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A Monte Carlo Simulation Comparison of Some Nonparametric Survival Functions for Incomplete Data

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Abstract

This article compares a new piecewise exponential estimator (NPEE) of a survival function for censored data with other three famous estimators Kaplan-Meier estimator (KME), Nelson estimator (NE), and an empirical Bayes type estimator (EBE) found in the literature. The NPEE, which is continuous on $[0, \infty)$, retains the spirit of the KME and provides an exponential tail with a hazard rate determined by a novel nonparametric consideration while the other three estimators have limited usage because of their various shortcomings. In our simulation study, we employed absolute bias and relative efficiency as measures of quality of the models. We chose three levels of censoring and two sample sizes and did comparisons at various quantiles. It is found that the NPEE, which is asymptotically equivalent to the KME, is shown to be better than the other three estimators for finite samples.

Keywords: survival function, modified Nelson estimator, Nelson estimator, Kaplan-Meier estimator, empirical Bayes type estimator, censored data

1. Introduction

We consider the problem of a comparison of a new piecewise exponential estimator (NPEE) with the famous nonparametric estimators- the Kaplan-Meier estimator (KME), the Nelson estimator (NE), and the empirical Bayes type estimator (EBE) of a continuous distribution function (DF), F, with the survival function (SF), $S = 1 - F$, under random right censoring. (Malla and Mukerjee, 2010) introduced NPEE as a continuous nonparametric estimator of the SF for censored data on $[0, \infty)$, which also provides a new method of handling the censoring. The (Nelson,1969, 1972) estimator, also called the Nelson-Aalen estimator, of the SF, can be constructed from the NPEE by considering the estimate of the hazard rate equals zero in the first interval, and the estimate of the hazard rate in the i^{th} interval is equal to the estimate of the hazard rate of the NPEE in the $(i-1)^{th}$ interval, $2 \leq i \leq m$, where $m = \#$ of uncensored observations in the data set. The NE may be considered as the smallest step function majoring the piecewise exponential NPEE with jumps at the uncensored observations. (Kaplan and Meier, 1958) introduced a nonparametric estimator of the SF when censored data are present, which is undefined beyond the last observation even in the uncensored case. (Susarla and Van Ryzin,1976, 1978) derived a Bayesian nonparametric estimator in the same setting by using a Dirichlet process prior under squared error loss and analyzed its asymptotic properties. (Rai, Susarla, and Van Ryzin, 1980) compared an empirical Bayes-type version of this estimator with the KME using various types of norms. This case has some similarities with NPEE.

(Malla and Mukerjee, 2010) has shown that NPEE is asymptotically equivalent to the KME. The large sample properties of the NE have been studied by (Breslow and Crowley, 1974), and that of KME have been studied by (Breslow and Crowley,1974), (Meier, 1975), and (Phadia and Van Ryzin, 1980). (Susarla and Van Ryzin, 1976, 1978) have studied the large sample properties of EBE. The summary of the results of these papers is that the pointwise relative asymptotic efficiencies of these estimators are unity. Hence, from a large sample theory point of view, the estimators NPEE, KME, and EBE are equivalent.

In this paper, we examine whether or not the estimator NPEE has any advantages over the other estimators for small finite samples. In a similar study, (Rai, Susarla, and Ryzin, 1980) has shown that the advantages of the EBE increase dramatically as the degree of censoring increases while the relative advantage of the EBE to the KME decreases as the sample size increases. In this paper, we introduce NPEE and compare it by simulation with the totally nonparametric estimators KME, NE, and EBE in two cases: (i) the case when the true survival curve is exponential, and (ii) the case when the true survival curve is Gamma. The simulation comparisons are made by using the absolute bias and the efficiency, small sample sizes of 15 and 30, and for three percentages of censoring (15, 50, and 75 percent) at five quantiles: $q_{.1}$, $q_{.2}$, $q_{.5}$, $q_{.8}$, and $q_{.9}$. Generally speaking, the NPEE, which is defined on all of $[0, \infty)$, and uses more of the information from the censored data, is an improved estimator over the KME and NE, and is better or as good as the EBE. In general terms, the advantages of the NPEE over the KME increase considerably but its advantages over the other two estimators are nominal as the degree of censoring increases.

The estimators are introduced and briefly discussed in Section 2. The details of the simulations and resulting comparisons are given in Section 3. In Section 4, we summarize the results and include some concluding remarks.

2. The Estimators

Let $X_1, X_2, \ldots, X_n$ be a random sample from S and let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a DF, C, with no mass at 0. We observe only $\{(T_i, \delta_i) : 1 \leq i \leq n\}$, $0 < T_1 \leq T_2 \leq \cdots \leq T_n$, where $T_i = \min\{X_i, Y_i\}$ and

$$\delta_i = \begin{cases} 1 & \text{if } X_i \leq Y_i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We assume that the m failures or uncensored observations are at $d_1 < d_2 < \cdots < d_m$ with $d_1 > 0$, and we let $d_0 = 0$ and $d_{m+1} = \infty$. Thus, there are no ties among the uncensored observations, which occur with probability 1. Tied censored observations are ordered arbitrarily among themselves. If there are ties between censored observations and an uncensored one, the uncensored one precedes the censored ones except for the last observation and where the the censored observations precede the uncensored one.

2.1 The New Piecewise Exponential Estimator

From here on we write λ and μ for λ_F and μ_F, respectively. We assume that $\mu < \infty$ and that $\hat{S}^{KM}(T_n) = 0$, which is equivalent to the assumption that the last observation is uncensored and $d_m = T_n$. Let

$$0 \equiv a_0 < a_1 \leq a_2 \leq \cdots \leq a_m \text{ be the jumps of } \hat{S}^{KM} \text{ in magnitude}$$

at $d_0, d_1, d_2, \ldots, d_m$, respectively, and let

$$A_0 = 0 \text{ and } A_k = \textstyle\sum_{i=1}^{k} a_i \text{ for } 1 \leq k \leq m.$$

Note that $\hat{S}^{KM}(t) = 1 - A_{k-1}$ for $d_{k-1} \leq t < d_k,\ 1 \leq k \leq m$.

Our new piecewise exponential estimator, $\hat{S}^{MN}$, of S on $[0, d_m)$ is obtained by defining the naive hazard rate, $\hat{\lambda}$, by

$$\hat{\lambda}(t) = \frac{a_k/(d_k - d_{k-1})}{1 - A_{k-1}}, \quad d_{k-1} \leq t < d_k, \quad 1 \leq k \leq m, \tag{2}$$

and then defining $\hat{S}^{MN}$ on $[0, d_m]$ using this hazard rate:

$$\begin{aligned} \hat{S}^{MN}(t) &= \hat{S}^{MN}(d_{k-1}) e^{-\int_{d_{k-1}}^{t} \frac{a_k}{(1-A_{k-1})(d_k - d_{k-1})} du} \\ &= e^{-\hat{\Lambda}_{k-1}} e^{-\frac{a_k(t-d_{k-1})}{(1-A_{k-1})(d_k-d_{k-1})}}, \quad d_{k-1} \leq t < d_k, \quad 1 \leq k \leq m, \end{aligned} \tag{3}$$

where

$$\hat{\Lambda}_k = \sum_{i=1}^{k} \frac{a_i}{1 - A_{i-1}} = \int_0^{d_k} \hat{\lambda}(u) du, \quad 1 \leq k \leq m, \text{ and } \Lambda_0 \equiv 0. \tag{4}$$

Note that $\hat{S}^{MN}(d_m) > 0$. (Malla and Mukerjee, 2010) showed that

$$\int_0^{d_m} \hat{S}^{MN}(t)\, dt < \hat{\mu}_F^{KM} \equiv \int_0^{d_m} \hat{S}^{KM}(t)\, dt, \text{ the estimator of } \mu \text{ using the KME.} \tag{5}$$

They have extended $\hat{S}^{MN}$ by adding a conditionally exponential tail on $[d_m, \infty)$ so that $\hat{\mu}_F^{MN} \equiv \int_0^{\infty} \hat{S}^{MN}(t)\, dt = \hat{\mu}^{KM}$. Their estimate of S for $t \in [d_m, \infty)$ is $\hat{S}^{MN}(t) = e^{-\hat{\Lambda}_m} e^{-\hat{\lambda}_{tail}(t-d_m)}$ where, $\hat{\lambda}_{tail}$ is given by:

$$\hat{\lambda}_{tail} = e^{-\hat{\Lambda}_m} / \sum_{i=1}^{m} (I_k - J_k), \tag{6}$$

for $1 \leq k \leq m$,

$$\begin{aligned} I_k &= \int_{d_{k-1}}^{d_k} \hat{S}^{KM}(t)\, dt = (1 - A_{k-1})(d_k - d_{k-1}), \text{and} \\ J_k &= \int_{d_{k-1}}^{d_k} \hat{S}^{MN}(t)\, dt = e^{-\hat{\Lambda}_{k-1}} \frac{(1 - A_{k-1})(d_k - d_{k-1})}{a_k} \left[1 - e^{-\frac{a_k}{1-A_{k-1}}}\right]. \end{aligned}$$

2.2 The Kaplan-Meier Estimator

The Kaplan-Meier estimator is a non-parametric statistic that is used to estimate the survival function from lifetime data. It is one of the most frequently used methods both in medical and non-medical fields to examine recovery rates, the probability of death, the effectiveness of treatment, etc. It is based on the idea of conditional probability.

Its mathematical formulation for our setting is as below.

Let $N(t) = \#\{T_i > t\}$. Under the assumption of no ties among the uncensored observations, the Kaplan-Meier (1958) estimator of S is given by

$$\hat{S}^{KM}(t) = \frac{N(t)}{n} \prod_{T_i \le t} \left(\frac{n-i+1}{n-i} \right)^{1-\delta_i}, \quad 0 \le t \le T_n; \tag{7}$$

for $t > T_n$, $\hat{S}^{KM}(t) = 0$ if $d_m = T_n$ and it is undefined if $d_m < T_n$. In the latter case, (Efron, 1967) (among others) suggested the estimator, $\hat{S}^{KM}_E$, by setting the tail estimate to 0 on $t > T_n$, which is equivalent to assuming that the last observation is uncensored whether it is censored or not, i.e., $d_m = T_n$, while (Gill, 1980) suggested the defective estimator, $\hat{S}^{KM}_G$, by setting $\hat{S}^{KM}_G(t) = \hat{S}^{KM}(T_n)$ for $t > T_n$.

2.3 The Nelson Estimator

The (Nelson,1969, 1972) estimator, also called the Nelson-Aalen estimator, is a famous non-parametric estimator of the cumulative hazard function. It is used to estimate the cumulative number of expected events within a certain period of time.

Its mathematical formulation for our setting is as below.

$$\hat{S}^{N}(t) = e^{-\hat{\Lambda}_{k-1}} \quad \text{for} \quad d_{k-1} \le t < d_k,\ 1 \le k \le m+1. \tag{8}$$

Recall that $\hat{\Lambda}_0 = 0$ and $d_{m+1} = \infty$. For the computational purpose, we define Nelson estimator here in terms of the NPEE. The Nelson estimator on $[0, d_m]$ can be obtained by setting the hazard rate to 0 in $[0, d_1)$, and then shifting NPEE's hazard rate in $[d_{k-1}, d_k)$ to $[d_k, d_{k+1})$, $1 \le k \le m-1$. The Nelson estimator uses the tail on $[d_m, \infty)$ using the hazard rate in $[d_{m-1}, d_m)$, ignoring all censored observations beyond d_m. We note here that we did not use our novel estimate of the hazard rate for the tail $[d_m, \infty)$ derived in equation (6) and kept the estimate proposed by (Nelson-Aalen, 1969). The KME tends to overestimate the SF, especially for high censoring. The Nelson estimator is 1 in $[0, d_1]$, but then it's strictly higher than the KME.

2.4 An Empirical Bayes Type Estimator

(Susarla and Van Ryzin, 1976, 1978) derived the following nonparametric Bayes estimate of S using a Dirichlet process prior under squared error loss and analyzed its asymptotic properties:

$$\hat{S}^{B}(t) = \frac{\alpha(t,\infty) + N(t)}{\alpha(0,\infty) + n} \prod_{T_i \le t} \left[\frac{\alpha(T_i,\infty) + (n-i+1)}{\alpha(T_i,\infty) + (n-i)} \right]^{1-\delta_i}, \tag{9}$$

where α is a finite positive measure on $(0, \infty)$ and $N(t) = \#\{T_i > t\}$. The value of $\alpha(0, \infty)$ is chosen to reflect the strength of one's belief in the prior relative to n, and the estimator becomes the KME if $\alpha(0, \infty) = 0$. The case they analyzed specifically was when $\alpha(t, \infty)/\alpha(0, \infty) = e^{-\lambda_0 t}$ for some $\lambda_0 > 0$. For the simulation study, (Rai, Susarla, and Van Ryzin, 1980) has chosen two cases: $\alpha(0, \infty) = \sqrt{n}$ and $\alpha(0, \infty) = n$, in which their empirical Bayes type estimator is mean-squared consistent, and inconsistent respectively. See (Rai, Susarla, and Van Ryzin, 1980) for the detailed discussions of these choices and see the pioneering paper of (Ferguson,1973) for the theory of Dirichlet processes and their applications in nonparametric statistical inference. They have compared their empirical Bayes type versions of the estimator with the KME by extensive simulations and showed uniform improvement over the KME by using three types of norms: L_1 norm, L_2 norm and sup-norm. For our study, we choose their mean-squared consistent empirical Bayes type estimator:

$$\hat{S}^{B}(t) = \frac{\sqrt{n}e^{-\hat{\lambda}t} + N(t)}{\sqrt{n} + n} \prod_{T_i \le t} \left[\frac{\sqrt{n}e^{-\hat{\lambda}T_i} + (n-i+1)}{\sqrt{n}e^{-\hat{\lambda}T_i} + (n-i)} \right]^{1-\delta_i}, \tag{10}$$

where $\hat{\lambda} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{m} T_i}$ is the unique maximum likelihood estimator of λ for the exponential distribution, the prior guess, under censoring (Bartholomew, 1957).

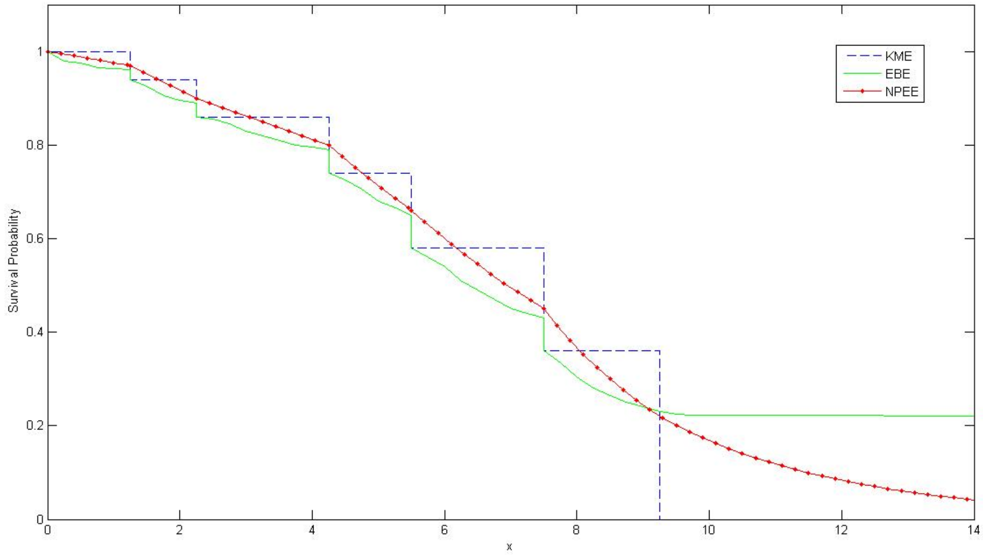

Figure 1. A graphical display of the estimators of a SF for a sample data

3. A Monte Carlo Simulation Study

In this section, we present the results of a simulation study comparing four nonparametric estimators of the survival function S. The first estimator is the new peace-wise exponential estimator (NPEE) which is given by equations (2) to (6). The 2nd estimator is the Kaplan-Meier estimator (KME) given by equation (7). The third estimator is the Nelson estimator (NE), given by (8). The fourth estimator is the empirical Bayes-type estimator given by (9). These estimators are compared by simulation using the measures of the absolute bias and the relative efficiency, the ratio of mean square errors as given below.

Let q_p be the pth, $0 \leq p \leq 1$, quantile of the variable.

1. The Bias: An estimate of the Bias of the estimator $\hat{S}(q_p)$ at the quantile q_p of the survival function $S(q_p)$ for n iterations $= \frac{\sum_{i=1}^{n} \hat{S}(q_p)_i}{n} - S(q_p)$.

2. The Relative Efficiency: The relative efficiency of any estimator $\hat{S}_1(q_p)$ with respect to another estimator $\hat{S}_2(q_p)$ at the quantile q_p for n iterations is given by

$$e(\hat{S}_1(q_p), \hat{S}_2(q_p)) = \frac{\sum_{i=1}^{n}(\hat{S}_1(q_p)_i - S(q_p))^2}{\sum_{i=1}^{n}(\hat{S}_2(q_p)_i - S(q_p))^2}.$$

Clearly, $e(\hat{S}_1(q_p), \hat{S}_2(q_p)) > 1$ implies that the estimator $\hat{S}_2$ is more efficient than the estimator $\hat{S}_1$ at the quantile q_p.

One of the underlying distributions is the exponential distribution with mean survival time = 1, i.e., $(1/\lambda = 1)$. The simulation results for this case are given in Tables 1-3. We note that this is the case in which the estimator NPEE and

Baye's prior guess of the estimator EBE have the correct parametric form. The second set of simulations, Tables 4-6, compares the same four estimators for the case where the underlying distribution is Gamma density $(t/\theta^2)e^{-t/\theta}$ with $\theta = 0.5$, which also has a mean of 1. We again note that this is the case in which the estimator NPEE and Bayes prior guess of the estimator EBE has the incorrect parametric form. The simulations in Tables 1- 6 are for the sample sizes n = 10 and n = 30. The absolute bias of the estimators and the relative efficiency of the NPEE with respect to the other three estimators (NE, KME, and EBE) are calculated at five quantiles namely $q_{.1}$, $q_{.2}$, $q_{.5}$, $q_{.8}$, and $q_{.9}$. To study the effect of censoring, each comparison is carried out at three different percentages of censoring: 15, 50, and 75 percent. The censoring distribution is exponential with parameter λ. By choosing the suitable values of the λ, one can get different censoring percentages for the simulations considered.

Each simulation result calculated for the estimators of the SF used 50,000 samples satisfying the conditions of the table entry. Each entry for the absolute bias gives the mean of these 50,000 absolute biases calculated. The relative efficiency, the ratio of the MSEs, gives the means of the 50,000 estimated values of the two estimators under comparison. Further detailed findings from Tables 1-6 are given in Section 4.

4. Discussion

After a careful examination of Tables 1-6, we come to the following conclusions. First, the continuous new piecewise exponential estimator which is defined on $[0, \infty)$ is better or at least as good as the other estimators both in terms of the absolute bias and relative efficiency in every case simulated. When the underlying distribution is exponential (Tables 1-3), our NPEE is better than EBE, despite the fact that EBEs prior distribution is exponential, KME, and NE except that the NPEE and NE are tied in terms of efficiency. The NE is as good as the NPEE in terms of efficiency but NE is the worst in terms of absolute bias. When the underlying distribution is Gamma distribution, EBE is the most improved estimator among all the estimators considered (Tables 4-6). For this case, the NPEE appears as good as the EBE both in terms of absolute bias and efficiency, but NPEE is better than the KME and the NE except that the NPEE and the NE are again tied in terms of efficiency.

Second, all the estimators considered for this study show improvement(deterioration) with an increase (decrease) in the sample size (censoring percentage). The KME is the most improved (deteriorated) estimator with an increase in the sample size (censoring percent). In other words, the large sample advantages of the KME over other estimators may not show up, particularly with heavy censoring until fairly large samples are taken. The advantages of NPEE over the other two estimators are nominal as the degree of censoring increases.

Based on these simulation findings, we feel that the estimator NPEE is an improved estimator over the KME and NE and is at least as good as the estimator EBE.

Finally, in closing, we remark that it will be interesting to see a comparison between the NPEE and EBE, EBE with a more flexible parametric family as its prior and $\alpha(0, \infty)$ as a function of the observed data. We also plan to show the weak convergence equivalence of the NPEE with the KME and use NPEE in place of the SF in the definition of the mean residual life function (MRL) to derive a continuous estimator of the MRL and study its properties.

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Table 1. Comparison of the Absolute Biases and Efficiencies of the Estimators at the Quantiles (q) for Exponential Survival with Sample Size n, Censoring Percentage = 15%, Based on 50,000 Iterations

q	$\lvert Bias(NPEE)\rvert$		$\lvert Bias(EBE)\rvert$		$\lvert Bias(KME)\rvert$		$\lvert Bias(NE)\rvert$		$\frac{MSE(EBE)}{MSE(NPEE)}$		$\frac{MSE(NE)}{MSE(NPEE)}$		$\frac{MSE(KME)}{MSE(NPEE)}$	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$
0.1	0.0036	0.0028	0.0045	0.0037	0.0057	0.0041	0.0066	0.0056	0.998	0.999	1.010	1.009	1.033	1.025
0.2	0.0032	0.0024	0.0039	0.0033	0.0056	0.0047	0.0062	0.0060	1.008	1.006	1.003	1.004	1.027	1.022
0.5	0.0030	0.0023	0.0036	0.0028	0.0055	0.0042	0.0067	0.0060	1.008	1.003	0.995	1.001	1.026	1.019
0.8	0.0033	0.0024	0.0037	0.0030	0.0054	0.0047	0.0066	0.0061	1.009	0.997	1.000	1.000	1.037	1.030
0.9	0.0036	0.0033	0.0042	0.0037	0.0058	0.0041	0.0067	0.056	1.005	1.007	1.004	1.007	1.040	1.031

Table 2. Comparison of the Absolute Biases and Efficiencies of the Estimators at the Quantiles (q) for Exponential Survival with Sample Size n, Censoring Percentage = 50%, Based on 50, 000 Iterations

q	$\lvert Bias(NPEE)\rvert$		$\lvert Bias(EBE)\rvert$		$\lvert Bias(KME)\rvert$		$\lvert Bias(NE)\rvert$		$\frac{MSE(EBE)}{MSE(NPEE)}$		$\frac{MSE(NE)}{MSE(NPEE)}$		$\frac{MSE(KME)}{MSE(NPEE)}$	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$
0.1	0.0038	0.0036	0.0047	0.0034	0.0060	0.0050	0.0066	0.0053	1.000	0.996	1.011	1.009	1.040	1.024
0.2	0.0034	0.0031	0.0043	0.0038	0.0059	0.0051	0.0067	0.0052	1.008	1.005	1.000	0.999	1.038	1.026
0.5	0.0035	0.0028	0.0038	0.0035	0.0061	0.0047	0.0069	0.0051	1.005	1.000	1.001	0.996	1.036	1.023
0.8	0.0036	0.0030	0.0042	0.0032	0.0056	0.0051	0.0070	0.0062	1.003	1.001	0.998	1.006	1.047	1.036
0.9	0.0039	0.0034	0.0044	0.0040	0.0060	0.0054	0.0065	0.0063	0.996	1.002	1.005	1.007	1.046	1.038

Table 3. Comparison of the Absolute Biases and Efficiencies of the Estimators at the Quantiles (q) for Exponential Survival with Sample Size n, Censoring Percentage = 75%, Based on 50, 000 Iterations

q	$\lvert Bias(NPEE)\rvert$		$\lvert Bias(EBE)\rvert$		$\lvert Bias(KME)\rvert$		$\lvert Bias(NE)\rvert$		$\frac{MSE(EBE)}{MSE(NPEE)}$		$\frac{MSE(NE)}{MSE(NPEE)}$		$\frac{MSE(KME)}{MSE(NPEE)}$	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$
0.1	0.0043	0.0040	0.0047	0.0043	0.0067	0.0052	0.0072	0.0052	1.003	1.003	1.007	1.012	1.054	1.040
0.2	0.0040	0.0036	0.0047	0.0040	0.0064	0.0054	0.0071	0.0055	1.006	1.000	1.002	0.997	1.047	1.041
0.5	0.0043	0.0035	0.0047	0.0042	0.0060	0.0053	0.0067	0.0057	1.000	0.999	1.001	0.998	1.040	1.033
0.8	0.0045	0.0042	0.0049	0.0041	0.0060	0.0055	0.0072	0.0061	1.001	1.005	0.994	1.000	1.049	1.034
0.9	0.0040	0.0040	0.0045	0.0040	0.0066	0.0053	0.0067	0.0061	0.999	1.001	1.009	1.006	1.049	1.042

Table 4. Comparison of the Absolute Biases and Efficiencies of the Estimators at the Quantiles (q) for Gamma Survival with Sample Size n, Censoring Percentage = 15%, Based on 50, 000 Iterations

q	$\lvert Bias(NPEE)\rvert$		$\lvert Bias(EBE)\rvert$		$\lvert Bias(KME)\rvert$		$\lvert Bias(NE)\rvert$		$\frac{MSE(EBE)}{MSE(NPEE)}$		$\frac{MSE(NE)}{MSE(NPEE)}$		$\frac{MSE(KME)}{MSE(NPEE)}$	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$
0.1	0.0053	0.0044	0.0056	0.0047	0.0063	0.0059	0.0072	0.0063	0.999	1.001	1.010	1.013	1.035	1.033
0.2	0.0049	0.0043	0.0049	0.0042	0.0051	0.0059	0.0066	0.0067	1.000	1.003	1.002	1.001	1.033	1.030
0.5	0.0044	0.0042	0.0041	0.0039	0.0051	0.0059	0.0060	0.0057	1.005	1.000	1.000	1.000	1.036	1.037
0.8	0.0043	0.0037	0.0040	0.0040	0.0055	0.0054	0.0062	0.0056	0.999	0.999	1.000	0.998	1.038	1.031
0.9	0.0046	0.0038	0.0048	0.0043	0.0061	0.0040	0.0070	0.0102	1.004	0.999	1.009	1.005	1.044	1.040

Table 5. Comparison of the Absolute Biases and Efficiencies of the Estimators at the Quantiles (q) for Gamma Survival with Sample Size n, Censoring Percentage = 50%, Based on 50, 000 Iterations

q	$\lvert Bias(NPEE)\rvert$		$\lvert Bias(EBE)\rvert$		$\lvert Bias(KME)\rvert$		$\lvert Bias(NE)\rvert$		$\frac{MSE(EBE)}{MSE(NPEE)}$		$\frac{MSE(NE)}{MSE(NPEE)}$		$\frac{MSE(KME)}{MSE(NPEE)}$	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$
0.1	0.0057	0.0050	0.0060	0.0052	0.0067	0.0062	0.0070	0.0067	1.005	1.000	1.010	1.011	1.036	1.034
0.2	0.0056	0.0047	0.0061	0.0050	0.0071	0.0057	0.0072	0.0067	1.000	1.004	0.997	1.004	1.040	1.040
0.5	0.0049	0.0040	0.0048	0.0037	0.0068	0.0058	0.0062	0.0065	1.002	0.999	1.000	0.999	1.025	1.022
0.8	0.0050	0.0042	0.0049	0.0040	0.0073	0.0052	0.0076	0.0062	0.999	1.003	1.000	1.000	1.033	1.025
0.9	0.0056	0.0047	0.0058	0.0045	0.0065	0.0061	0.0071	0.0070	1.000	0.998	1.003	1.008	1.044	1.035

Table 6. Comparison of the Absolute Biases and Efficiencies of the Estimators at the Quantiles (q) for Gamma Survival with Sample Size n, Censoring Percentage = 75%, Based on 50, 000 Iterations

q	$\lvert Bias(NPEE)\rvert$		$\lvert Bias(EBE)\rvert$		$\lvert Bias(KME)\rvert$		$\lvert Bias(NE)\rvert$		$\frac{MSE(EBE)}{MSE(NPEE)}$		$\frac{MSE(NE)}{MSE(NPEE)}$		$\frac{MSE(KME)}{MSE(NPEE)}$	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$	$n = 10$	$n = 30$
0.1	0.0063	0.0058	0.0064	0.0062	0.0073	0.0068	0.0072	0.0071	1.004	1.003	1.008	1.011	1.040	1.034
0.2	0.0059	0.0051	0.0058	0.0050	0.0072	0.0063	0.0077	0.0070	0.997	1.003	0.999	0.994	1.045	1.040
0.5	0.0050	0.0047	0.0052	0.0050	0.0064	0.0064	0.0072	0.0068	1.003	1.000	1.000	0.999	1.035	1.025
0.8	0.0052	0.0055	0.0050	0.0049	0.0067	0.0065	0.0077	0.0072	1.000	0.999	1.003	1.000	1.044	1.039
0.9	0.0050	0.0046	0.0046	0.0042	0.0062	0.0056	0.0073	0.0070	0.999	1.000	1.009	0.999	1.046	1.038

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