^{-x+y}\sum_{n=0}^{+\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = e^{-x+y}.E_\alpha(t^\alpha) \end{array}\right.$$

or $E_\alpha(t^\alpha)$ is the Mittag-Leffler function

At step $k = 2$, let us calculate $N_1(v^1(x,y,t), w^1(x,y,t)); N_2(u^1(x,y,t), w^1(x,y,t))$ and $N_3(u^1(x,y,t), v^1(x,y,t))$

$$\begin{array}{lcl} N_1(v^1(x,y,t), w^1(x,y,t)) &=& \frac{\partial v^1(x,y,t)}{\partial x}\times\frac{\partial w^1(x,y,t)}{\partial y} - \frac{\partial v^1(x,y,t)}{\partial y}\times\frac{\partial w^1(x,y,t)}{\partial x} \\ &=& e^{x-y}.E_\alpha(t^\alpha)\times e^{-x+y}.E_\alpha(t^\alpha) - e^{x-y}.E_\alpha(t^\alpha)\times e^{-x+y}.E_\alpha(t^\alpha) \\ &=& e^{x-y-x+y}.(E_\alpha(t^\alpha))^2 - e^{x-y-x+y}.(E_\alpha(t^\alpha))^2 \\ &=& (E_\alpha(t^\alpha))^2 - (E_\alpha(t^\alpha))^2 \\ &=& 0 \end{array}$$

$$\begin{aligned}
N_2(u^1(x,y,t),w^1(x,y,t)) &= \frac{\partial u^1(x,y,t)}{\partial x}\times\frac{\partial w^1(x,y,t)}{\partial y}+\frac{\partial u^1(x,y,t)}{\partial y}\times\frac{\partial w^1(x,y,t)}{\partial x}\\
&= e^{-x+y}.E_\alpha(-t^\alpha)\times e^{x+y}.E_\alpha(t^\alpha)-e^{x+y}.E_\alpha((-t)^\alpha)\times e^{-x+y}.E_\alpha(t^\alpha)\\
&= e^{-x+y+x+y}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)-e^{x+y-x+y}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)\\
&= e^{2y}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)-e^{2y}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
N_3(u^1(x,y,t),v^1(x,y,t)) &= \frac{\partial u^1(x,y,t)}{\partial x}\times\frac{\partial v^1(x,y,t)}{\partial y}+\frac{\partial u^1(x,y,t)}{\partial y}\times\frac{\partial v^1(x,y,t)}{\partial x}\\
&= -e^{x+y}.E_\alpha(-t^\alpha)\times e^{x-y}.E_\alpha(t^\alpha)+e^{x+y}.E_\alpha(-t^\alpha)\times e^{x-y}.E_\alpha(t^\alpha)\\
&= -e^{x+y+x-y}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)+e^{x+y+x-y}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)\\
&= -e^{2x}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)+e^{2x}.E_\alpha(t^\alpha).E_\alpha(-t^\alpha)\\
&= 0
\end{aligned}$$

so the algorithm at step $k=2$ is the same as at step $k=1$.

from where

$$\begin{cases}
u^2(x,y,t) &= u^1(x,y,t)=e^{x+y}.E_\alpha(-t^\alpha)\\
v^2(x,y,t) &= v^1(x,y,t)=e^{x-y}.E_\alpha(t^\alpha)\\
w^2(x,y,t) &= w^1(x,y,t)=e^{-x+y}.E_\alpha(t^\alpha)
\end{cases}$$

recursively we have:

$$\begin{cases}
u^k(x,y,t) &= e^{x+y}.E_\alpha(-t^\alpha)\\
v^k(x,y,t) &= e^{x-y}.E_\alpha(t^\alpha) \quad ; \quad t\geq 1\\
w^k(x,y,t) &= e^{-x+y}.E_\alpha(t^\alpha)
\end{cases}$$

$$\begin{cases}
u(x,y,t) &= \lim_{k\to+\infty} u^k(x,y,t)=e^{x+y}.E_\alpha(-t^\alpha)\\
v(x,y,t) &= \lim_{k\to+\infty} v^k(x,y,t)=e^{x-y}.E_\alpha(t^\alpha)\\
w(x,y,t) &= \lim_{k\to+\infty} w^k(x,y,t)=e^{-x+y}.E_\alpha(t^\alpha)
\end{cases}$$

The exact solution of the system (18) for $\alpha=1$ is:

$$\begin{cases}
u(x,y,t) &= e^{x+y-t}\\
v(x,y,t) &= e^{x-y+t}\\
w(x,y,t) &= e^{-x+y+t}
\end{cases}$$

5.2 Example 2

Consider the simplified fractional order equation of the Navier-Stokes equation in dimension 2:

$$\begin{cases}
\dfrac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}+u(x,y,t)\dfrac{\partial u(x,y,t)}{\partial x}=-v(x,y,t)\dfrac{\partial u(x,y,t)}{\partial y}+\dfrac{\eta}{\rho}\left(\dfrac{\partial^2 u(x,y,t)}{\partial x^2}+\dfrac{\partial^2 u(x,y,t)}{\partial y^2}\right)\\
\dfrac{\partial^\alpha v(x,y,t)}{\partial t^\alpha}+u(x,y,t)\dfrac{\partial v(x,y,t)}{\partial x}=-v(x,y,t)\dfrac{\partial v(x,y,t)}{\partial y}+\dfrac{\eta}{\rho}\left(\dfrac{\partial^2 v(x,y,t)}{\partial x^2}+\dfrac{\partial^2 v(x,y,t)}{\partial y^2}\right)\\
\dfrac{\partial u(x,y,t)}{\partial x}+\dfrac{\partial v(x,y,t)}{\partial y}=0\\
u(x,y,0)=-e^{x+y}\\
v(x,y,0)=e^{x+y}
\end{cases} \tag{24}$$

With t the time variable; η dynamic viscosity; ρ the density; $\mu=\dfrac{\eta}{\rho}$ the viscosity of the kinematics. In our example we will use Cartesian coordinates (x,y) and assume that the liquid is at rest, i.e. that the pressure is zero for the numerical resolution and $0<\alpha\leq 1$.

Let us pose:

$$\begin{cases} L_t u(x,y,t) = \dfrac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}; \\ L_t v(x,y,t) = \dfrac{\partial^\alpha v(x,y,t)}{\partial t^\alpha} \\ Ru(x,y,t) = \dfrac{\eta}{\rho}\left(\dfrac{\partial^2 u(x,y,t)}{\partial x^2} + \dfrac{\partial^2 u(x,y,t)}{\partial y^2}\right) \\ Rv(x,y,t) = \dfrac{\eta}{\rho}\left(\dfrac{\partial^2 v(x,y,t)}{\partial x^2} + \dfrac{\partial^2 v(x,y,t)}{\partial y^2}\right) \\ N_1(u(x,y,t),v(x,y,t)) = u(x,y,t)\dfrac{\partial u(x,y,t)}{\partial x} + v(x,y,t)\dfrac{\partial u(x,y,t)}{\partial y} \\ N_2(u(x,y,t),v(x,y,t)) = u(x,y,t)\dfrac{\partial v(x,y,t)}{\partial x} + v(x,y,t)\dfrac{\partial v(x,y,t)}{\partial y} \end{cases}$$

we have:

$$\begin{cases} L_t u(x,y,t) + N_1(u(x,y,t),v(x,y,t)) = Ru(x,y,t) \\ L_t v(x,y,t) + N_2(u(x,y,t),v(x,y,t)) = Rv(x,y,t) \end{cases} \tag{25}$$

Applying $L_t^{-1} = I_0{}^\alpha(.)$ to $L_t u(x,y,t)$ and $L_t v(x,y,t)$ to (25); we get

$$\begin{cases} u(x,y,t) = u(x,y,0) + I_0^\alpha(R(u(x,y,t))) - I_0^\alpha(N_1(u(x,y,t),v(x,y,t))) \\ v(x,y,t) = v(x,y,0) + I_0^\alpha(R(v(x,y,t))) - I_0^\alpha(N_2(u(x,y,t),v(x,y,t))) \end{cases} \tag{26}$$

Let's apply the method of successive approximations to (26)

$$\begin{cases} u^k(x,y,t) = -e^{x+y} + I_0^\alpha(R(u^k(x,y,t))) - I_0^\alpha(N_1(u^{k-1}(x,y,t),v^{k-1}(x,y,t))); k \geq 1 \\ v^k(x,y,t) = e^{x+y} + I_0^\alpha(R(v^k(x,y,t))) - I_0^\alpha(N_2(u^{k-1}(x,y,t),v^{k-1}(x,y,t))); k \geq 1 \end{cases} \tag{27}$$

From (27), we obtain the following SBA algorithm (Bassono, F., 2013; Zongo,G., So, O., & Pare, Y., 2016; Pare, Y., 2010; Pare, Y., Bassono, F. & Some, B., 2012; Pare, Y. (2021).).

$$\begin{cases} \begin{cases} u_0^k(x,y,t) = -e^{x+y} - I_0^\alpha(N_1(u^{k-1}(x,y,t),v^{k-1}(x,y,t))); k \geq 1 \\ u_{n+1}^k(x,y,t) = I_0^\alpha(R(u_n^k(x,y,t))); n \geq 0 \end{cases} \\ \begin{cases} v_0^k(x,y,t) = e^{x+y} - I_0^\alpha(N_2((u^{k-1}(x,y,t),v^{k-1}(x,y,t))); k \geq 1 \\ v_{n+1}^k(x,y,t) = I_0^\alpha(R(v_n^k(x,y,t))); n \geq 0 \end{cases} \end{cases} \tag{28}$$

At step k=1, we apply Picard's principle, we choose u^0 and v^0 such that

$N_1(u^0,v^0) = N_1(u^0,v^0) = 0$ so we take $u^0 = v^0 = 0$, the above algorithm becomes:

$$\begin{cases} \begin{cases} u_0^1(x,y,t) = -e^{x+y} \\ u_{n+1}^1(x,y,t) = I_0^\alpha(R(u_n^1(x,y,t))); n \geq 0 \end{cases} \\ \begin{cases} v_0^1(x,y,t) = e^{x+y} \\ v_{n+1}^1(x,y,t) = I_0^\alpha(R(v_n^1(x,y,t))); n \geq 0 \end{cases} \end{cases} \tag{29}$$

For $n = 0$, we get:

$$u_1^1(x,y,t) = I_0^\alpha\left(\frac{\eta}{\rho}\left(\frac{\partial^2 u_0^1(x,y,t)}{\partial x^2} + \frac{\partial^2 u_0^1(x,y,t)}{\partial y^2}\right)\right) = \frac{-2\eta}{\rho}\frac{t^\alpha}{\Gamma(\alpha+1)}e^{x+y}$$

$$v_1^1(x,y,t) = I_0^\alpha\left(\frac{\eta}{\rho}\left(\frac{\partial^2 v_0^1(x,y,t)}{\partial x^2} + \frac{\partial^2 v_0^1(x,y,t)}{\partial y^2}\right)\right) = \frac{2\eta}{\rho}\frac{t^\alpha}{\Gamma(\alpha+1)}e^{x+y}$$

For $n = 1$, we get:

$$u_2^1(x,y,t) = I_0^\alpha\left(\frac{\eta}{\rho}\left(\frac{\partial^2 u_1^1(x,y,t)}{\partial x^2} + \frac{\partial^2 u_1^1(x,y,t)}{\partial y^2}\right)\right) = \frac{-\left(\frac{2\eta}{\rho}t^\alpha\right)^2}{\Gamma(2\alpha+1)}e^{x+y}$$

$$v_2^1(x,y,t) = I_0^\alpha\left(\frac{\eta}{\rho}\left(\frac{\partial^2 v_1^1(x,y,t)}{\partial x^2} + \frac{\partial^2 v_1^1(x,y,t)}{\partial y^2}\right)\right) = \frac{\left(\frac{2\eta}{\rho}t^\alpha\right)^2}{\Gamma(2\alpha+1)}e^{x+y}$$

For $n = 2$, we get:

$$u_3^1(x,y,t) = I_0^\alpha\left(\frac{\eta}{\rho}\left(\frac{\partial^2 u_2^1(x,y,t)}{\partial x^2} + \frac{\partial^2 u_2^1(x,y,t)}{\partial y^2}\right)\right) = \frac{-\left(\frac{2\eta}{\rho}t^\alpha\right)^3}{\Gamma(3\alpha+1)}e^{x+y}$$

$$v_3^1(x,y,t) = I_0^\alpha\left(\frac{\eta}{\rho}\left(\frac{\partial^2 v_2^1(x,y,t)}{\partial x^2} + \frac{\partial^2 v_2^1(x,y,t)}{\partial y^2}\right)\right) = \frac{\left(\frac{2\eta}{\rho}t^\alpha\right)^3}{\Gamma(3\alpha+1)}e^{x+y}$$

Recursively, we have:

$$\left\{\begin{array}{lcl} u_0^1(x,y,t) &=& -e^{x+y} \\ u_1^1(x,y,t) &=& \frac{-\left(\frac{2\eta}{\rho}t^\alpha\right)}{\Gamma(\alpha+1)}e^{x+y} \\ u_2^1(x,y,t) &=& \frac{-\left(\frac{2\eta}{\rho}t^\alpha\right)^2}{\Gamma(2\alpha+1)}e^{x+y} \\ u_3^1(x,y,t) &=& \frac{-\left(\frac{2\eta}{\rho}t^\alpha\right)^3}{\Gamma(3\alpha+1)}e^{x+y} \\ \vdots &=& \vdots \\ u_n^1(x,y,t) &=& \frac{-\left(\frac{2\eta}{\rho}t^\alpha\right)^n}{\Gamma(n\alpha+1)}e^{x+y} \end{array}\right. ; \quad \left\{\begin{array}{lcl} v_0^1(x,y,t) &=& e^{x+y} \\ v_1^1(x,y,t) &=& \frac{\left(\frac{2\eta}{\rho}t^\alpha\right)}{\Gamma(\alpha+1)}e^{x+y} \\ v_2^1(x,y,t) &=& \frac{\left(\frac{2\eta}{\rho}t^\alpha\right)^2}{\Gamma(2\alpha+1)}e^{x+y} \\ v_3^1(x,y,t) &=& \frac{\left(\frac{2\eta}{\rho}t^\alpha\right)^3}{\Gamma(3\alpha+1)}e^{x+y} \\ \vdots &=& \vdots \\ v_n^1(x,y,t) &=& \frac{\left(\frac{2\eta}{\rho}t^\alpha\right)^n}{\Gamma(n\alpha+1)}e^{x+y} \end{array}\right.$$

the solution at step $k = 1$ is:

$$\left\{\begin{array}{l} u^1(x,y,t) = -e^{x+y}\sum_{n=0}^{+\infty}\frac{\left(\frac{2\eta}{\rho}t^\alpha\right)^n}{\Gamma(n\alpha+1)} = -e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \\ v^1(x,y,t) = e^{x+y}\sum_{n=0}^{+\infty}\frac{\left(\frac{2\eta}{\rho}t^\alpha\right)^n}{\Gamma(n\alpha+1)} = e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \end{array}\right.$$

with $E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)$ the Mittag-Leffler function At step $k = 2$, we have:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} u_0^2(x,y,t) = -e^{x+y} - I_0^\alpha(N_1(u^1(x,y,t), v^1(x,y,t))); k \geq 1 \\ u_{n+1}^2(x,y,t) = I_0^\alpha(R(u_n^2(x,y,t))); n \geq 0 \end{array}\right. \\ \left\{\begin{array}{l} v_0^2(x,y,t) = e^{x+y} - I_0^\alpha(N_2((u^1(x,y,t), v^1(x,y,t))); k \geq 1 \\ v_{n+1}^2(x,y,t) = I_0^\alpha(R(v_n^2(x,y,t))); n \geq 0 \end{array}\right. \end{array}\right. \tag{30}$$

let's calculate $N_1(u^1(x,y,t), v^1(x,y,t))$ et $N_2(u^1(x,y,t), v^1(x,y,t))$

$$\begin{aligned} N_1(u^1(x,y,t), v^1(x,y,t)) &= u^1(x,y,t)\frac{\partial u^1(x,y,t)}{\partial x} + v^1(x,y,t)\frac{\partial u^1(x,y,t)}{\partial y} \\ &= -e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\left(-e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right) + \\ &\quad e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\left(-e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right) \\ &= \left[e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right]^2 - \left[e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right]^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned}
N_2(u^1(x,y,t), v^1(x,y,t)) &= u^1(x,y,t)\frac{\partial v^1(x,y,t)}{\partial x} + v^1(x,y,t)\frac{\partial v^1(x,y,t)}{\partial y} \\
&= -e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\left(e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right) + \\
&\quad e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\left(e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right) \\
&= -\left[e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right]^2 + \left[e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right)\right]^2 \\
&= 0
\end{aligned}$$

so the algorithm at step $k = 2$ is the same as at step $k = 1$

hence

$$\begin{cases} u^2(x,y,t) = u^1(x,y,t) = -e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \\ v^2(x,y,t) = v^1(x,y,t) = e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \end{cases}$$

recursively we have:

$$\begin{cases} u^k(x,y,t) = -e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \\ v^k(x,y,t) = e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \end{cases} ; t \geq 1$$

$$\begin{cases} u(x,y,t) = \lim_{k\to+\infty} u^k(x,y,t) = -e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \\ v(x,y,t) = \lim_{k\to+\infty} v^k(x,y,t) = e^{x+y}E_\alpha\left(\frac{2\eta}{\rho}t^\alpha\right) \end{cases}$$

The exact solution of the system (24) for $\alpha = 1$ is

$$\begin{cases} u(x,y,t) = -e^{x+y+\frac{2\eta t}{\rho}} \\ v(x,y,t) = e^{x+y+\frac{2\eta t}{\rho}} \end{cases}$$

6. Conclusion

In this work, we have proposed a new approach of the Some Blaise Abbo method (SBA) for systems of nonlinear partial differential equations of fractional order. then we proved the convergence of the algorithm and the uniqueness of the solution. Finally, we successfully used this new approach to solve two examples.

References

Bassono, F. (2013). Rsolution numrique compare de quelques quations fonctionnelles par la mthode SBA, dcompositionnelle dAdomian et perturbation, Thse de Doctorat unique, Universit de Ouagadougou.

Harrat, A. (2018). Systmes dvolution abstraits dordres fractionnaires et leurs applications (Doctoral dissertation).

Khalouta, A. (2019). Rsolution des quations aux drives partielles linaires et non-linaires moyennant des approches analytiques: extension aux cas dEDP dordre fractionnaire (Doctoral dissertation).

Nebie, A. W. (2022). Rsolution de quelques quations fonctionnelles dordre fractionnaire au sens de Caputo par la mthode SBA et par la mthode HPM, Thse de Doctorat unique, Universit Joseph Ki-Zerbo, Soutenue le 11 mars 2022, UFR SEA.

Ouedraogo, S., Abbo, B., Yaro, R., & Pare, Y. (2020). Convergence of a new approach of Adomians method for solving volterra fractional integral of second kind, *127*(2), 107-120. https://doi.org/10.17654/MS127020107

Pare, Y. (2010). Resolution de quelques equations fonctionnelles par la numerique SBA (Some Blaise-Abbo), Thse de Doctorat unique. Universit de Ouagadougou.

Pare, Y., Bassono, F., & Some, B. (2012). A new technique for numerical resolution of non linear integral equations of Fredholm by SBA method, Far East. *J. Appl. Math., 70*(1).

Pare, Y. (2021). Equations intgrales et intgro-diffrentielles, Generis publishing.

Sahadevan, R., & Bakkyaraj, T. (2015). Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. *Fractional Calculus and Applied Analysis, 18*(1), 146-162. https://doi.org/10.1515/fca-2015-0010

Tellab, B., & Haouam, K. (2018). Rsolution des quations diffrentielles fractionnaires (Doctoral dissertation).

Zongo,G., So, O., & Pare, Y. A. (2016). comparaison of Adomian Method and SBA method on the Nonlinear Schrodingers equation, October 8.