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An Analytic Form for Riemann Zeta Function at Integer Values

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Abstract

An original definition of the generalized Euler-Mascheroni constants allowed us to prove that their infinite sum converges to the number $(1 - Ln2)$. By considering this number is the Lebesgue measure of a set defined as the difference between the area of the square unit and the area under the curve $y = 1/x \ 1 \leq x \leq 2$; we introduced a partition of this set such that each Lebesgue measure of the subsets can be related to values of Riemann zeta function at integers. From this relationship, we proved that the Lambert W function can produce all $\zeta(s)$ values whatever is the parity of s. Finally, by considering that $\zeta(s)$ values allow calculation of the probability, for s integers chosen in an interval $[1, n]\ n \in \mathbb{N}$, to be coprime; we proved that Lambert W function can describe prime numbers distribution.

Keywords: zeta function, odd integers, Euler-Mascheroni constant, Apéry's constant, Lambert W function, prime numbers

1. Introduction

In mathematics, some problems which look trivial remain unsolved for many centuries even if the greatest genius spent a lot of efforts trying to solve them. Among these problems, values of Riemann zeta function at odd integers i.e., $\{\zeta(2k + 1); k = 1,2, \dots\}$ is certainly one of the greatest as reported by many authors (van der Poorten, 1979; Srivastava, 1999; Lagarias, 2013; Delplace, 2019).

As an illustration, it is quite easy to calculate the integral $\int_1^{+\infty} \frac{1}{x^3} dx$ giving the rational value $1/2$, but calculation of the series: $\zeta(3) = \sum_{k=1}^{+\infty} \frac{1}{k^3}$ has been, until now, not achieved. Using a geometrical interpretation, as proposed in the following Figure 1, we show, it is much easier to calculate the area under the quite complex shape curve of function $f(x) = 1/x^3 \ x \geq 1$, rather than calculating the sum of all rectangular areas corresponding to an upper Darboux sum.

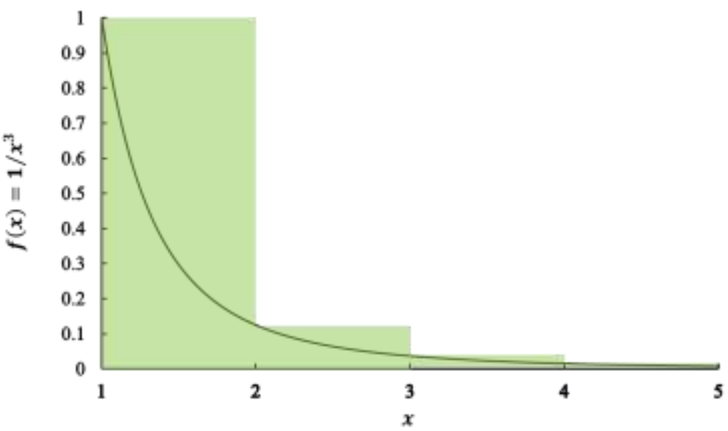

Figure 1. The curve of function $f(x) = 1/x^3 \ 1 \leq x \leq 5$ and rectangles of upper Darboux sum

From the French mathematician Roger Apéry's work (Apéry, 1979), we know that this series, corresponding to $\zeta(3)$, and often called Apéry's constant, is an irrational number which is always considered an amazing result (van der Poorten, 1979). But for other odd values, i.e. $\zeta(5), \zeta(7), \dots$ we have no idea of the deep nature of these numbers.

For even powers, the problem was solved by Léonard Euler around 1735 (Ayoub, 1974), starting with famous Basel problem and giving the incredible result $\zeta(2) = \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. The genius Euler established then the general closed form $\zeta(2k) = |B_{2K}| \frac{(2\pi)^{2k}}{2(2k)!}$ with B_{2k} the Bernoulli numbers. It allows to obtain results such as: $\zeta(4) = \frac{\pi^4}{90}$; $\zeta(6) = \frac{\pi^6}{945}$; $\zeta(8) = \frac{\pi^8}{9450}$; ...

In the same way of thinking, i.e., calculation of the upper Darboux sum and the area under the curve by integration, Euler introduced the famous Euler-Mascheroni constant (Lagarias, 2013) as:

$$\gamma = \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x} dx\right) = \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - Ln(n)\right) \quad (1)$$

As shown in Figure 1, the function f is, in that case, $f(x) = 1/x$ and upper Darboux sum corresponds to the harmonic series. In that amazing case, both the series and the integral diverge; but their difference has an asymptotic behavior giving famous γ value:

$\gamma = 0.577\ 215\ 664\ 901\ 532\ 860\ 606\ 512\ 090\ 082\ ...$

In the next section, the often-called generalized Euler-Mascheroni constants will be introduced in order to study more deeply the relationship between γ and $\zeta(2k+1)$ values. Then, by considering a set of Lebesgue measure value $(1 - Ln2)$, we will introduce a partition of this set able to recover all $\zeta(s)$ values at integers whatever their parity is. Finally, in the discussion, we will consider a possible closed form for $\zeta(s)$ values and the consequences in number theory for prime numbers repartition.

2. Material Studied, Generalized Euler-Mascheroni Constant and $\zeta(2k+1)$ Values

A generalized Euler-Mascheroni constant can be defined as follows (Euler's constant - Wikipedia):

Definition 1:

$$\forall s \geq 1;\ s \in \mathbb{N} \quad \gamma_s = \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k^s} - \int_1^n \frac{1}{x^s} dx\right) \quad (2)$$

From a geometrical point of view, any γ_s value is a measurement of the area of light green zone in the following Figure 2.

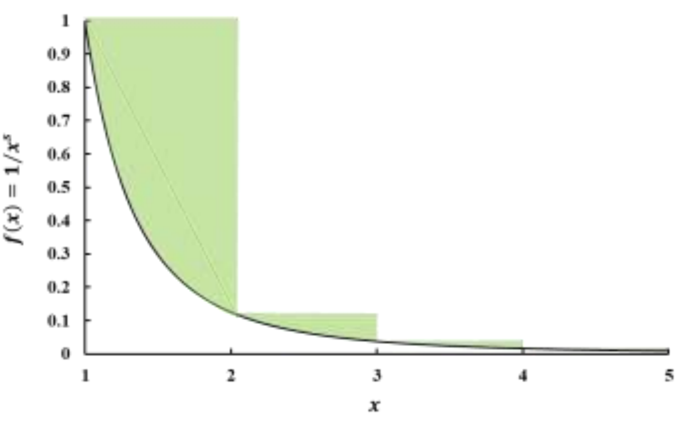

Figure 2. Geometrical meaning of γ_s values i.e., the area of light green zones

The case $s = 1$ corresponds to Euler-Mascheroni constant as reported in the introduction:

$$\gamma_1 = \gamma = \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x} dx\right)$$

For $s > 1$, both the series and the integral converge, and we have: $\gamma_s = \zeta(s) - \frac{1}{s-1}$. For each value of s, $\zeta(s)$ is a measurement of the total area of upper Darboux sum rectangles and $\frac{1}{s-1}$ is a measurement of the area under the curve of equation $y = 1/x^s$ (according to Figure 2). We have, of course, $\lim_{s\to+\infty} \frac{1}{s-1} = 0$ following the evolution of $y = 1/x^s$ curve shape as s increases and $\lim_{s\to+\infty} \zeta(s) = 1$ in agreement with the first square area in Figure 2 $(1 \leq x \leq 2)$.

Because of these trivial considerations, we have the following Theorem 1.

Theorem 1:

$$\lim_{n\to+\infty} \sum_{s=1}^{n} \gamma_s = +\infty$$

In order to avoid divergence of the above series, we propose to define the generalized Euler-Mascheroni constants as followed.

Definition 2:

$$\forall s \geq 1;\ s \in \mathbb{N} \quad \gamma_s^* = \lim_{n\to+\infty} \left(\sum_{k=2}^{n} \frac{1}{k^s} - \int_2^n \frac{1}{x^s} dx \right) \tag{3}$$

This definition means we suppress the first light green surface in Figure 2.

It allows the following theorem (trivial) to be written.

Theorem 2:

$$\gamma_1^* = \gamma + Ln2 - 1 \tag{4}$$

Proof:

From Definition 2, we have:

$$\gamma_1^* = \lim_{n\to+\infty} \left(\sum_{k=2}^{n} \frac{1}{k} - \int_2^n \frac{1}{x} dx \right)$$

$$\Rightarrow \gamma_1^* = \lim_{n\to+\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k} - 1 \right) - \left(\int_1^n \frac{1}{x} dx - \int_1^2 \frac{1}{x} dx \right) \right]$$

$$\Rightarrow \gamma_1^* = \lim_{n\to+\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x} dx \right) + \int_1^2 \frac{1}{x} dx - 1 \right]$$

$$\Rightarrow \gamma_1^* = \int_1^2 \frac{1}{x} dx - 1 + \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x} dx \right)$$

$$\Rightarrow \gamma_1^* = Ln2 - 1 + \gamma$$

For $s > 1$, we can propose the following theorem.

Theorem 3:

$$\forall s > 1;\ s \in \mathbb{N} \quad \gamma_s^* = (\zeta(s) - 1) - \frac{1}{s-1} 2^{1-s} \tag{5}$$

Proof:

From Definition 2, we have:

$$\gamma_s^* = \lim_{n\to+\infty}\left(\sum_{k=2}^{n}\frac{1}{k^s} - \int_2^n \frac{1}{x^s}dx\right)$$

$$\Rightarrow\ \gamma_s^* = \lim_{n\to+\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k^s} - 1\right) - \left(\int_1^n \frac{1}{x^s}dx - \int_1^2 \frac{1}{x^s}dx\right)\right]$$

$$\Rightarrow\ \gamma_s^* = \lim_{n\to+\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k^s} - \int_1^n \frac{1}{x^s}dx\right) + \int_1^2 \frac{1}{x^s}dx - 1\right]$$

$$\Rightarrow\ \gamma_s^* = \int_1^2 \frac{1}{x^s}dx - 1 + \lim_{n\to+\infty}\left(\sum_{k=1}^{n}\frac{1}{k^s} - \int_1^n \frac{1}{x^s}dx\right)$$

Using Definition 1, we can write:

$$\Rightarrow\ \gamma_s^* = \frac{1}{s-1}(1-2^{1-s}) - 1 + \gamma_s$$

$$\Rightarrow\ \gamma_s^* = (\zeta(s)-1) - \frac{1}{s-1}2^{1-s}$$

Because of the above Theorems 2 & 3 we can write the following Corollary 1.

Corollary 1: $\forall s \geq 1;\ s \in \mathbb{N}$ the measurements of the areas given by γ_s^* values depend on mathematical constants γ, Ln2 and $\zeta(s)$.

We can now study the convergence of the series $\lim_{n\to+\infty}\sum_{s=2}^{n}\gamma_s^*$. This approach gave rise to the following theorem.

Theorem 4: the series

$$\lim_{n\to+\infty}\sum_{s=2}^{n}\gamma_s^* = \sum_{s=2}^{+\infty}\gamma_s^*$$

converges to the transcendental number $(1-\text{Ln}2)$

Proof:

From Theorem 3, we can write:

$$\lim_{n\to+\infty}\sum_{s=2}^{n}\gamma_s^* = \lim_{n\to+\infty}\left[\sum_{s=2}^{n}\left((\zeta(s)-1) - \frac{1}{s-1}2^{1-s}\right)\right] = \lim_{n\to+\infty}\left[\sum_{s=2}^{n}(\zeta(s)-1)\right] - \lim_{n\to+\infty}\left[\sum_{s=2}^{n}\left(\frac{1}{s-1}2^{1-s}\right)\right]$$

It is well known (Particular values of the Riemann zeta function - Wikipedia) that,

$$\lim_{n\to+\infty}\left[\sum_{s=2}^{n}(\zeta(s)-1)\right] = \sum_{s=2}^{+\infty}(\zeta(s)-1) = 1$$

For the other sum,

$$\sum_{s=2}^{n}\left(\frac{1}{s-1}2^{1-s}\right)$$

we can write:

$$\forall s \in \mathbb{N} \ \ s \geq 2 \ \int_{2}^{+\infty} \frac{1}{x^s} dx = \frac{1}{s-1} \ 2^{1-s}$$

$$\Rightarrow \ \sum_{s=2}^{n} \left(\frac{1}{s-1} 2^{1-s} \right) = \sum_{s=2}^{n} \int_{2}^{+\infty} \frac{1}{x^s} dx$$

Using integral linearity gives:

$$\sum_{s=2}^{n} \left(\frac{1}{s-1} 2^{1-s} \right) = \int_{2}^{+\infty} \sum_{s=2}^{n} \frac{1}{x^s} dx$$

The series in the right-hand side converges as a geometric series, giving:

$$\lim_{n \to +\infty} \left[\sum_{s=2}^{n} \left(\frac{1}{s-1} 2^{1-s} \right) \right] = \int_{2}^{+\infty} \sum_{s=2}^{+\infty} \frac{1}{x^s} dx = \int_{2}^{+\infty} \frac{1}{x(x-1)} \ dx$$

Considering that:

$$\int_{2}^{+\infty} \frac{1}{x(x-1)} \ dx = \lim_{n \to +\infty} \left[\int_{2}^{n} \frac{1}{x(x-1)} \ dx \right]$$

Expanding the rational fraction under the integrand gives:

$$\int_{2}^{n} \frac{1}{x(x-1)} dx = \int_{2}^{n} \frac{1}{x-1} dx - \int_{2}^{n} \frac{1}{x} dx = Ln \left(\frac{n-1}{n} \right) + Ln2$$

And,

$$\lim_{n \to +\infty} \left[Ln \left(\frac{n-1}{n} \right) + Ln2 \right] = Ln2$$

Giving the result,

$$\lim_{n \to +\infty} \sum_{s=2}^{n} \gamma_s^* = \lim_{n \to +\infty} \left[\sum_{s=2}^{n} (\zeta(s) - 1) \right] - \lim_{n \to +\infty} \left[\sum_{s=2}^{n} \left(\frac{1}{s-1} 2^{1-s} \right) \right] = \sum_{s=2}^{+\infty} \gamma_s^* = 1 - Ln2$$

Finally, we know, from (Baker, 1990), that $Ln2$ is a transcendental number and then the series converges to a transcendental number.

Because we know, from Theorem 1 that $\gamma_1^* = \gamma + Ln2 - 1$, we obtain directly the following corollary.

Corollary 2: The series

$$\sum_{s=1}^{+\infty} \gamma_s^*$$

converges to the Euler-Mascheroni constant γ.

From these results and by use of Theorems 2 & 3 it is possible to calculate Riemann zeta function at odd integer values in the following way (the considered example is calculation of Apéry’s constant):

$$\gamma_1^* + \gamma_2^* + \gamma_3^* = \gamma - \gamma_4^* - \gamma_5^* - \gamma_6^* - \cdots = \alpha$$

$$\Rightarrow \ \gamma + Ln2 - 1 + (\zeta(2) - 1) - \frac{1}{2} + (\zeta(3) - 1) - \frac{1}{8} = \alpha$$

$$\Rightarrow \ \zeta(3) = 3 - \zeta(2) - \left(\gamma + Ln2 - \frac{5}{8} - \alpha\right)$$

By use of a matrix approach, (Delplace, 2017) proposed the following expression for Apéry's constant.

$$\zeta(3) = 3 - \zeta(2) - \sum_{k=2}^{+\infty} \frac{1}{k^3(k-1)}$$

This result can also be obtained by expanding the rational fraction $\frac{1}{k^3(k-1)}$ as followed:

$$\frac{1}{k^3(k-1)} = \frac{1}{k-1} - \frac{1}{k} - \frac{1}{k^2} - \frac{1}{k^3}$$

$$\Rightarrow \ \sum_{k=2}^{+\infty} \frac{1}{k^3(k-1)} = \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} - \sum_{k=2}^{+\infty} \frac{1}{k^2} - \sum_{k=2}^{+\infty} \frac{1}{k^3}$$

The first series in the right-hand side is a telescopic series of value equal to 1, giving:

$$\sum_{k=2}^{+\infty} \frac{1}{k^3(k-1)} = 1 - (\zeta(2) - 1) - (\zeta(3) - 1) = 3 - \zeta(2) - \zeta(3)$$

Giving the above researched result.

As reported in (Mathar, 2009), this result can be generalized by induction giving the following theorem.

Theorem 5:

$$\forall s > 1; \ s \in \mathbb{N} \quad \sum_{k=2}^{+\infty} \frac{1}{k^s(k-1)} = s - \sum_{k=2}^{s} \zeta(k) \tag{6}$$

Identification of above expressions obtained for Apéry constant:

$$\zeta(3) = 3 - \zeta(2) - \left(\gamma + Ln2 - \frac{5}{8} - \alpha\right)$$

And,

$$\zeta(3) = 3 - \zeta(2) - \sum_{k=2}^{+\infty} \frac{1}{k^3(k-1)}$$

We obtain,

$$\sum_{k=2}^{+\infty} \frac{1}{k^3(k-1)} = \gamma + Ln2 - \frac{5}{8} - \alpha$$

Numerically we have:

$$\alpha = 0.492\ 353\ 815\ 469\ 298\ 891\ 895\ 897\ 539\ 698 \ldots$$

Giving

$$\sum_{k=2}^{+\infty}\frac{1}{k^3(k-1)} = 0.153\ 009\ 029\ 992\ 179\ 278\ 127\ 846\ 671\ 842\ 524\ 820\ \ldots$$

A recent paper (Delplace, 2019), cited in (Srivastava, 2019), also showed that Apéry's constant can be calculated as followed:

$$\zeta(3) = \frac{\pi^3}{28} + \frac{16}{7}\sum_{k=1}^{+\infty}\frac{1}{(4k-1)^3} \quad (7)$$

We can then consider that the quantity $\pi^3/28$ must be introduce in the previous equations. The quantity $(\alpha + 5/8)$ is strictly greater than $\pi^3/28$ then, by defining quantity $\tau = (\alpha + 5/8) - \pi^3/28$, we obtain the following theorem.

Theorem 6:

$$\zeta(3) = 3 - \frac{\pi^2}{6} + \frac{\pi^3}{28} - \gamma - Ln2 + \tau \quad (8)$$

With, $\tau = 0.009\ 986\ 791\ 172\ 876\ 742\ 771\ 743\ 430\ 158\ 716\ 636\ \ldots$

As an interesting result, τ appears not very far from the rational number $1/100$.

This approach can be generalized to all $\zeta(2k+1)$ $k = 1,2,\ldots$ values showing the deep link between Riemann zeta function values at odd integers and the Euler-Mascheroni constant which was the objective of this paragraph. But despite of these efforts, the undetermined quantity we called τ does not allow a closed form for Apéry constant to be obtain even if this quantity is quite close to a rational number. Another time, the problem of Riemann zeta function values at odd integers shows its great complexity and we decided to develop a new approach. In above Theorem 4, we found that the transcendental number $(1 - Ln2)$ is a measurement of the total area defined by γ_s^* $s = 2,3,\ldots$ And these areas are clearly identified by the corresponding light green zones in Figure 2.

We will now consider that $(1 - Ln2)$ is also a Lebesgue measure of the area defined as the difference between the measure of the square unit and the curve $y = 1/x$ $1 \leq x \leq 2$.

3. Results

Using Theorems 2 & 3 of the previous chapter, we can calculate numerical values γ_s^* for $s \geq 1$. The following Figure 3 shows values found for $s = 1,\ldots,5$

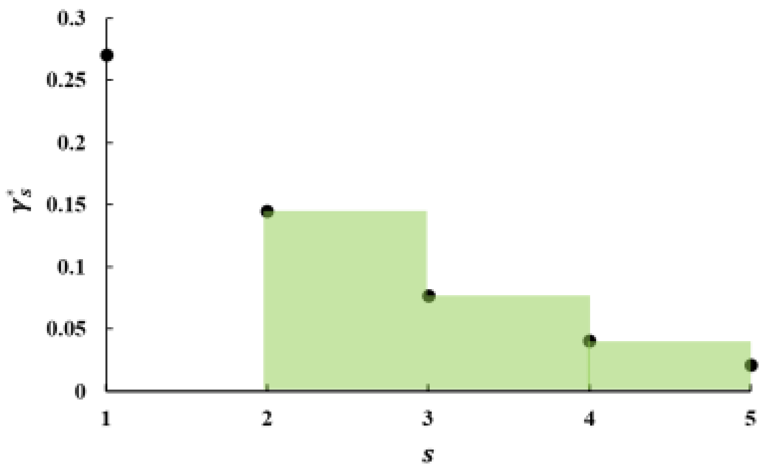

Figure 3. γ_s^* values for $s = 1,\ldots,5$ and light green rectangles A_k $k \geq 2$

From Theorem 4, we know that the infinite sum of measurements of light green rectangles in above Figure 3 is $(1 - Ln2)$. Let us consider that each rectangle is a subset A_k of a set E of Lebesgue measure $\mu(E) = (1 - Ln2)$. We have, $E = \bigcup_{k=2}^{+\infty} A_k$ with $A_j \cap A_k = \emptyset$ $j \neq k$ which means that the A_k are a partition of the set E. We have then $\forall k \geq 2, k \in \mathbb{N}$ $\mu(A_k) = \gamma_k^*$ and $\mu(E) = \sum_{k=2}^{+\infty}\mu(A_k) = \sum_{k=2}^{+\infty}\gamma_k^* = (1 - Ln2)$.

As proposed at the end of the second paragraph of this paper, we introduce now the set E' corresponding to the area difference between the square unit and the curve $y = 1/x$ $1 \leq x \leq 2$. This set is illustrated in Figure 4.

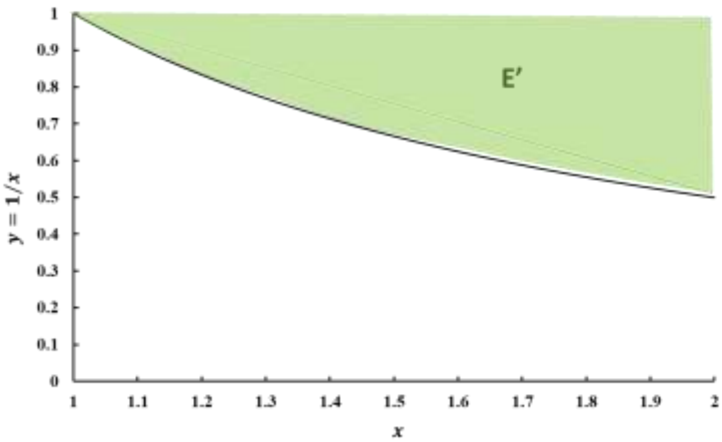

Figure 4. The curve $y = 1/x \ 1 \leq x \leq 2$ and the set E'

The set E' has the same Lebesgue measure than the set E: $\mu(E') = \mu(E) = (1 - Ln2)$.

We introduce now a mapping of the subsets A_k of E in the set E' giving the subsets B_k as a partition of E' illustrated in Figure 5.

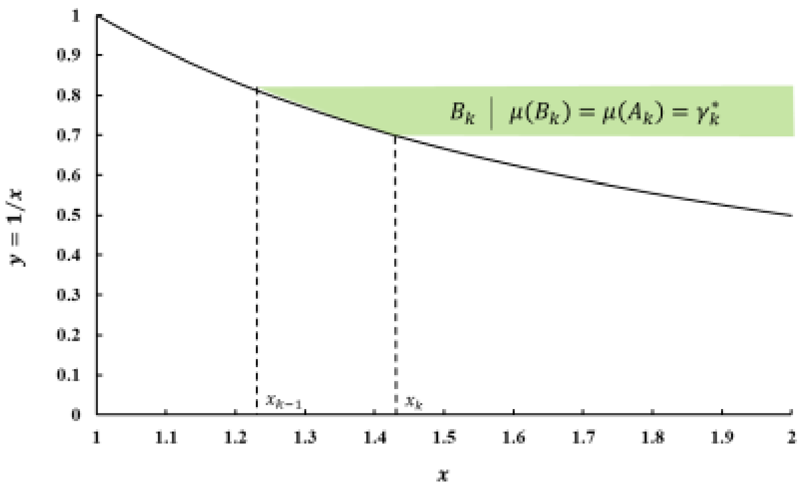

Figure 5. The subsets B_k such as $\mu(B_k) = \mu(A_k) = \gamma_k^*$

The measure $\mu(B_k)$ can be calculated as followed using notations of Figure 5.

$$\mu(B_k) = (2 - x_{k-1})\frac{1}{x_{k-1}} - \int_{x_{k-1}}^{x_k} \frac{1}{x} dx - (2 - x_k)\frac{1}{x_k}$$

Giving,

$$\mu(B_k) = \left(Lnx_{k-1} + \frac{2}{x_{k-1}}\right) - \left(Lnx_k + \frac{2}{x_k}\right)$$

The first term starts at $k = 2$ with $x_1 = 1$ according to Figure 5. From above relationship and using $\mu(B_k) = \mu(A_k) = \gamma_k^*$ we can write the following terms:

$$\mu(B_2) = 2 - Lnx_2 - \frac{2}{x_2} = \zeta(2) - \frac{3}{2}$$

$$\mu(B_3) = \left(Lnx_2 + \frac{2}{x_2}\right) - \left(Lnx_3 + \frac{2}{x_3}\right) = \zeta(3) - \frac{9}{8}$$

$$\mu(B_4) = \left(Lnx_3 + \frac{2}{x_3}\right) - \left(Lnx_4 + \frac{2}{x_4}\right) = \zeta(4) - \frac{25}{24}$$

…

Giving the following theorem as a recursive formula.

Theorem 7:

$$\forall k \geq 2, k \in \mathbb{N} \quad Lnx_k + \frac{2}{x_k} = 2 + \sum_{n=2}^{k}\left(\frac{1}{n-1}2^{1-n}\right) - \sum_{n=2}^{k}(\zeta(n) - 1) = R_k \tag{9}$$

If we find a continuous function able to give all values of x_k, then we can obtain an analytic form for all $\zeta(k)$ values, whatever is the parity of integer k. Due to the mathematical form of transcendental equation (9):

$$Lnx_k + \frac{2}{x_k} = R_k \tag{10}$$

The Lambert W function (Lambert W function - Wikipedia) appeared the right candidate giving the following theorem.

Theorem 8:

Solutions x_k of equation

$$Lnx_k + \frac{2}{x_k} = R_k \; with \; R_k \in \mathbb{R}$$

Take the form,

$$x_k = exp(W(-2e^{-R_k}) + R_k) = \frac{-2}{W(-2e^{-R_k})} \tag{11}$$

Proof:

By replacing the proposed solution (11) in the equation (10), we obtain,

$$W(-2e^{-R_k}) + R_k + \frac{2}{exp(W(-2e^{-R_k}) + R_k)} = W(-2e^{-R_k}) + R_k + \frac{2e^{-R_k}}{exp\left(W(-2e^{-R_k})\right)}$$

Because Lambert W function is the inverse function of $y = xe^x$, we have the identity $e^{W(x)} = x/W(x)$, giving the following equality,

$$W(-2e^{-R_k}) + R_k + \frac{2e^{-R_k}}{exp\left(W(-2e^{-R_k})\right)} = W(-2e^{-R_k}) + R_k + \frac{2e^{-R_k}e^{R_k}}{-2}W(-2e^{-R_k}) = R_k$$

From the definition of B_k subsets, values of $x_k \in [1,2]$ meaning the following corollary must be true.

Corollary 3:

$$\lim_{k\to+\infty} exp(W(-2e^{-R_k}) + R_k) = 2$$

Proof:

From previous results given in chapter 2 of this paper, we have,

$$\lim_{k\to+\infty} R_k = 2 + Ln2 - 1 = 1 + Ln2$$

Giving,

$$\lim_{k\to+\infty} exp(W(-2e^{-R_k}) + R_k) = exp\left(W\left(-2e^{-(1+Ln2)}\right) + (1 + Ln2)\right) = e^{(1+Ln2)}exp\left(W\left(-2e^{-(1+Ln2)}\right)\right)$$

Using the identity $e^{W(x)} = x/W(x)$,

$$\lim_{k\to+\infty} exp(W(-2e^{-R_k}) + R_k) = e^{(1+Ln2)}\frac{-2e^{-(1+Ln2)}}{W(-2e^{-(1+Ln2)})} = \frac{-2}{W(-2e^{-(1+Ln2)})}$$

We also have,

$$-2e^{-(1+Ln2)} = -\frac{1}{e}$$

And taking $x = -1$ in equality $(xe^x) = x$, gives $W(-1/e) = -1$

Finally, we obtain the expected result,

$$\lim_{k\to+\infty} exp(W(-2e^{-R_k}) + R_k) = 2$$

4. Discussion

The Lambert W function cannot be expressed in terms of elementary functions. It is a multivalued special function with two real branches called W_0 and W_{-1}. The branch W_0 corresponds to real numbers $x \in [-1/e\,, +\infty[$ and the other branch W_{-1} to real numbers $x \in [-1/e\,, 0[$ meaning that the function gives two values in this last interval.

From the definition of subsets B_k, we know that $1 < x_k \le 2$. By considering that $(1 + Ln2) \le R_k \le 7/2 - \zeta(2)$, it is straightforward that W_{-1} is the right branch which satisfy Theorem 8.

Because Lambert W function is a special function, and according to (Chow, 2018), it appears not possible to say that x_k are closed-form numbers even if this problem is known complex and always the subject of debates. But, at least, these numbers are given by a well-defined continuous function (W_{-1}), and therefore, they are analytical values.

In order to illustrate calculations, values of $R_k, x_k, \zeta(k)$ from theorem 7 and $,\zeta(k)$ from Euler formula (even integers) or from OEIS (odd integers) are given in the following Table 1.

Table 1. Values of $R_k, x_k, \zeta(k)$ for $2 \le k \le 7$

k	R_k	x_k	$\zeta(k)$ theorem 7	$\zeta(k)$ Euler or OEIS
2	$\frac{7}{2} - \zeta(2)$ 1.855065933…	$-2/W_{-1}\left(-2e^{\zeta(2)-\frac{7}{2}}\right)$ 1.189230998…	$\frac{7}{2} - \left(Lnx_2 + \frac{2}{x_2}\right)$ 1.644934067…	$\frac{\pi^2}{6}$ 1.644934067…
3	$\frac{37}{8} - \zeta(2) - \zeta(3)$ 1.778009029…	$-2/W_{-1}\left(-2e^{\zeta(2)+\zeta(3)-\frac{37}{8}}\right)$ 1.360184737…	$\frac{9}{8} + Ln\left(\frac{x_2}{x_3}\right) + \left(\frac{2}{x_2} - \frac{2}{x_3}\right)$ 1.202056903…	1.202056903…
4	$\frac{17}{3} - \zeta(2) - \cdots - \zeta(4)$ 1.737352462…	$-2/W_{-1}\left(-2e^{\zeta(2)+\cdots+\zeta(4)-\frac{17}{3}}\right)$ 1.506578656…	$\frac{25}{24} + Ln\left(\frac{x_3}{x_4}\right) + \left(\frac{2}{x_3} - \frac{2}{x_4}\right)$ 1.082323234…	$\frac{\pi^4}{90}$ 1.082323234…
5	$\frac{1283}{192} - \zeta(2) - \cdots - \zeta(5)$ 1.716049707…	$-2/W_{-1}\left(-2e^{\zeta(2)+\cdots+\zeta(5)-\frac{1283}{192}}\right)$ 1.626605862…	$\frac{65}{64} + Ln\left(\frac{x_4}{x_5}\right) + \left(\frac{2}{x_4} - \frac{2}{x_5}\right)$ 1.036927755…	1.036927755…
6	$\frac{7381}{960} - \zeta(2) - \cdots - \zeta(6)$ 1.704956645…	$-2/W_{-1}\left(-2e^{\zeta(2)+\cdots+\zeta(6)-\frac{7381}{960}}\right)$ 1.721679673…	$\frac{161}{160} + Ln\left(\frac{x_5}{x_6}\right) + \left(\frac{2}{x_5} - \frac{2}{x_6}\right)$ 1.017343062…	$\frac{\pi^6}{945}$ 1.017343062…
7	$\frac{16687}{1920} - \zeta(2) - \cdots - \zeta(7)$ 1.699211535…	$-2/W_{-1}\left(-2e^{\zeta(2)+\cdots+\zeta(7)-\frac{16687}{1920}}\right)$ 1.794993948…	$\frac{385}{384} + Ln\left(\frac{x_6}{x_7}\right) + \left(\frac{2}{x_6} - \frac{2}{x_7}\right)$ 1.008349277…	1.008349277…

As reported before, R_k values decrease as k increase to reach the limit value $(1 + Ln2) = 1.693147181$ ... Then, $\zeta(k)$ values are calculated with recursive formula of theorem 7 and implicit values given by W_{-1}. It is interesting to notice that as R_k values vary in a very narrow range (from $(1+Ln2)=1.693147181\ldots$ to $(7/2-\zeta(2))=1.855065933\ldots$), quantities $(-2e^{-R_k})$ also vary in a very narrow range (from $-1/e=-0.367879441\ldots$ to $-2e^{\zeta(2)-7/2}=-0.312885255\ldots$). Finally, theorem 7 gives $\zeta(k)$ varying in the range 1 to $\pi^2/6$.

We can then say that $\zeta(k)$ values at integers are fully determined by Lambert W function whatever is the parity of integers k. This function appears to be a more general form than the closed form found by Euler at even integers.

This result could be important in number theory due to the link between Riemann zeta function values and the probability that k integers chosen at random in any interval $[1,n]$ will be coprime. Famous Cesàro theorem (Théorème de Cesàro (théorie des nombres) — Wikipédia (wikipedia.org) showed that this probability is $p_k=1/\zeta(k)$ when n tends toward infinity. Because we demonstrated above that all $\zeta(k)$ come from Lambert W function values we can write the following theorem.

Theorem 9:

The probability that k integers chosen at random in any interval $[1,\mathrm{n}]$ will be coprime is fully determined by values of Lambert W function: $W_{-1}(x)$ with $x\in\left[-1/e\,,-2e^{\zeta(2)-7/2}\right]$.

This strong result is linked to the partition of the set E' of measure $\mu(E')=(1-Ln2)$ we introduced in paragraph 3 of this paper. We tried without success, to find other partitions of this set and other sets having the same Lebesgue measure. Of course, research about this type of sets would be interesting.

5. Conclusion

As reported in the introduction of this paper, the enigma of Riemann zeta function at odd integers remains an interesting subject for many reasons. The first one, probably the most fundamental in mathematics, is the understanding of the deep nature of numbers $\zeta(3),\zeta(5),\zeta(7),\ldots$ Euler himself was surprised by the value $\pi^2/6$ he found for the Basel problem, and he said: "I discovered against all odds, an elegant expression of the sum of the series: squaring of the circle. I discovered that six times the sum of this series was equal to the square of the length of the circumference whose diameter is one".

Another one concern the link between Riemann zeta function and prime numbers repartition which is always a great mystery known to be linked to famous Riemann conjecture. But knowledge at integers, and a deep understanding of the function, associated to probability of integers chosen at random to be coprime, is also of importance. Among others, applications in physics are also of major interest. Values at odd integers can be found in quantum electrodynamics (Berestetskii et al., 1982) or in continuous media mechanics (Delplace, 2019).

This article is an attempt to find a response to this mystery by use of generalized Euler-Mascheroni constants. The readers must know that for most of the duration of this work, we had no idea of a continuous function able to give access to all $\zeta(s)$ values. It is research of a set of Lebesgue measure $(1-Ln2)$ and the way to obtain a partition able to give all $\zeta(s)$ values that oriented us toward the Lambert W function.

It is amazing that the partition of such a very simple set defined as the difference between the area of square unit and the area under the curve $y=1/x\ \ 1\le x\le 2$ contains all $\zeta(s)$ values. This partition being given by Lambert W function values. Of course, this deep link is important for a better understanding of prime numbers distribution, and this subject had been already discussed in (Visser, 2013). This author showed that prime counting function has upper and lower bonds that can be related to Lambert W function. Moreover, the numerous applications of this function in physics make also the role of Riemann zeta function in physics more evident.

To conclude, new perspectives in both number theory and physics could be opened by the link between Riemann zeta function and Lambert W function established in this paper.

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