Extended Sine and Cosine Functions for Scalene Triangles

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Abstract

This article pushes the role of sine and cosine functions beyond the traditional purpose of determining the sides of a right triangle, into the realm of determining the lengths of the sides of any triangle with practically the same ease. Extended functions are formulated dependent on two angles (instead of the traditional one) — $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ — that allow (via direct application) the computation of the lengths of the two shorter sides of a scalene triangle, as a result of the angular projection (from reference angle α and a variable obtuse angle γ) of the longer side or *extended hypotenuse* (for right triangles, the obtuse angle is fixed to $\gamma = 90$ deg, allowing only the variation of α — a significant limitation). When integrated into larger more complex mathematical formula, the extended sine and cosine functions add greater flexibility and open the door for the mathematician or scientist to explore possibilities that are non-orthogonal. Solved exercises are provided at the end, with the purpose of illustrating the robustness and advantage of the application of these new extended sine and cosine functions to determine the normalized sides of a scalene triangle — a requirement that is present virtually in any technical discipline.

Keywords: Trigonometry, sine, cosine, formula, scalene, right, triangle

1. Introduction

Sine function $\sin(\alpha)$ and cosine function $\cos(\alpha)$ are used systematically by mathematicians when composing larger more complex formula (Berndt et al 1997, Liu et al 2021). In science, these functions are used to define solutions for differential equations that govern physical systems — e.g., vibration theory (Timoshenko & Young 1974, Gomes 2011), electrical engineering (Hughes, 2006), telecommunications (Staelin 2014), acoustics (Rayleigh, 1945) and fluid dynamics (Houghton & Carpenter, 2000), etc. They provide the ability to convert an angle into a projected normalized length within the framework of a right triangle (a triangle that has an orthogonal obtuse angle $\gamma = 90$ deg). Once applied, this orthogonal dependency is automatically imprinted into the larger and more complex formula — an implicit limitation — restricting its field of application. Imagine now that those particular right triangles morph into more generic scalene triangles. Here, this implicit limitation of orthogonality is invalid. One example of such numerical mathematics involving scalene triangles is the field of computational fluid dynamics, in particular the making of triangular meshes that are used to compute highly complex flow fields (Tomac and Eller, 2014) [Figure 1a]. The question becomes, what happens to the sine $\sin(\alpha)$ and cosine $\cos(\alpha)$ functions when conditions arise where the implicit assumption of orthogonality (i.e., $\gamma = 90$ deg) is no longer valid (i.e., what is $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$)[Figure 1b]?

2. Hypothesis

There must be an version of the sine and cosine functions extended to scalene triangles — $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ — that would replace the original in providing the correct normalized project lengths of the longer to the two shorter sides (of which a right triangle is a particular case).

3. Theory

Such extended functions would be very useful as they would open the realm of possibilities of complex formula containing trigonometric functions to be applicable in environments other than orthogonal. Both the Pythagoras theorem and trigonometry (in general) form part of most secondary education curricula around the world, including the Canadian Curriculum (Canadian Ministry of Education, 2020), which makes this paper of interest to both students and professionals.

3.1 Formulating the Extended Functions

Let us begin by defining the problem geometrically, which will subsequently allow us to establish mathematical relations between angles and distances. Following on the original foundation presented by Euclid in classical trigonometry (Euclid

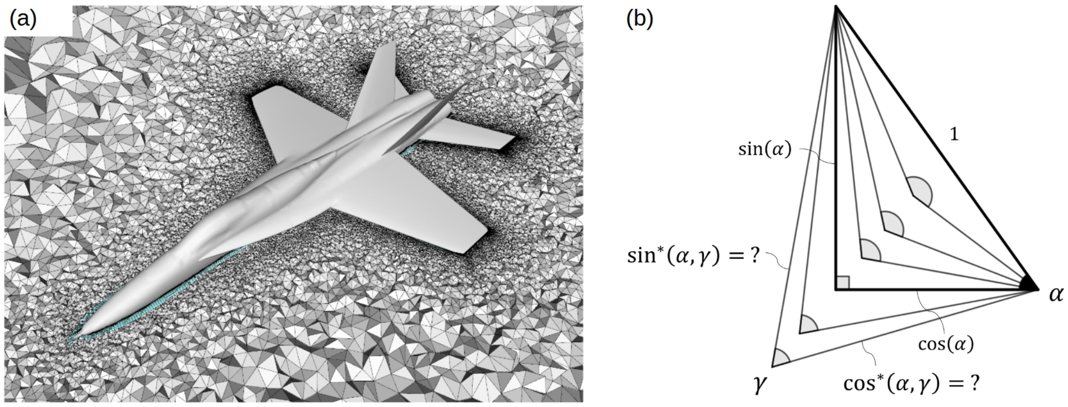

Figure 1. (a) Numerical mesh for fluid dynmics computations and (b) Extending the applicability of sine and cosine functions to scalene triangles

et al, 1908), start by considering the scalene triangle $\triangle ABC$ enclosed within the right-angled triangle $\triangle ADC$ in Figure 2. They both have a variable internal angle $\angle CAB = \angle CAD = \alpha$, and have a different internal reference angle (opposing the unit longest side $z = 1$) D $\angle ABC = \gamma$ for the scalene triangle $\triangle ABC$, and angle $\angle ADC = \pi/2$ for the right-angled triangle $\triangle ADC$. When $\gamma = 90$ deg, the scalene triangle becomes the right-angled triangle $\triangle ADC$.

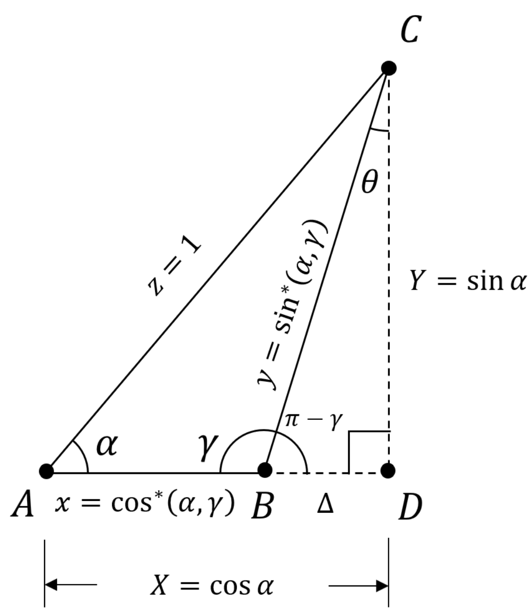

Figure 2. Parameterization of the projections of a scalene triangle ABC from a right-angled triangle ADC

Just as the vertical and horizontal projections of the hypotenuse $z = AC = 1$ are $X = AD = \cos(\alpha)$ and $Y = CD = \sin(\alpha)$ for the right-angled triangle $\triangle ADC$, the corresponding oblique and horizontal projections of the hypotenuse $z = AC = 1$ for the scalene triangle $\triangle ABC$ are $x = AB = \cos^*(\alpha, \gamma)$ and $y = BC = \sin^*(\alpha, \gamma)$. The traditional sine and cosine functions for the particular case of a right-angled triangle (governed by the internal angles α and $\gamma = \pi/2$) are commonly used in mathematics and science (Curtis 2010, Howard & Workman 2018, Parisher & Rhea 2012, Rawlings 2000), hence it is expected that the new expressions for the generic case of a scalene triangle (encompassing two variable internal angles — α and γ) will be equaly useful. We start by finding the generalized sine function $y = BC = \sin^*(\alpha, \gamma)$. The vertical projection $Y = DC$ of the hypotenuse is given as

$$Y = y\cos(\theta) \tag{1}$$

Which is further expanded with $Y = \sin(\alpha)$ and $y = \sin^*(\alpha, \gamma)$ [according to Fig.(2)] as

$$\sin(\alpha) = \sin^*(\alpha, \gamma)\cos(\theta) \tag{2}$$

Knowing that the angle $\angle CBD$ is $\pi - \gamma$, and from the sum of angles within triangle $\triangle CBD$ gives the following relation

$$\theta = \gamma - \frac{\pi}{2} \tag{3}$$

Applying the angle difference identity for cosine gives

$$\cos(\theta) = \cos\left(\gamma - \frac{\pi}{2}\right) = \cos(\gamma)\cos\left(\frac{\pi}{2}\right) + \sin(\gamma)\sin\left(\frac{\pi}{2}\right) = \sin(\gamma) \tag{4}$$

Substituting Eq.(4) back into Eq.(2), while re-arranging, gives the desired extended sine function as

$$\sin^*(\alpha, \gamma) = \frac{\sin(\alpha)}{\sin(\gamma)} \tag{5}$$

We move our attention to defining the extended cosine function $x = AB = \cos^*(\alpha, \gamma)$. The horizontal projection $X = AD$ (of the hypotenuse $z = AC = 1$) is given as

$$X = x + \Delta \tag{6}$$

The value of Δ is found by projecting the oblique side of triangle $\triangle ABC$ – i.e., $y = BC = \sin^*(\alpha, \gamma)$ – onto the horizontal axis, forming $\Delta = BD$ as

$$\Delta = \sin^*(\alpha, \gamma)\sin(\theta) \tag{7}$$

Replacing Eq.(7) into Eq.(6), while knowing that $X = \cos(\alpha)$ and $x = \cos^*(\alpha, \gamma)$ [according to Figure 2], gives

$$\cos(\alpha) = \cos^*(\alpha, \gamma) + \frac{\sin(\alpha)}{\sin(\gamma)}\sin(\theta) \tag{8}$$

Re-arranging Eq.(8) gives

$$\cos^*(\alpha, \gamma) = \cos(\alpha) - \frac{\sin(\alpha)}{\sin(\gamma)}\sin(\theta) \tag{9}$$

Applying the angle difference identity for sine to $\theta = \gamma - \pi/2$ [from Eq.(3)] gives

$$\sin(\theta) = \sin\left(\gamma - \frac{\pi}{2}\right) = \sin(\gamma)\cos\left(\frac{\pi}{2}\right) - \cos(\gamma)\sin\left(\frac{\pi}{2}\right) = -\cos(\gamma) \tag{10}$$

Substituting Eq.(10) in Eq.(9), and re-arranging, gives the desired extended cosine function as

$$\cos^*(\alpha, \gamma) = \cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha) \tag{11}$$

Making $\sin(\gamma)$ a common denominator, and using the angle sum identity $\sin(\alpha + \gamma) = \cos(\alpha)\sin(\gamma) + \sin(\alpha)\cos(\gamma)$ further reduces the expression to

$$\cos^*(\alpha, \gamma) = \frac{\sin(\gamma)\cos(\alpha) + \cos(\gamma)\sin(\alpha)}{\sin(\gamma)} = \frac{\sin(\alpha + \gamma)}{\sin(\gamma)} \tag{12}$$

The formulation of a generalized tangent function is obtained by dividing $\sin^*(\alpha, \gamma)$ in Eq.(5) by $\cos^*(\alpha, \gamma)$ in Eq.(11), resulting in

$$\tan^*(\alpha,\gamma) = \frac{\sin^*(\alpha,\gamma)}{\cos^*(\alpha,\gamma)} = \frac{\frac{\sin(\alpha)}{\sin(\gamma)}}{\frac{\sin(\alpha+\gamma)}{\sin(\gamma)}} = \frac{\sin(\alpha)}{\sin(\alpha+\gamma)} \tag{13}$$

3.2 Proving Their Universal Validity

The law of cosines [written below in Eq.(14)] is a broad expression that relates the lengths of the three sides x, y and z of any triangle (Maor 2007, Pickover 2012), which not only covers the particular case of the Pythagoras theorem $x^2+y^2=z^2$ (where orthogonality defines the obtuse angle $\gamma = \pi/2$ resulting in a right-angled triangle), but also all other possibilities for values of γ, that result in a scalene triangle.

$$z^2 = y^2 + x^2 - 2xy\cos(\gamma) \tag{14}$$

The particular case of the Pythagoras theorem is satisfied by replacing $x = \cos(\alpha)$, $y = \sin(\alpha)$, $z = 1$ and $\gamma = \pi/2$. We will now prove that the general case of the law of cosines in is fullfilled by the extended expressions of sine or $\sin^*(\alpha,\gamma)$ [given by Eq.(5)] and cosine or $\cos^*(\alpha,\gamma)$ [given by Eq.(11)], which together imply a scalene triangle with the following sides x, y and z.

$$x = \sin^*(\alpha,\gamma) \quad ; \quad y = \cos^*(\alpha,\gamma) \quad ; \quad z = 1 \tag{15}$$

Begin by first conviniently expanding separetely — based on the expressions given by Eq.(5) and Eq.(11) — each term in Eq.(14). Squaring x and substituting $\cos^*(\alpha,\gamma)$ [from Eq.(11)] results in

$$x^2 = \cos^*(\alpha,\gamma)^2 = \left(\cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\right)^2 \tag{16}$$

Note that the longer expanded version of $\cos^*(\alpha,\gamma)$ is applied here for conveninence. The above equation expands further to

$$x^2 = \cos^2(\alpha) + 2\cos(\alpha)\frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha) + \left(\frac{\cos(\gamma)}{\sin(\gamma)}\right)^2\sin^2(\alpha) \tag{17}$$

In a similarly manner, squaring y and substituting $\sin^*(\alpha,\gamma)$ [from Eq.(5)] gives

$$y^2 = \sin^*(\alpha,\gamma)^2 = \frac{\sin(\alpha)^2}{\sin(\gamma)^2} \tag{18}$$

The remainder term [from Eq.(14)] $-2xy\cos(\gamma)$ becomes

$$-2xy\cos(\gamma) = -2[\sin^*(\alpha,\gamma)\cos^*(\alpha,\gamma)]\cos(\gamma) \tag{19}$$

Which is expanded using Eq.(5) and Eq.(11) to

$$-2xy\cos(\gamma) = -2\left(\frac{\sin(\alpha)}{\sin(\gamma)}\right)\left(\cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\right)\cos(\gamma)$$

Elaborating further gives

$$-2\frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\cos(\alpha) - 2\frac{1}{\sin(\gamma)}\sin^2(\alpha)\frac{\cos(\gamma)}{\sin(\gamma)}\cos(\gamma) \tag{20}$$

This concludes in the final expression

$$-2xy\cos(\gamma) = -2\frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\cos(\alpha) - 2\left(\frac{\cos(\gamma)}{\sin(\gamma)}\right)^2\sin^2(\alpha) \tag{21}$$

The terms given Eq.(17), Eq.(18) and Eq.(21) are respectively replaced in the law of cosines [given by Eq.(14)] resulting in

$$\begin{aligned} z^2 = y^2 + x^2 - 2xy\cos(\gamma) \quad &= \quad \frac{\sin^2(\alpha)}{\sin(\gamma)^2} + \cos^2(\alpha) + 2\cos(\alpha)\frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha) + \left(\frac{\cos(\gamma)}{\sin(\gamma)}\right)^2 \sin^2(\alpha) - \\ &- \quad 2\frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\cos(\alpha) - 2\left(\frac{\cos(\gamma)}{\sin(\gamma)}\right)^2 \sin^2(\alpha) \end{aligned} \tag{22}$$

Grouping all terms that multiply $\sin^2(\alpha)$ in Eq.(22) gives

$$\sin^2(\alpha) \times \left[\frac{1}{\sin^2(\gamma)} - 2\frac{\cos^2(\gamma)}{\sin^2(\gamma)} + \frac{\cos^2(\gamma)}{\sin^2(\gamma)}\right] \tag{23}$$

The term between brackets simplifies to

$$\frac{1 - 2\cos^2(\gamma) + \cos^2(\gamma)}{\sin^2(\gamma)} = \frac{1 - \cos^2(\gamma)}{\sin^2(\gamma)} = 1 \tag{24}$$

Grouping all terms that multiply $\sin(\alpha)\cos(\alpha)$ in Eq.(22) gives

$$\sin(\alpha)\cos(\alpha) \times \left[-2\frac{\cos(\gamma)}{\sin(\gamma)} + 2\frac{\cos(\gamma)}{\sin(\gamma)}\right] = 0 \tag{25}$$

Replacing the above simplifications from Eq.(23) and Eq.(25) back into Eq.(22) results in

$$1^2 = [1] \times \sin^2(\alpha) + [0] \times \sin(\alpha)\cos(\alpha) + \cos^2(\alpha) \tag{26}$$

That further reduces to the following true relation, valid for any value of α and γ

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \tag{27}$$

This completes the proof, confirming that the extended sine function $\sin^*(\alpha, \gamma)$ [given by Eq.(5)], the extended cosine function $\cos^*(\alpha, \gamma)$ [given by Eq.(11)] and the normalized extended hypotenuse, are given respectively as

$$x = \cos^*(\alpha, \gamma) = \cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha) = \frac{\sin(\alpha + \gamma)}{\sin(\gamma)} \quad ; \quad y = \sin^*(\alpha, \gamma) = \frac{\sin(\alpha)}{\sin(\gamma)} \quad ; \quad z = 1 \tag{28}$$

successfully satisfy the law of cosines $z^2 = y^2 + x^2 - 2xy\cos(\gamma)$. In a practical sense, these expressions determine the normalized (that is, when $z = 1$) lengths of the sides x and y of any scalene triangle, when the two internal angles — reference α and obtuse γ — are known. It is worth noting that, since $x^2 + y^2 = 1$ yields the known relation $\sin^2(\alpha) + \cos^2(\alpha) = 1$ (often usefull in simplifying trigonometric algebra), the law of cosines [re-written as $y^2 - 2xy\cos(\gamma) + x^2 = z^2$] yields an extended version of the same relation as

$$\sin^*(\alpha, \gamma)^2 - 2\sin^*(\alpha, \gamma)\cos^*(\alpha, \gamma)\cos(\gamma) + \cos^*(\alpha, \gamma)^2 = 1 \tag{29}$$

4. Exercises

4.1 Direct Application

To demonstrate the robustness of the extended sine and cosine equations, various scalene triangles are drawn in Figure 3 superimposed on a common *extended hypotenuse* as side AB. Since the term "hypotenuse" is commonly associated with the longest side of a right triangle, the denomination for the equivalent longest side in a scalene triangle is by extension hereforth defined as "extended hypotenuse". The drawing was created with the open-source software Geogebra (Eaton et al, 2021), with all the sides being measured by the inbuilt tool as per Figure 3. The obtuse angle γ for all triangles is defined in purple (being $\angle ACB$ that for the right triangle), and the reference internal angle α in green (being $\angle CBA$ that for the right triangle). It will be shown that by applying the values of these angles and the common extended hypotenuse to Eqs.(28) that all the smaller sides of the various scalene triangles can be calculated with one application of the formulas. This is the equivalent application to a scalene triangle, of the use of the sine and cosine functions to determine the sides of a right triangle.

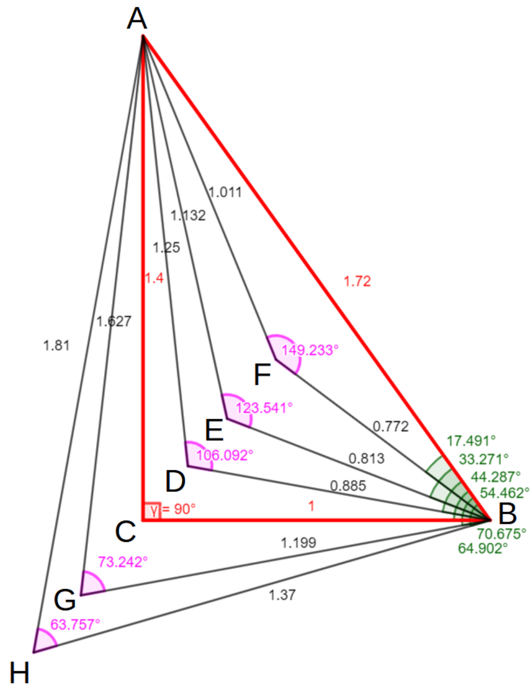

Figure 3. Various scalene triangles and their respective side lengths and internal angles

The following calculations are to be made using a calculator or any other means of computation (e.g. tablet, laptop, etc), with the formation of a spreadsheet in open-source LibreOffice Calc as one possibility. Starting from the top right to the bottom left, for the scalene triangle $\triangle AFB$ the sides AF and FB are computed as follows

$$\triangle AFB = \begin{cases} x = FB = 1.72\cos^*(17.491, 149.233) = 1.72\frac{\sin(17.491+149.233)}{\sin(149.233)} = 0.772 \\ y = AF = 1.72\sin^*(17.491, 149.233) = 1.72\frac{\sin(17.491)}{\sin(149.233)} = 1.011 \end{cases} \tag{30}$$

$$\triangle AEB = \begin{cases} x = EB = 1.72\cos^*(33.271, 123.541) = 1.72\frac{\sin(33.271+123.541)}{\sin(123.541)} = 0.813 \\ y = AE = 1.72\sin^*(33.271, 123.541) = 1.72\frac{\sin(33.271)}{\sin(123.541)} = 1.132 \end{cases} \tag{31}$$

$$\triangle ADB = \begin{cases} x = DB = 1.72\cos^*(44.287, 106.092) = 1.72\frac{\sin(44.287+106.092)}{\sin(106.092)} = 0.885 \\ y = AD = 1.72\sin^*(44.287, 106.092) = 1.72\frac{\sin(44.287)}{\sin(106.092)} = 1.250 \end{cases} \tag{32}$$

$$\triangle ACB = \begin{cases} x = CB = 1.72\cos^*(54.462, 90) = 1.72\frac{\sin(54.462+90)}{\sin(90)} = 1.000 \\ y = AC = 1.72\sin^*(54.462, 90) = 1.72\frac{\sin(54.462)}{\sin(90)} = 1.400 \end{cases} \tag{33}$$

$$\triangle AGB = \begin{cases} x = GB = 1.72\cos^*(64.902, 73.242) = 1.72\frac{\sin(64.902+73.242)}{\sin(73.242)} = 1.199 \\ y = AG = 1.72\sin^*(64.902, 73.242) = 1.72\frac{\sin(64.902)}{\sin(73.242)} = 1.627 \end{cases} \tag{34}$$

$$\triangle AHB = \begin{cases} x = HB = 1.72\cos^*(70.675, 63.757) = 1.72\frac{\sin(70.675+63.757)}{\sin(63.757)} = 1.369 \\ y = AH = 1.72\sin^*(70.675, 63.757) = 1.72\frac{\sin(70.675)}{\sin(63.757)} = 1.810 \end{cases} \tag{35}$$

4.2 Using a Circle to Find Projections

It's common practice to determine sine and cosine values from a triangle by drawing a circle with the radius at a given angle α to the horizontal axis, and measuring the vertically and horizontally projected lengths of its hypotenuse/radius (Figure 4a). For example, in electrical engineering this is known as a phase vector diagram assessment applicable to transformers, generatos and motors (Hava et al, 1999). However, until now this was restricted to an orthogonal axes system (i.e., when $\gamma = 90$ deg). The extended sine and cosine functions allow one to corroborate the measured projections drawn in a system of axes other than orthogonal (like for example, in Figure 4b for $\gamma = 120$ deg). Let us start by looking at the conventional application of Figure 4a.

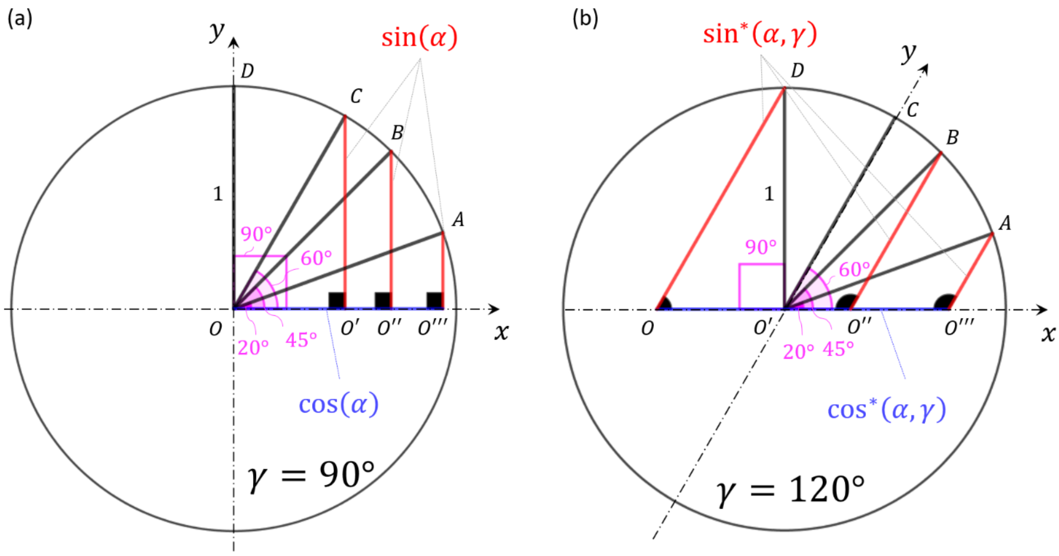

Figure 4. Circle approach to find sine and cosine values: (a) traditional [$\gamma = 90$ deg] and (b) extended [$\gamma = 120$ deg]

The conventional process of inscribing a right triangle into a circle, where the hypotenuse is the radius of the circle, allows measurement of its sides. Such system enables a visual comparison of the sides of the different triangles, and the effect of rotating the hypotenuse around the circle (i.e., as triangle's internal reference angle changes) has on the triangle's shape. The tipical result shown in Figure 4a for an orthogonal base system (i.e., when $\gamma = 90$ deg) is computed as follows

$$\gamma = 90 \text{ deg} \begin{cases} x = \cos(20) = 0.940 & y = \sin(20) = 0.342 & \triangle AOO''' \\ x = \cos(45) = 0.707 & y = \sin(45) = 0.707 & \triangle BOO'' \\ x = \cos(60) = 0.5 & y = \sin(60) = 0.866 & \triangle COO' \\ x = \cos(90) = 0 & y = \sin(90) = 1 & \triangle DOO \end{cases} \tag{36}$$

When the axis system changes, where the y-axis rotates to 120 degree (for example) in quadrant II and IV, the sides of the triangles also change accordingly. When thinking of the usefullness of such a system, one must consider the recently published extended versions of the Pythagoras theorem (using triangles [Teia, 2021a] and using hexagons [Teia, 2021b]) and how these versions depart from the original Pythagoras theorem. If the triangles in Figure 4a are governed by the Pythagoras theorem (i.e., $\gamma = 90$ deg), then the triangles in Figure 4b are governed by the extended Pythagoras theorem using triangles (i.e., $\gamma = 120$ deg). As discussed in a previous publication (Teia, 2021a), the extended version differs from the original in that it has a coupling area given by the term xy. In science, such a coupling term translates into an energetic buffer that generates a lag in transformation of kinetic into potential energy and vice versa [e.g. refer to the workings of the mass-spring system (Rayleigh 1945) and RLC electrical circuit (Rawlins 2000)]. These are extended topics which will be discussed in a following publication. For the present study, the axes system of $\gamma = 120$ deg in Figure 4b, the sides of the triangles are found the same way as above, except the functions used are Eqs.(28). Measuring the sides of the triangles would result in the predicted values of

$$\gamma = 120 \text{ deg} \begin{cases} x = \cos^*(20, 120) = \frac{\sin(20+120)}{\sin(120)} = 0.742 & y = \sin^*(20, 120)\frac{\sin(20)}{\sin(120)} = 0.395 & \triangle AO'O''' \\ x = \cos^*(45, 120) = \frac{\sin(45+120)}{\sin(120)} = 0.299 & y = \sin^*(45, 120)\frac{\sin(45)}{\sin(120)} = 0.816 & \triangle BO'O'' \\ x = \cos^*(60, 120) = \frac{\sin(60+120)}{\sin(120)} = 0 & y = \sin^*(60, 120)\frac{\sin(60)}{\sin(120)} = 1 & \triangle CO'O' \\ x = \cos^*(90, 120) = \frac{\sin(90+120)}{\sin(120)} = -0.577 & y = \sin^*(90, 120)\frac{\sin(90)}{\sin(120)} = 1.155 & \triangle DO'O \end{cases} \tag{37}$$

5. Conclusion

Trigonometry is — in its great majority — implicitly conditioned by orthogonality. This occurs via the inherent exclusive connection of sine and cosine functions to the lengths of the two shorter sides of the (hypotenuse) normalized right-angled triangles (defined by reference angle α and a fixed obtuse angle $\gamma = \pi/2$), and no others. This *orthogonal* expression of trigonometry precludes the existance of a more general *non-orthogonal* counter-part of these same functions. Thus, the goal of this article is to open doors for a new more expanded foundation by providing a wider perspective of the role and definition of the functions sine and cosines, as they are extended to scalene triangles (defined by reference angle α and a variable obtuse angle γ) — this in turn leads also to the formulation of extended expressions for their derivatives, extended exponential functions, extended identity rules, extended hyperbolic sine and cosine functions, etc. From an applied mathematics perspective, the extended sine and cosine functions and their solutions of the general governing equation (of which the Pythagoras theorem is a particular case), allows a wider interpretation and modeling of cornerstone physical systems like the spring-mass-damper system in mechanical engineering (encompassing vibration theory), and the RLC (Resistance-Induction-Capacitance) circuit in electrical engineering — both being taught as part of the syllabus in engineering at Universities worldwide. The usefullness of such extended functions reaches beyond these two systems into other field of science that employ sine and cosine functions and right triangles. Some other possibilities are now briefly described:

- In civil engineering, structures that are not perfectly vertical (e.g., a tower of a suspenssion bridge that inclined with time) offer an ideal example where traditional sine and cosine functions cease to be valid (i.e., angle to horizontal is no longer 90 degrees), and must be replaced by their extended sine and cosine versions.
- In aeronautics engineering, an horizontally flying aircraft being affected by a cross wind has a true air speed vector tha differs from its ground speed vector. These three velocity components often form a scalene triangle from which the normalized true and relative aircraft speed can be computed via their angles by employing the extended sine and cosine functions. While if the absolute components could be determine trigonometrically otherwise, the extended sine and cosine inherent their unique usefullness from the traditional sine and cosine in that they provide a normalized relative size (which can then be scaled based on the *extended hypotenuse*) solely dependent on the internal angles of the scalene triangle.
- In turbomachinery, velocity vector diagrams are tools used by aerospace engineers to design a turbine stage on an aircraft engine. These diagrams link the relative and absolute air velocity vector exiting a stator to the spinning velocity of the rotor, altogether forming a vectorial scalene triangle whose sides are governed by the Law of Cosines, and can be determined by the direct application of the extended sine and cosine functions (when the magnitude of the longest side is known along with two angles, alowing the determination of the projected smaller sides).
- In telecommunications, beamforming (and beam steering) of an antenna array is a capability computed via a wave interference pattern that results in a signal directional high sensitivity or gain lobe. This mathematical process comprises of sine functions measuring the time delay between antenna elements (via a right triangle of distances) for beam forming, and also includes an artificial time delay for beam steering. The extended sine function could replace this right triangle by a scalene triangle, whose difference to the aformentioned right triangle, would be quantified by the departure of the angle γ from 90 degrees, which is the artificial delay quantified as an angle.
- In relativity physics, time dialation is expressed by an equation derived from the application of the Pythagoras theorem to the example of a light clock on a train being withnessed by a stationary bystander. This establishes a quadratic relation between the speed of light, the speed of the train and the relative stretched time interval, because the train is displacing perpendicularly to the "clock"s beam of light. If the vertical beam and train's horizontal velocity vectors can be altered, then a new variable γ (different from 90 degrees) can be added to the study of this important problem, whose assessment can be readily assisted by the extended sine and cosine functions.
- In orbital mechanics, the relative position of three satellites (forming a scalene triangle between them) can be tracked to have an optimum relative distance (which is known in its normalized form) by measuring their relative angles using internal optical sensors (to be determined by employing the extended sine and cosine functions).
- In signal processing, the aerial transmission of data streams is often employed using quaternary signaling schemes, like the quadrature phase-shift keying (QPSK). These make use of an in-phase signal I and a quadrature signal Q, which are fixed at a phase of $\gamma = 90$ degrees to each other. By providing the ability to alter this angle γ between the axis (to other than orthogonal), the present work provides the mathematical tools to alter the relative Q-I phase to any angle, allowing therefore another degree of freedom that could insert more symbols per transmission, thus

increasing the numbers of bits that can be combined (i.e., boost transmission bit rate without affecting bandwidth requirements).

- In optical physics, Snell's Law equates the ratio of refraction indexes of two mediums (i.e., n_1/n_2) to the ratio of sines of the angles of an incident beam passing from one to the next (i.e., $\sin(\theta_2)/\sin(\theta_1)$). In the new extended trigonometric context, the ratio of sines could be combined into a single extended sine function [i.e., the ratio of sine $\sin(\theta_2)/\sin(\theta_1) = \sin^*(\theta_2, \theta_1)$], where each refraction index is relatable to the internal acute and obtuse angles of a scalene triangle.

Some of these examples are to be presented further in a separate article that is to be published in an engineering journal, possibly The Journal of Open Engineering. The present paper offers the potential start of a new upgrade process of such trigonometric functions (governed by the Pythagoras theorem — a particular case) into more general, and thus more powerful and versatile versions of themselves (governed by the Law of Cosines — a general case). The present author is on-track to present some of these upgrades, starting with the extended angle sum and difference identity rule for the extended sine and cosine functions, followed by the explanation of the gamma derivative (which is a generalized version of the classical derivative applicable to the extended sine and cosine functions). Both topics are expected to be published in this journal, as extended articles in series to the present one.

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