

Extended Angle Sum and Difference Identity Rules for Scalene Triangles

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Abstract

The present article extends the applicability of the angle sum and difference identity rules $\sin(A \pm B)$ and $\cos(A \pm B)$ beyond the particular case of a right-angled triangle into the general case of scalene triangles as $\sin^*(A \pm B, \gamma)$ and $\cos^*(A \pm B, \gamma)$, adding the effect of independently varying both the reference angle $\alpha = A \pm B$ and the obtuse angle γ . Accompanied by appropriate theorems and proofs, the mathematical end result are four updated equations that supersede the traditional expressions $\sin(A \pm B) = \sin(A) \cos(b) \pm \cos(A) \sin(B)$ and $\cos(A \pm B) = \cos(a) \cos(B) \mp \sin(A) \sin(B)$, where the conventional $\sin(\alpha)$ and $\cos(\alpha)$ functions are replaced by the [already proven] extended versions $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ enclosing modifications including two angles α and γ . An open-source program scripted in Octave is provided for the verification of the derived expressions, including plotting the geometric results as a figure for both the cases of angle summation and subtraction.

Keywords: Trigonometry, angle, entities, sum, difference, rule, scalene, triangle

1. Introduction

The discovery of the identity rules is intimately tied to the development of Trigonometry — a branch of mathematics that studies the relationships between lengths and angles of sides of triangles. The etymology of the word trigonometry is Trigonon (triangle) plus Metron (measure), both expressions originating from Latin derivatives of Greek words (Johnson, 2016). The angle sum and difference identity rules essentially establish an arithmetic expression between the normalized sides (i.e. expressed in terms of sines and cosine functions) of two triangles superimposed in such a way that their individual reference angles A and B add to form the reference angle $A + B$ of the larger triangle resulting from the superposition. The opposite is true when the angles are subtracted. Ptolomy theorem offers probably one of the oldest and most well-known proof of the angle sum and difference formulas for sines and cosines (Joyce 2013). The identity rules are a mathematical cornerstone in the Canadian educational system (Canadian Ministry of Education, 2020), making this subject of interest to students and professionals alike. Other popular proofs are also widely available (Ren 1999, Kung 2008, Smiley 2018, Smiley et al 2018), some presenting all six trigonometric angle sum and difference identities in one drawing (Nelsen 2000). Research in this topic is still ongoing in modern mathematics, with new geometrical developments of these trigonometric angle sum and difference identities being presented today (Ollerton 2018). For a right-angled triangle, the angle sum and difference identity rules for $\alpha = A \pm B$ are

$$\sin(A \pm B) = \sin(A) \cos(B) \pm \cos(A) \sin(B) \tag{1}$$

$$\cos(A \pm B) = \cos(A) \cos(B) \mp \sin(A) \sin(B) \tag{2}$$

All existing proofs share one common restrictive assumption — they all apply only to right-angle triangles comprising of one variable reference angle α and one fixed obtuse angle of $\gamma = \pi/2$, invoking inherently the traditional forms of $\sin(\alpha)$ and $\cos(\alpha)$ functions to establish a relation between the sides [governed by the Pythagoras theorem $\sin^2(\alpha) + \cos^2(\alpha) = 1$]. On the other hand, the sides of a scalene triangle (Figure 1) are interrelated by the extended expressions for the $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ that were proved to be

$$\sin^*(\alpha, \gamma) = \frac{\sin(\alpha)}{\sin(\gamma)} \quad ; \quad \cos^*(\alpha, \gamma) = \cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(\alpha) = \frac{\sin(\alpha + \gamma)}{\sin(\gamma)} \tag{3}$$

satisfying the more general governing equation that is the Law of Cosines $\sin^*(\alpha, \gamma)^2 - 2 \sin^*(\alpha, \gamma) \cos^*(\alpha, \gamma) \cos(\gamma) + \cos^*(\alpha, \gamma)^2 = 1$ (of which the Pythagoras theorem is a particular case with $\gamma = \pi/2$). From this outcome, the inher-

ent question to ask is, how will the angle sum and difference identity rules look like when applied to scalene triangles (governed by reference angle $\alpha = A \pm B$ and obtuse angle γ)?

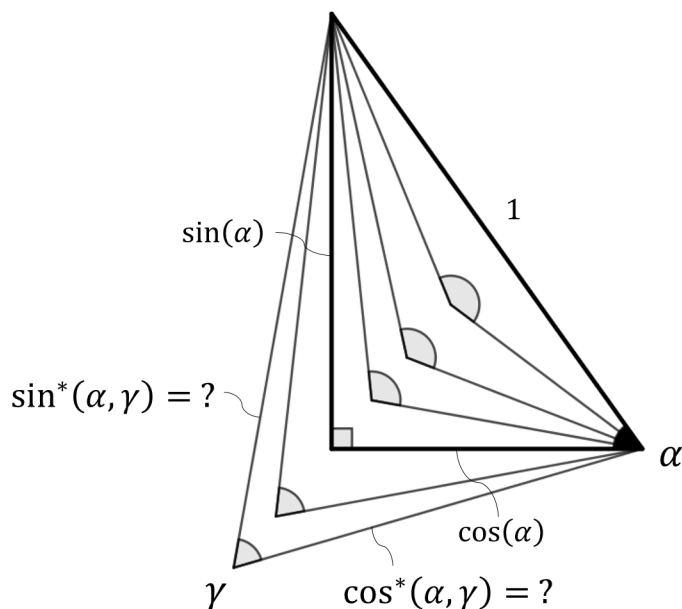


Figure 1. Extending the applicability of sine and cosine functions to scalene triangles [1].

2. Hypothesis

Just as there are angle sum and difference identity rules for sine and cosine functions that govern (the particular case of) right-angled triangles, there must exist an equivalent set of identity rules for the extended sine and cosines functions governing (the general case of) scalene triangles, which should naturally be derived from evolutive modifications into the existing proofs of angle sum and difference identity rules.

3. Theory

Start with geometrical relations to define mathematical equations. Prove that they reduce back to the original form. First this is done for the sum of angles, and second for the difference between angles.

3.1 Extended Angle Sum Identity Rule

Theorem 1 (Angle Sum Identity Rule for Scalene Triangles). *If the extended $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ functions are the normalized projected side lengths of a scalene triangle — where angle α is the reference angle formed between the unit side and the projected extended diagonal or vertical sine side of the triangle, and the angle γ is the obtuse angle formed between the two projected sine and cosine sides of the triangle — then, for the particular case where the reference angle α is the result of the sum between smaller angles A and B such that $\alpha = A + B$, the relationship between lengths given for this particular case by the extended sine and cosine are given as*

$$\sin^*(A + B, \gamma) = \sin^*(A, \gamma) \cos^*(B, \gamma) + \cos^*(A, \pi - \gamma) \sin^*(B, \gamma) \tag{4}$$

$$\cos^*(A + B, \gamma) = \cos^*(A, \gamma) \cos^*(B, \gamma) - \sin^*(A, \pi - \gamma) \sin^*(B, \gamma) \tag{5}$$

Proof. In addition to proving Theorem 1, it will also be shown that the extended functions $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ given by Eqs.(4-5) reduce back, for the particular case of a right-angled triangle (where $\gamma = \pi/2$), to the original form and $\sin(\alpha)$ given $\cos(\alpha)$ by Eqs.(3). Let us start by formulating the extended sine function $\sin^*(A + B, \gamma)$ from first principles.

Consider the scalene triangle $\triangle ADB$ in Figure 2 that presents a reference angle $\angle BAD \equiv \alpha = A$ and an obtuse angle $\angle ADB \equiv \gamma$. A second scalene triangle $\triangle ABC$ is also seen on top of the first $\triangle ADB$ having as a reference angle $\angle CAB \equiv \alpha = B$ and obtuse angle $\angle ABC \equiv \gamma$. Here, the side AB of triangle $\triangle ABC$ equals the longest side of triangle $\triangle ADB$. Together, the triangles $\triangle ABC$ and $\triangle ADB$ define the larger triangle $\triangle AFC$, which has a reference angle $\angle CAF \equiv \alpha = A + B$ and obtuse angle $\angle AFC \equiv \gamma$. The longest side AC of triangle $\triangle AFC$ has a unit length. Overall, all the diagrams in this article were drawn with the open-source software Geogebra (Feng 2014). Concerning triangle $\triangle ABC$, the side adjacent to

$\alpha = B$ is $AB = \cos^*(B, \gamma)$, while its opposing side is $BC = \sin^*(B, \alpha)$. For triangle $\triangle ADB$, the side adjacent to $\alpha = A$ is $AD = \cos^*(B, \gamma) \cos^*(A, \gamma)$, while its opposing side is

$$EF = BD = \cos^*(B, \gamma) \sin^*(A, \gamma) \tag{6}$$

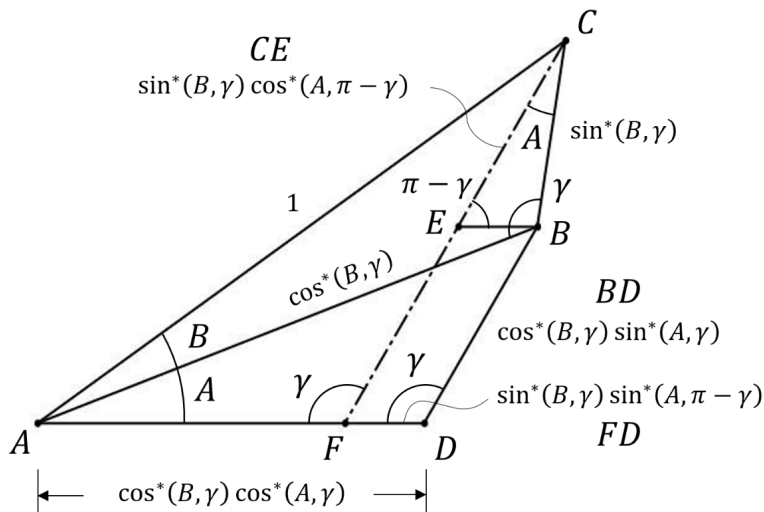


Figure 2. An angle summation within a scalene triangle and associated projections.

An important triangle formed by the interference between $\triangle ADB$ and $\triangle ABC$ is that of $\triangle CEB$ at the top right corner, which possesses an internal reference angle $\angle ECB \equiv A$ and opposing angle between CE and EB of $\angle CEB \equiv \angle \equiv \pi - \gamma$. For triangle $\triangle ABC$, the side opposing angle $\alpha = B$ is $CB = \sin^*(B, \gamma)$. This means that the triangle $\triangle CEB$, which is defined by angles $\alpha = A$ and $\pi - \gamma$, the length of the side CE adjacent to angle $\alpha = A$ is

$$CE = \sin^*(B, \gamma) \cos^*(A, \pi - \gamma) \tag{7}$$

Now, the projection of the longest side AC of the triangle $\triangle AFC$ opposing angle $\alpha = A + B$ is

$$CF = \sin^*(A + B, \gamma) = EF + CE \tag{8}$$

This means that when replacing the aforementioned expressions for CF [in Eq.(8)], CE [in Eq.(7)] and $EF = BD$ [in Figure 2 to the right], while at the same time re-arranging, the resulting expression for the extended angle sum identity rule for sine — previously defined in Eq.(4) — is given as

$$\sin^*(A + B, \gamma) = \sin^*(A, \gamma) \cos^*(B, \gamma) + \cos^*(A, \pi - \gamma) \sin^*(B, \gamma) \tag{9}$$

where the extended functions $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ are defined in Eq.(3). This completes the first part of the proof. It will now be shown that Eq.(9) reduces back to the already proven form of Eq.(3) with the reference angle being replaced directly with $\alpha = A + B$, or

$$\sin^*(A + B, \gamma) = \frac{\sin(A + B)}{\sin(\gamma)} \tag{10}$$

Start by defining the extended sine function $\sin^*(\alpha, \gamma)$ for each of the two angles A and B

$$\sin^*(A, \gamma) = \frac{\sin(A)}{\sin(\gamma)} \quad ; \quad \sin^*(B, \gamma) = \frac{\sin(B)}{\sin(\gamma)} \tag{11}$$

At the same time, define the extended cosine function $\cos^*(\alpha, \gamma)$ for angle B

$$\cos^*(B, \gamma) = \cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) \tag{12}$$

Both Eq.(11) and Eq.(12) will be replaced in Eq.(9) at a later stage. For triangle $\triangle CEB$, the extended cosine function for angle reference A and obtuse angle is $\pi - \gamma$ has the expression

$$\cos^*(A, \pi - \gamma) = \cos(A) + \frac{\cos(\pi - \gamma)}{\sin(\pi - \gamma)} \sin(A) \tag{13}$$

where the angle difference identities between π and γ for sine and cosine are given by the following relations

$$\sin(\pi - \gamma) = \sin(\pi) \cos(\gamma) - \cos(\pi) \sin(\gamma) = + \sin(\gamma) \tag{14}$$

$$\cos(\pi - \gamma) = \cos(\pi) \cos(\gamma) + \sin(\pi) \sin(\gamma) = - \cos(\gamma) \tag{15}$$

Substituting these in Eq.(13) simplifies to

$$\cos^*(A, \pi - \gamma) = \cos(A) - \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \tag{16}$$

Replacing Eq.(11), Eq.(12) and Eq.(16) into Eq.(9) gives

$$\sin^*(A + B, \gamma) = \left[\frac{\sin(A)}{\sin(\gamma)} \right] \left[\cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) \right] + \left[\cos(A) - \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \right] \left[\frac{\sin(B)}{\sin(\gamma)} \right] \tag{17}$$

Expanding the terms between brackets

$$\frac{1}{\sin(\gamma)} \sin(A) \cos(B) + \frac{1}{\sin(\gamma)} \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) \sin(A) + \frac{1}{\sin(\gamma)} \cos(A) \sin(B) - \frac{1}{\sin(\gamma)} \sin(A) \sin(B) \frac{\cos(\gamma)}{\sin(\gamma)} \tag{18}$$

And re-arranging, results in

$$\frac{1}{\sin(\gamma)} \left[\sin(A) \cos(B) + \cos(A) \sin(B) \right] + \frac{1}{\sin(\gamma)} \frac{\cos(\gamma)}{\sin(\gamma)} \left[\sin(B) \sin(A) - \sin(B) \sin(A) \right] \tag{19}$$

The second term between brackets vanishes, reducing Eq.(17) to

$$\sin^*(A + B, \gamma) = \frac{1}{\sin(\gamma)} \left[\sin(A) \cos(B) + \cos(A) \sin(B) \right] \tag{20}$$

Knowing that the angle sum identity for sine is , further simplifies this expression to

$$\sin^*(A + B, \gamma) = \frac{\sin(A + B)}{\sin(\gamma)} \tag{21}$$

which is by definition the extended sine function $\sin^*(\alpha, \gamma)$ with $\alpha = A + B$ [as stated in Eq.(10)]. The reduction of Eq.(9) to Eq.(21) — a particular case of Eq.(3) that already proved to be true — implies that Eq.(9) is also inherently true. This completes the first part of the proof concerning the extended sine function only.

Let us now advance to the extended cosine expression $\cos^*(A + B, \gamma)$ and in formulating its expression from first principles. From Figure 2, the triangle $\triangle CEB$ has a side opposing angle $\alpha = A$ equal to $EB = FD = \sin^*(B, \gamma) \sin^*(A, \pi - \gamma)$. The projection of the longest side AC of the triangle $\triangle AFC$ adjacent to angle $\alpha = A + B$ is $AF = \cos^*(A + B, \gamma)$, which is also the difference $AF = AD - FD$. Replacing the aforementioned expressions for AD [in Figure 2 at the bottom] and FD [in Figure 2 to the right], while re-arranging, results in the required Eq.(5) [here conveniently renumbered as Eq.(22)]

$$\cos^*(A + B, \gamma) = \cos^*(A, \gamma) \cos^*(B, \gamma) - \sin^*(A, \pi - \gamma) \sin^*(B, \gamma) \tag{22}$$

The extended cosine functions $\cos^*(A, \gamma)$ and $\cos^*(B, \gamma)$, and the extended sine function $\sin^*(B, \gamma)$, are defined in Eq.(12) and Eq.(11), respectively. The remainder unknown term is the extended sine function $\sin^*(A, \pi - \gamma)$, which is defined by the triangle $\triangle CEB$ whose obtuse angle is $\pi - \gamma$ (instead of the frequent γ), resulting in

$$\sin^*(A, \pi - \gamma) = \frac{\sin(A)}{\sin(\pi - \gamma)} = \frac{\sin(A)}{\sin(\gamma)} \tag{23}$$

Substituting Eq.(11), Eq.(12) and Eq.(23) into Eq.(22) gives

$$\cos^*(A + B, \gamma) = \left[\cos(A) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \right] \left[\cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) \right] - \left[\frac{\sin(A)}{\sin(\gamma)} \right] \left[\frac{\sin(B)}{\sin(\gamma)} \right] \tag{24}$$

Expanding the terms between brackets

$$\left[\cos(A) \cos(B) + \cos(A) \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \cos(B) + \frac{\cos^2(\gamma)}{\sin^2(\gamma)} \sin(A) \sin(B) \right] - \left[\frac{\sin(A) \sin(B)}{\sin^2(\gamma)} \right] \tag{25}$$

And with further re-arranging

$$\left[\cos(A) \cos(B) + \cos(A) \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \cos(B) + \frac{\cos^2(\gamma)}{\sin^2(\gamma)} \sin(A) \sin(B) \right] - \left[\frac{\sin(A) \sin(B)}{\sin^2(\gamma)} \right] \tag{26}$$

Remembering that the sum traditional identity rule $\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$ and from the Pythagoras Theorem $\cos^2(\gamma) - 1 = -\sin^2(\gamma)$, the above expression further simplifies to

$$\cos(A) \cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A + B) - \sin(A) \sin(B) \tag{27}$$

Knowing the other sum traditional identity rule $\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$ simplifies Eq.(27), with the end result for Eq.(24) to be the desired outcome

$$\cos^*(A + B, \gamma) = \cos(A + B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A + B) \tag{28}$$

which is by definition the extended cosine function [as defined previously in Eq.(3)] for $\alpha = A + B$. The reduction of Eq.(22) to Eq.(28) — a particular case of Eq.(3) already proven to be true — implies that Eq.(22) is also inherently true. This completes the proof.

3.2 Extended Angle Difference Identity Rule

Theorem 2 *Angle Difference Identity Rule for Scalene Triangles*. If the extended $\sin^*(\alpha, \gamma)$ and $\cos^*(\alpha, \gamma)$ functions are the normalized projected side lengths of a scalene triangle — where angle α is the reference angle formed between the unit side and the projected extended diagonal or vertical sine side of the triangle, and the angle γ is the obtuse angle formed between the two projected sine and cosine sides of the triangle — then, for the particular case where the reference angle is the result of the difference between smaller angles A and B such that $\alpha = A - B$, the relationship between lengths given for this particular case by the extended sine and cosine are given as

$$\sin^*(A - B, \gamma) = \sin^*(A, \gamma) \cos^*(B, \gamma) - \cos^*(A - \pi + 2\gamma, \pi - \gamma) \sin^*(B, \gamma) \tag{29}$$

$$\cos^*(A - B, \gamma) = \cos^*(A, \gamma) \cos^*(B, \gamma) + \sin^*(A - \pi + 2\gamma, \pi - \gamma) \sin^*(B, \gamma) \tag{30}$$

Proof. Let us start with the extended sine function $\sin^*(A - B, \gamma)$. Consider the scalene triangle $\triangle ABD$ of reference angle $\angle BAD \equiv \alpha = A$ and obtuse angle $\angle ADB \equiv \gamma$ in Figure 3. As seen before in Figure 2, the second scalene triangle $\triangle ABC$ of reference angle $\angle BAC \equiv \alpha = B$ and obtuse angle $\angle ABC \equiv \gamma$ is formed on top of the first — this time inverted — such that its smaller length AB equals the longest length of triangle $\triangle ABD$.

That is, the new triangle is the same as in Figure 2, except that it is mirrored about length AB . Together, their subtraction defines the smaller triangle $\triangle AFC$ of reference angle $\angle CAF \equiv \alpha = A - B$ and obtuse angle $\angle AFC \equiv \gamma$. The important

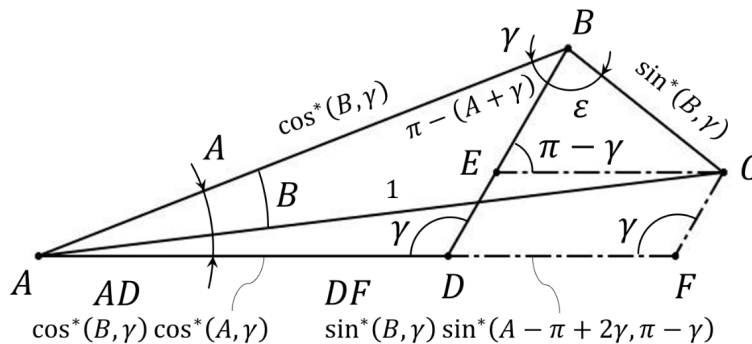


Figure 3. An angle subtraction within a scalene triangle and associated projections.

difference from Figure 2 is that the internal angles of the triangle $\triangle BEC$ have changed with the inversion, becoming $\epsilon = A - \pi + 2\gamma$ for the internal reference angle $\angle CBE$ and $\pi - \gamma$ for the obtuse angle $\angle BEC$. The term obtuse is used here to identify the angle playing the role of $\pi/2$ in a right-angled triangle. For a scalene triangle, this will change and will not always be literally the case, but once the proof is complete it will show that Eq.(29) and Eq.(30) are both true for any combination of $\alpha, \gamma \in \mathfrak{R}$. In Figure 3, the oblique projection of the longest side AC of the triangle $\triangle AFC$ is $CF = \sin^*(A - B, \gamma)$, which is also the difference $CF = BD - BE$.

$$\sin^*(A - B, \gamma) = \sin^*(A, \gamma) \cos^*(B, \gamma) - \cos^*(A - \pi + 2\gamma, \pi - \gamma) \sin^*(B, \gamma) \tag{31}$$

We will now prove that Eq.(31) holds true for any reference angle $\alpha = A - B$ and obtuse angle γ where $\alpha, \gamma \in \mathfrak{R}$. The process is the same as before in section 3.1, except for the change of sign between the two products in Eq.(9) and the extended cosine term in the second product that changes from $\cos^*(A, \pi - \gamma)$ to $\cos^*(A - \pi + 2\gamma, \pi - \gamma)$, as discussed due to the changes in the angles of triangle $\triangle BEC$. This modified cosine term is first expanded using Eq.(3) to

$$\cos^*(A - \pi + 2\gamma, \pi - \gamma) = \cos(A - \pi + 2\gamma) + \frac{\cos(\pi - \gamma)}{\sin(\pi - \gamma)} \sin(A - \pi + 2\gamma) \tag{32}$$

Replacing expressions for $\sin(\pi - \gamma)$ from Eq.(14) and for $\cos(\pi - \gamma)$ from Eq.(15) gives

$$\cos^*(A - \pi + 2\gamma, \pi - \gamma) = \cos(A - \pi + 2\gamma) - \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A - \pi + 2\gamma) \tag{33}$$

The terms $\sin(A - \pi + 2\gamma)$ and $\cos(A - \pi + 2\gamma)$ are further elaborated — where the angle $\alpha = A - \pi + 2\gamma$ is conveniently re-written as the difference between two angles — by the traditional angle sum and difference identity rules [given by Eq.(1) and Eq.(2)]. Starting with $\sin(A - \pi + 2\gamma)$, it gives

$$\sin(A - \pi + 2\gamma) = \sin(A) \cos(\pi - 2\gamma) - \cos(A) \sin(\pi - 2\gamma) \tag{34}$$

The terms with argument $\pi - 2\gamma$ are expanded into

$$\sin(\pi - 2\gamma) = \sin(\pi) \cos(2\gamma) - \cos(\pi) \sin(2\gamma) = + \sin(2\gamma) \tag{35}$$

$$\cos(\pi - 2\gamma) = \cos(\pi) \cos(2\gamma) + \sin(\pi) \sin(2\gamma) = - \cos(2\gamma) \tag{36}$$

Substituting into Eq.(34) further simplifies it to

$$\sin(A - \pi + 2\gamma) = - \sin(A) \cos(2\gamma) - \cos(A) \sin(2\gamma) \tag{37}$$

Moreover, it is known that

$$\sin(2\gamma) = \sin(\gamma) \cos(\gamma) + \cos(\gamma) \sin(\gamma) = 2 \sin(\gamma) \cos(\gamma) \tag{38}$$

$$\cos(2\gamma) = \cos(\pi) \cos(2\gamma) + \sin(\pi) \sin(2\gamma) = \cos^2(\gamma) - \sin^2(\gamma) \tag{39}$$

Which when replaced back into Eq.(37) gives

$$\sin(A - \pi + 2\gamma) = -\sin(A) [\cos^2(\gamma) - \sin^2(\gamma)] - \cos(A) [2 \sin(\gamma) \cos(\gamma)] \tag{40}$$

The same approach is used to determine $\cos(A - \pi + 2\gamma)$. The classical identity rule gives

$$\cos(A - \pi + 2\gamma) = \cos(A) \cos(\pi - 2\gamma) + \sin(A) \sin(\pi - 2\gamma) \tag{41}$$

Replacing $\sin(\pi - 2\gamma)$ from Eq.(35) and $\cos(\pi - 2\gamma)$ from Eq.(36) further simplifies Eq.(41) to

$$\cos(A - \pi + 2\gamma) = -\cos(A) \cos(2\gamma) + \sin(A) \sin(2\gamma) \tag{42}$$

The terms $\sin(2\gamma)$ and $\cos(2\gamma)$ are expanded with Eq.(38) and Eq.(39) giving

$$\cos(A - \pi + 2\gamma) = -\cos(A) [\cos^2(\gamma) - \sin^2(\gamma)] + \sin(A) [2 \sin(\gamma) \cos(\gamma)] \tag{43}$$

Substituting the individual expressions for $\sin(A - \pi + 2\gamma)$ in Eq.(40) and $\cos(A - \pi + 2\gamma)$ in Eq.(43) into the original expression of $\cos^*(A - \pi + 2\gamma, \pi - \gamma)$ in Eq.(33) results in

$$\begin{aligned} \cos^*(A - \pi + 2\gamma, \pi - \gamma) &= I + II = \\ &= -\cos(A) [\cos^2(\gamma) - \sin^2(\gamma)] + \sin(A) [2 \sin(\gamma) \cos(\gamma)] - \\ &\quad - \frac{\cos(\gamma)}{\sin(\gamma)} \{ -\sin(A) [\cos^2(\gamma) - \sin^2(\gamma)] - \cos(A) [2 \sin(\gamma) \cos(\gamma)] \} \end{aligned} \tag{44}$$

For convenience, this expression is decomposed into two parts. Let us start by simplifying Part *I*

$$\text{Part } I = -\cos(A) [\cos^2(\gamma) - \sin^2(\gamma)] + \sin(A) [2 \sin(\gamma) \cos(\gamma)] \tag{45}$$

which can be rewritten as

$$\text{Part } I = -\cos(A) \cos^2(\gamma) + \cos(A) \sin^2(\gamma) + 2 \sin(\gamma) \cos(\gamma) \sin(A) \tag{46}$$

Concerning Part *II*, it can be further simplified as

$$\text{Part } II = -\frac{\cos(\gamma)}{\sin(\gamma)} \{ -\sin(A) [\cos^2(\gamma) - \sin^2(\gamma)] - \cos(A) [2 \sin(\gamma) \cos(\gamma)] \} \tag{47}$$

Knowing that $\cos^2(\gamma) - \sin^2(\gamma) = 1 - 2 \sin^2(\gamma)$ further expands this — with some simplification — to

$$\frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) [1 - 2 \sin^2(\gamma)] + 2 \cos^2(\gamma) \cos(A) \tag{48}$$

which expands to

$$\text{Part } II = \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) - 2 \cos(\gamma) \sin(\gamma) \sin(A) + 2 \cos^2(\gamma) \cos(A) \tag{49}$$

Replacing back into the latest expression of $\cos^*(A - \pi + 2\gamma, \pi - \gamma)$ [given by Eq.(44)] both Part *I* [in Eq.(46)] and Part *II* [in Eq.(49)] results in

$$\begin{aligned} \cos^*(A - \pi + 2\gamma, \pi - \gamma) &= I + II = \\ &= -\cos(A) \cos^2(\gamma) + \cos(A) \sin^2(\gamma) + 2 \sin(\gamma) \cos(\gamma) \sin(A) + \\ &+ \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) - 2 \cos(\gamma) \sin(\gamma) \sin(A) + 2 \cos^2(\gamma) \cos(A) \end{aligned} \tag{50}$$

The two terms with the common multiplier $\sin(\gamma) \cos(\gamma)$ cancel, and those with $\cos^2(\gamma)$ simplify to

$$\cos(A) \sin^2(\gamma) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) + \cos^2(\gamma) \cos(A) \tag{51}$$

Which can be re-arranged as

$$\cos(A) \left[\sin^2(\gamma) + \cos^2(\gamma) \right] + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \tag{52}$$

Ultimately, this reduces Eq.(50) to

$$\cos^*(A - \pi + 2\gamma, \pi - \gamma) = \cos(A) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) = \cos^*(A, \gamma) \tag{53}$$

Note that this allows the identity rule for the extended sine with angles $\alpha = A - B$ and γ — given initially by Eq.(31) — to be rewritten as

$$\sin^*(A - B, \gamma) = \sin^*(A, \gamma) \cos^*(B, \gamma) - \cos^*(A, \gamma) \sin^*(B, \gamma) \tag{54}$$

Despite Eq.(54) being more convenient, for the purpose of future comparison the former version [Eq.(31)] will be used during the remainder of this paper. Expanding the individual terms in Eq.(59) with Eq.(11) and Eq.(12) results in

$$\sin^*(A - B, \gamma) = \left[\frac{\sin(A)}{\sin(\gamma)} \right] \left[\cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) \right] - \left[\cos(A) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \right] \left[\frac{\sin(B)}{\sin(\gamma)} \right] \tag{55}$$

which can be further elaborated to

$$\frac{1}{\sin(\gamma)} \sin(A) \cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \frac{1}{\sin(\gamma)} \sin(A) \sin(B) - \frac{1}{\sin(\gamma)} \cos(A) \sin(B) - \frac{\cos(\gamma)}{\sin(\gamma)} \frac{1}{\sin(\gamma)} \sin(A) \sin(B) \tag{56}$$

By cancelling the terms with the common multiplier $\sin(A) \sin(B)$ simplifies the original expression for $\sin^*(A - B, \gamma)$ [in Eq.(31)] to

$$\sin^*(A - B, \gamma) = \frac{1}{\sin(\gamma)} \left[\sin(A) \cos(B) - \cos(A) \sin(B) \right] \tag{57}$$

And knowing that the angle difference identity for sine is $\sin(A - B) = \sin(A) \cos(B) - \sin(B) \cos(A)$, the final expression becomes

$$\sin^*(A - B, \gamma) = \frac{\sin(A - B)}{\sin(\gamma)} \tag{58}$$

which is by definition the extended sine function $\sin^*(\alpha, \gamma)$ with $\alpha = A - B$. The reduction of Eq.(31) to Eq.(58) — a particular case of Eq.(3) that already proved to be true — implies that Eq.(31) is also inherently true. This completes the first part of the proof concerning the extended sine function only.

Let us now advance to the cosine expression $\cos^*(A - B, \gamma)$. In Figure 3, the horizontal projection of the unit side AC of the triangle $\triangle AFC$ given by $AF = \cos^*(A - B, \gamma)$ is given as the sum of two segments $AF = AD + DF$. According to Figure 3, the segment AD results from a double projection (adjacent to angles A and B) of $AC = 1$ onto $AB = \cos^*(A, \gamma)$

that is then projected onto $AD = \cos^*(A, \gamma) \cos^*(B, \gamma)$. Similarly, the segment AF also results from a double projection (opposing angles A and $\epsilon = A - \pi + 2\gamma$) of $AC = 1$ onto $BC = \sin^*(B, \gamma)$ that is then projected onto $EC = DF = \sin^*(A - \pi + 2\gamma, \pi - \gamma) \sin^*(B, \gamma)$. Adding the segments AD and DF results in

$$\cos^*(A - B, \gamma) = \cos^*(A, \gamma) \cos^*(B, \gamma) + \sin^*(A - \pi + 2\gamma, \gamma) \sin^*(B, \gamma) \tag{59}$$

We will now prove that Eq.(59) holds true for any reference angle $\alpha = A - B$ and obtuse angle γ where $\alpha, \gamma \in \mathfrak{R}$. The process is still the same as before, except for a sign change in Eq.(24) and the modified term $\sin^*(A - \pi + 2\gamma, \gamma)$, which will now be expanded first as we reduce Eq.(59)

$$\sin^*(A - \pi + 2\gamma, \pi - \gamma) = \frac{\sin(A - \pi + 2\gamma)}{\sin(\pi - \gamma)} \tag{60}$$

The term $\sin(A - \pi + 2\gamma)$ was already determined and given by Eq.(40), here repeated for convenience

$$\sin(A - \pi + 2\gamma) = -\sin(A) [\cos^2(\gamma) - \sin^2(\gamma)] - \cos(A) [2 \sin(\gamma) \cos(\gamma)] \tag{61}$$

Substituting into Eq.(60) and replacing $\sin(\pi - \gamma) = \sin(\gamma)$ [from Eq.(14)] gives

$$\sin^*(A - \pi + 2\gamma, \pi - \gamma) = \frac{-\sin(A) [\cos^2(\gamma) - \sin^2(\gamma)] - \cos(A) [2 \sin(\gamma) \cos(\gamma)]}{\sin(\gamma)}$$

The relation $\cos^2(\gamma) - \sin^2(\gamma) = 1 - 2 \sin^2(\gamma)$ further reduces the right of Eq.(62) into

$$\frac{-\sin(A) [1 - 2 \sin^2(\gamma)]}{\sin(\gamma)} - 2 \cos(A) \cos(\gamma) \tag{62}$$

Further re-arranging results in the transformation of Eq.(62) into the resulting expression

$$\sin^*(A - \pi + 2\gamma, \pi - \gamma) = -\frac{\sin(A)}{\sin(\gamma)} + 2 \sin(A) \sin(\gamma) - 2 \cos(A) \cos(\gamma) \tag{63}$$

The various terms in Eq.(59) for the expanded angle subtraction identity rule or $\cos^*(A - B, \gamma)$ are expanded using the expressions in Eq.(64), Eq.(11) and Eq.(12) giving

$$\begin{aligned} \cos^*(A - B, \gamma) = I + II &= \left[\cos(A) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \right] \left[\cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) \right] + \\ &+ \left[-\frac{\sin(A)}{\sin(\gamma)} + 2 \sin(A) \sin(\gamma) - 2 \cos(A) \cos(\gamma) \right] \left[\frac{\sin(B)}{\sin(\gamma)} \right] \end{aligned} \tag{64}$$

For convenience, Eq.(65) is divided into two part. The gradual simplification process starts by expanding Part I as

$$\text{Part } I = \cos(B) \cos(A) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A) \cos(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(B) \cos(A) + \frac{\cos^2(\gamma)}{\sin^2(\gamma)} \sin(A) \sin(B) \tag{65}$$

Knowing that $\cos^2(\gamma) = 1 - \sin^2(\gamma)$ allows this to be re-written as

$$\cos(B) \cos(A) + \frac{\cos(\gamma)}{\sin(\gamma)} \left[\sin(A) \cos(B) + \sin(B) \cos(A) \right] + \frac{1 - \sin^2(\gamma)}{\sin^2(\gamma)} \sin(A) \sin(B) \tag{66}$$

With further simplifying, it becomes

$$\text{Part I} = \cos(B) \cos(A) - \sin(A) \sin(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \left[\sin(A) \cos(B) + \sin(B) \cos(A) \right] + \frac{1}{\sin^2(\gamma)} \sin(A) \sin(B) \tag{67}$$

Our attention moves to Part II of Eq.(65) given as

$$\text{Part II} = \left[-\frac{\sin(A)}{\sin(\gamma)} + 2 \sin(A) \sin(\gamma) - 2 \cos(A) \cos(\gamma) \right] \left[\frac{\sin(B)}{\sin(\gamma)} \right] \tag{68}$$

Expanding the product between terms in brackets

$$-\frac{\sin(A) \sin(B)}{\sin(\gamma) \sin(\gamma)} + 2 \sin(A) \sin(\gamma) \frac{\sin(B)}{\sin(\gamma)} - 2 \cos(A) \cos(\gamma) \frac{\sin(B)}{\sin(\gamma)} \tag{69}$$

Re-arranging gives

$$\text{Part II} = -\frac{1}{\sin^2(\gamma)} \sin(A) \sin(B) + 2 \sin(A) \sin(B) - 2 \frac{\cos(\gamma)}{\sin(\gamma)} \cos(A) \sin(B) \tag{70}$$

Replacing Part I from Eq.(68) and Part II from Eq.(71) into the expression for $\cos^*(A - B, \gamma)$ given by Eq.(65) yields

$$\begin{aligned} \cos^*(A - B, \gamma) &= \cos(B) \cos(A) - \sin(A) \sin(B) + \frac{1}{\sin^2(\gamma)} \sin(A) \sin(B) + \\ &+ \frac{\cos(\gamma)}{\sin(\gamma)} \left[\sin(A) \cos(B) + \sin(B) \cos(A) \right] - \frac{1}{\sin^2(\gamma)} \sin(A) \sin(B) + \\ &+ 2 \sin(A) \sin(B) - 2 \frac{\cos(\gamma)}{\sin(\gamma)} \cos(A) \sin(B) \end{aligned} \tag{71}$$

The terms multiplying $1/\sin^2(\gamma)$ cancel, and the terms multiplying $\cos(\gamma)/\sin(\gamma)$ and $\sin(A) \sin(B)$ also further simplify. Re-organizing the remainder of the terms gives

$$\cos^*(A - B, \gamma) = \cos(B) \cos(A) + \sin(A) \sin(B) + \frac{\cos(\gamma)}{\sin(\gamma)} \left[\sin(A) \cos(B) - \sin(B) \cos(A) \right] \tag{72}$$

Finally, replacing the traditional identity rule $\cos(A - B) = \cos(B) \cos(A) + \sin(A) \sin(B)$ and $\sin(A - B) = \sin(A) \cos(B) - \sin(B) \cos(A)$ gives the final expression

$$\cos^*(A - B, \gamma) = \cos(A - B) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(A - B) \tag{73}$$

which is by definition the extended cosine function $\cos^*(\alpha, \gamma)$ [as defined previously in Eq.(3)] with $\alpha = A - B$. As before, the reduction of Eq.(59) to Eq.(74) — a particular case of Eq.(12), already shown to be true — implies that Eq.(22) is also inherently true. This completes the proof.

4. Summary

The traditional angle sum and difference identities for sine function and cosine function — with reference angle and governed by the normalized Pythagoras Theorem — are

$$\sin(A \pm B) = \sin(A) \cos(B) \pm \cos(A) \sin(B) \tag{74}$$

$$\cos(A \pm B) = \cos(A) \cos(B) \mp \sin(A) \sin(B) \tag{75}$$

These equations serve a particular case, valid only for a right-angled triangle (i.e., that is when the triangles obtuse angle is $\gamma = \pi/2$). For a scalene triangle, the modified angle sum and difference identities for the extended sine function $\sin^*(\alpha, \gamma)$ and cosine function $\cos^*(\alpha, \gamma)$ — with reference angle $\alpha = A \pm B$ and obtuse angle γ [angle opposing the fixed unit side] being governed by the normalized Law of Cosines $\sin^*(\alpha, \gamma)^2 - 2 \sin^*(\alpha, \gamma) \cos^*(\alpha, \gamma) c \cos(\gamma) + \cos^*(\alpha, \gamma)^2 = 1$ — are for summation

$$\sin^*(A + B, \gamma) = \sin^*(A, \gamma) \cos^*(B, \gamma) + \cos^*(A, \pi - \gamma) \sin^*(B, \gamma) \tag{76}$$

$$\cos^*(A + B, \gamma) = \cos^*(A, \gamma) \cos^*(B, \gamma) - \sin^*(A, \pi - \gamma) \sin^*(B, \gamma) \tag{77}$$

and for subtraction

$$\sin^*(A - B, \gamma) = \sin^*(A, \gamma) \cos^*(B, \gamma) - \cos^*(A - \pi + 2\gamma, \pi - \gamma) \sin^*(B, \gamma) \tag{78}$$

$$\cos^*(A - B, \gamma) = \cos^*(A, \gamma) \cos^*(B, \gamma) + \sin^*(A - \pi + 2\gamma, \pi - \gamma) \sin^*(B, \gamma) \tag{79}$$

where the extended sine function $\sin^*(\alpha, \gamma)$ and extended cosine function $\cos^*(\alpha, \gamma)$ are defined as

$$\sin^*(\alpha, \gamma) = \frac{\sin(\alpha)}{\sin(\gamma)} \quad ; \quad \cos^*(\alpha, \gamma) = \cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)} \sin(\alpha) = \frac{\sin(\alpha + \gamma)}{\sin(\gamma)} \tag{80}$$

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Appendix A

To facilitate the usage of the new extended rules presented in this article — highlighted by the summary in Chapter 4 — a program was written with the open-source software Octave (Eaton et al 2021) and is enclosed below for users to easily employ and analyze these fundamental equations applicable to scalene triangles. An example (assuming angles $\gamma = 100$ deg, $A = 30$ deg and $B = 20$ deg) of the end result of this program is shown in Figure 4 for the extended angle sum identity rule, and in Figure 5 for the extended angle difference identity rule. These angles can be modified to the discretion of the user in the appropriate control lines (number 4, 5 and 7) of the program. The program allows changes to these reference angles A , B and γ , and computes and plots automatically the result as shown in Figure 4 and 5.

Please note that this program was written with Octave version 6.1.0 (x64) [which can be readily downloaded freely at <https://gnu.mirror.constant.com/octave/windows/>], but it does not seem to load properly in the latest version 7.2.0 (the

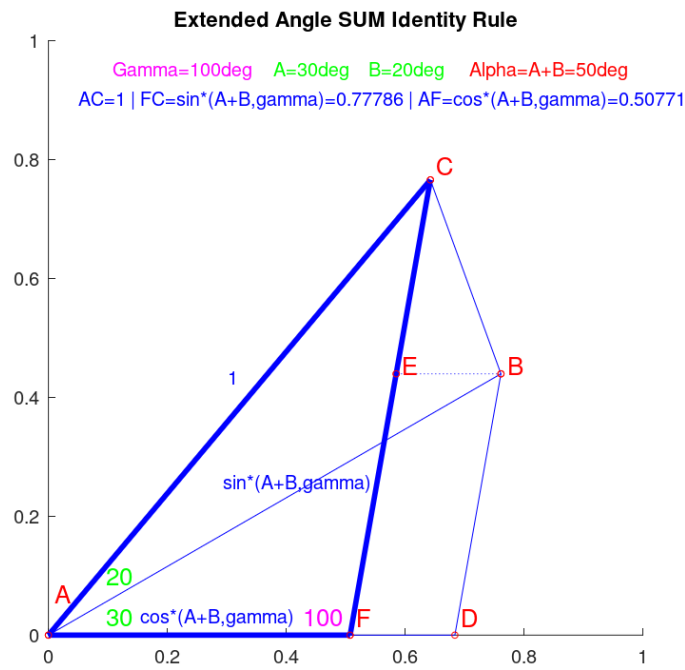


Figure 4. Example result of the program - the extended angle sum identity rule.

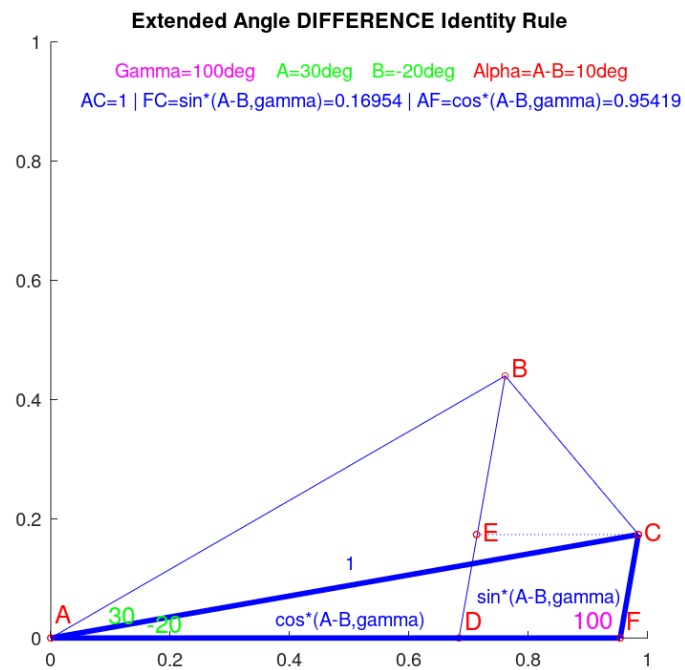


Figure 5. Example result of the program - the extended angle difference identity rule.

drawGraph function does not load in the 7.2.0). Also, doing direct copy and paste of the entire code from a PDF reader may replace the symbol ' by ´, which will prevent the program from running. This is probably a result of the pdf format exported by Miktex, and is simple to rectify. After copying and pasting the program into the Editor tab of Octave 6.1.0, right click the mouse on the text and select "Find and Replace". In the new pop-up window, copy and paste the ´ (that is seen in Octave) in the "Find what:" box, and in the "Replace with" box type the correct one ' from your keyboard. Then press the button "Replace All" to correct the entire program. The program should now run without any error. Also, when copying the entire code, you may inadvertently copy the page numbers 13 and 14; remember to delete them.

PROGRAM

```

clc, clear, format short
pkg load geometry;
pkg load matgeom;
disp('ANGLE VARIABLES')
disp('Reference Angle A', A=30 % CONTROL ANGLE A
disp('Reference Angle B', B=20 % CONTROL ANGLE B
disp('Combined Reference Angle Alpha=A+B', alpha=A+B
disp('Obtuse Angle gamma', gamma=100 % CONTROL ANGLE GAMMA
disp('_____')
disp('SINE AND COSINE')
SineA=sind(A); disp('sin(A)'); disp(SineA)
CosineA=cosd(A); disp('cos(A)'); disp(CosineA)
SineB=sind(B); disp('sin(B)'); disp(SineB)
CosineB=cosd(B); disp('cos(B)'); disp(CosineB)
disp('_____')
disp('ORIGINAL ANGLE SUM AND DIFFERENCE IDENTITY RULES')
% sin(A+B)=sin(A)cos(B)+cos(A)sin(B)
disp('sin(A+B)=sin(A)cos(B)+cos(A)sin(B)')
IdSinPlus=sind(A)*cosd(B)+cosd(A)*sind(B); disp(IdSinPlus)
% sin(A-B)=sin(A)cos(B)-cos(A)sin(B)
disp('sin(A-B)=sin(A)cos(B)-cos(A)sin(B)')
IdSinMinus=sind(A)*cosd(B)-cosd(A)*sind(B); disp(IdSinMinus)
% cos(A+B)=cos(A)cos(B)-sin(A)sin(B)
disp('cos(A+B)=cos(A)cos(B)-sin(A)sin(B)')
IdCosPlus=cosd(A)*cosd(B)-sind(A)*sind(B); disp(IdCosPlus)
% cos(A-B)=cos(A)cos(B)+sin(A)sin(B)
disp('cos(A-B)=cos(A)cos(B)+sin(A)sin(B)')
IdCosMinus=cosd(A)*cosd(B)+sind(A)*sind(B); disp(IdCosMinus)
disp('_____')
disp('EXTENDED SINE AND COSINE')
ExtSineA=sind(A)/sind(gamma); disp('sin*(A,gamma)'); disp(ExtSineA)
ExtCosineA=cosd(A)+cosd(gamma)/sind(gamma)*sind(A); disp('cos*(A,gamma)');
disp(ExtCosineA)
ExtSineB=sind(B)/sind(gamma); disp('sin*(B,gamma)'); disp(ExtSineB)
ExtCosineB=cosd(B)+cosd(gamma)/sind(gamma)*sind(B); disp('cos*(B,gamma)');
disp(ExtCosineB)
ExtSineAPigamma=sind(A)/sind(180-gamma); disp('sin*(A,pi-gamma)'); disp(ExtSineAPigamma)
ExtCosineAPigamma=cosd(A)+cosd(180-gamma)/sind(180-gamma)*sind(A);
disp('cos*(A,pi-gamma)'); disp(ExtCosineAPigamma)
ExtSineAPigammaMod=sind(A-180+2*gamma)/sind(180-gamma);
disp('sin*(A-pi+2*gamma,pi-gamma)'); disp(ExtSineAPigammaMod)
ExtCosineAPigammaMod=cosd(A-180+2*gamma)+cosd(180-gamma)/sind(180-gamma)*sind(A-180+2*gamma);
disp('cos*(A-pi+2*gamma,pi-gamma)');
disp(ExtCosineAPigammaMod)
disp('_____')
disp('EXTENDED ANGLE SUM AND DIFFERENCE IDENTITY RULES')
% sin*(A+B,gamma)=sin*(A,gamma)cos*(B,gamma)+cos*(A,gamma-gamma)sin*(B,gamma)
disp('sin*(A+B,gamma)=sin*(A,gamma)cos*(B,gamma)+cos*(A,pi-gamma)sin*(B,gamma)')

```

```

ExtIdSinPlus=ExtSineA*ExtCosineB+ExtCosineAPigamma*ExtSineB; disp(ExtIdSinPlus)
% cos*(A+B,gamma)=cos*(A,gamma)cos*(B,gamma)-sin*(A,gamma-gamma)sin*(B,gamma)
disp('cos*(A+B,gamma)=cos*(A,gamma)cos*(B,gamma)-sin*(A,pi-gamma)sin*(B,gamma)')
ExtIdCosPlus=ExtCosineA*ExtCosineB-ExtSineAPigamma*ExtSineB; disp(ExtIdCosPlus)
% sin*(A-B,gamma)=sin*(A,gamma)cos*(B,gamma)-cos*(A-pi+2gamma,pi-gamma)sin*(B,gamma)
disp('sin*(A-B,gamma)=sin*(A,gamma)cos*(B,gamma)-cos*(A-pi+2gamma,pi-gamma)sin*(B,gamma)')
ExtIdSinMinus=sind(A)*ExtCosineB-ExtCosineAPigammaMod*ExtSineB; disp(ExtIdSinMinus)
% cos*(A-B,gamma)red=cos*(A,gamma)cos*(B,gamma)+sin*(A-pi+2gamma,pi-gamma)sin*(B,gamma)
disp('cos*(A-B,gamma)=cos*(A,gamma)cos*(B,gamma)+sin*(A-pi+2*gamma,pi-gamma)sin*(B,gamma)')
ExtIdCosMinus=ExtCosineA*ExtCosineB+ExtSineAPigammaMod*ExtSineB; disp(ExtIdCosMinus)
figure(1)
% PLOTTING SUM RULE
subplot(1,2,1)
disp('-----')
disp('GRAPH NODES FOR PLOTTING EXTENDED SUM IDENTITY RULE')
xA=0, yA=0 % Point A
xB=ExtCosineB*cosd(A), yB=ExtCosineB*sind(A) % Point B
xC=IdCosPlus, yC=IdSinPlus % Point C
xD=ExtCosineA*ExtCosineB, yD=0 % Point D
xE=ExtCosineB*cosd(A)-ExtSineAPigamma*ExtSineB, yE=ExtCosineB*sind(A) % Point E
xF=ExtIdCosPlus, yF=0 % Point F
Nodes = [ xA, yA; xB, yB; xC, yC; xD, yD; xE, yE; xF, yF];
Edges = [1, 2; 1, 3; 1, 4; 2, 3; 2, 4];
g = drawGraph(Nodes, Edges);
Nodes1 = [0, 0; xC, yC; xF, yF]; Edges1 = [1, 2; 1, 3; 2, 3];
g1 = drawGraph(Nodes1, Edges1);
Nodes2 = [ xB, yB; xE, yE]; Edges2 = [1, 2];
g2 = drawGraph(Nodes2, Edges2);
set(g1, 'markerfacecolor', 'g', 'markersize', 50, 'linewidth', 5);
set(g2, 'linestyle', ':', 'Color', 'blue');
text(0.01,0.07,'A','Color','red','FontSize',26)
text(xB+0.01,1.03*yB,'B','Color','red','FontSize',26)
text(xC+0.01,1.03*yC,'C','Color','red','FontSize',26)
text(xD+0.01,0.03,'D','Color','red','FontSize',26)
text(xE+0.01,1.03*yE,'E','Color','red','FontSize',26)
text(xF+0.01,0.03,'F','Color','red','FontSize',26)
text(0.15*xC,0.04*yC, num2str(A), 'Color', 'green', 'FontSize', 26)
text(0.15*xC,0.13*yC, num2str(B), 'Color', 'green', 'FontSize', 26)
text(xF-0.08,0.03, num2str(gamma), 'Color', 'magenta', 'FontSize', 26)
if max(xD,xF)>1 top=max(xD,xF), else top=1 end
xlim([0 top]); ylim([0 top]), axis square
title('Extended Angle SUM Identity Rule','FontSize',14)
set(gca,'FontSize',20)
strAt=strcat(' A=',num2str(A),'deg');
strBt=strcat(' B=',num2str(B),'deg');
strAlphaP=strcat(' Alpha=A+B=',num2str(A+B),'deg');
strGammat=strcat(' Gamma=',num2str(gamma),'deg');
text(0.1,0.95,strGammat,'Color','magenta','FontSize',20)
text(0.37,0.95,strAt,'Color','green','FontSize',20)
text(0.53,0.95,strBt,'Color','green','FontSize',20)
text(0.7,0.95,strAlphaP,'Color','red','FontSize',20)
strIdSinPlus=strcat(' FC=sin*(A+B,gamma)=',num2str(ExtIdSinPlus),' ');
strIdCosPlus=strcat(' AF=cos*(A+B,gamma)=',num2str(ExtIdCosPlus));
strRulesP=strcat(' AC=1 ---',strIdSinPlus,strIdCosPlus)
text(0.05,0.9,strRulesP,'Color','blue','FontSize',20)
text(xF/2-0.1,0.03,'cos*(A+B,gamma)', 'Color', 'blue', 'FontSize', 20)
text(xF/2+0.04,yC/3,'sin*(A+B,gamma)', 'Color', 'blue', 'FontSize', 20)

```

```

text(xC/2-0.02,yC/2+0.05,'1','Color','blue','FontSize',20)
% PLOTTING DIFFERENCE RULE
subplot(1,2,2)
disp('_____')
disp('GRAPH NODES FOR PLOTTING EXTENDED DIFFERENCE IDENTITY RULE')
xAm=0, yAm=0 % Point A
xBm=ExtCosineB*cosd(A), yBm=ExtCosineB*sind(A) % Point B
xCm=IdCosMinus, yCm=IdSinMinus % Point C
xDm=ExtCosineA*ExtCosineB, yDm=0 % Point D
xEm=ExtCosineA*ExtCosineB-ExtIdSinMinus*cosd(gamma), yEm=IdSinMinus % Point E
xFm=ExtIdCosMinus, yFm=0 % Point F
NodesM = [ xAm, yAm; xBm, yBm; xCm, yCm; xDm, yDm; xEm, yEm; xFm, yFm];
EdgesM = [1, 2; 1, 3; 1, 4; 2, 3; 2, 4];
gm = drawGraph(NodesM, EdgesM);
Nodes1M = [ 0, 0; xCm, yCm; xFm, yFm];
Edges1M = [1, 2; 1, 3; 2, 3];
g1m = drawGraph(Nodes1M, Edges1M);
Nodes2M = [ xCm, yCm; xEm, yEm]; Edges2M = [1, 2];
g2m = drawGraph(Nodes2M, Edges2M);
set(g1m, 'markerfacecolor', 'g', 'markersize', 50, 'linewidth', 5);
set(g2m, 'linestyle', ':', 'Color', 'blue');
text(0.006,0.04,'A','Color','red','FontSize',26)
text(xBm+0.01,1.03*yBm,'B','Color','red','FontSize',26)
text(xCm+0.01,1.03*yCm,'C','Color','red','FontSize',26)
text(xDm+0.01,0.03,'D','Color','red','FontSize',26)
text(xEm+0.01,1.03*yEm,'E','Color','red','FontSize',26)
text(xFm+0.01,0.03,'F','Color','red','FontSize',26)
text(0.15*xC,0.05*yC, num2str(A),'Color','green','FontSize',26)
text(0.25*xC,0.03*yC, num2str(-B),'Color','green','FontSize',26)
text(xFm-0.08,0.03, num2str(gamma),'Color','magenta','FontSize',26)
if max(xD,xF)>1 top=max(xD,xF), else top=1 end
xlim([0 top]); ylim([0 top]), axis square
title('Extended Angle DIFFERENCE Identity Rule','FontSize',14)
set(gca,'FontSize',20)
strAt=strcat(' A=',num2str(A),'deg');
strBt=strcat(' B=',num2str(-B),'deg');
strAlphaM=strcat(' Alpha=A-B=',num2str(A-B),'deg');
strGammat=strcat(' Gamma=',num2str(gamma),'deg');
text(0.1,0.95,strGammat,'Color','magenta','FontSize',20)
text(0.37,0.95,strAt,'Color','green','FontSize',20)
text(0.53,0.95,strBt,'Color','green','FontSize',20)
text(0.7,0.95,strAlphaM,'Color','red','FontSize',20)
strIdSinMinus=strcat(' FC=sin*(A-B,gamma)=',num2str(ExtIdSinMinus),'—');
strIdCosMinus=strcat(' AF=cos*(A-B,gamma)=',num2str(ExtIdCosMinus));
strRulesM=strcat(' AC=1 —',strIdSinMinus,strIdCosMinus)
text(0.05,0.9,strRulesM,'Color','blue','FontSize',20)
text(xFm/2-0.1,0.03,'cos*(A-B,gamma)','Color','blue','FontSize',20)
text(xFm-0.24,yCm/3+0.01,'sin*(A-B,gamma)','Color','blue','FontSize',20)
text(xCm/2,yCm/2+0.04,'1','Color','blue','FontSize',20)

```

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