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Controllability and Hyers-Ulam Stability of Impulsive Integro-differential Equations in Banach Spaces via Iterative Methods

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Abstract

We investigate Hyers-Ulam stability and controllability of a system governed by impulsive integro-differential equations. Sufficient conditions for controllability and Hyers-Ulam stability are simultaneously established by using iterative methods. Moreover, we provide some examples for demonstrating the main result.

Keywords: Hyers-Ulam stability, controllability, impulsive integro-differential equations, iterative methods

1. Introduction

The Hyers-Ulam stability was originally studied for functional equations. The story began with the Ulam problem that posed at Wisconsin University in 1940 (Ulam, 1940). Hyers presented the first answer to the this problem that lead to the concept of stability for functional equations, (Hyers, 1941). After that, the stability of various functional equations and its applications have been intensively studied by many mathematicians, see (Rassias, 2014). In 1993, Obloza extended the Hyers-Ulam stability to ordinary linear differential equations in the sense of an approximate solution for the exact solution. In other words, a differential equation is stable in Hyers-Ulam sense if for every approximate solution, we can find an exact solution which is close to it, (Obloza, 1993). This is extremely useful in many applications. Thereafter, the stability in this sense of differential equations has been inspected by several mathematicians, for examples see (Choda,Takahasi, & Miura, 2001; Miura, 2002; Jung, 2006; Tripathy, 2021).

Impulsive integro-differential equations in Banach spaces have become important in recent years. They are used to explain the abrupt changes in the behaviour of a system which is extremely sophisticated. It can be said that they are a generalization of initial value problems. Many phenomena in the real-world such as engineering, medicine, control systems, biological models can be modelled by impulsive integro-differential equations, for examples see, (Lakshmikantham, Bainov, & Simeonov, 1989; Paul,& Anguraj, 2006; Castro, & Ramos, 2009 ; Burton, 2005; Jain, Reddy & Kadam, 2018).

In 2018, Kucche and Shikhare established Hyers-Ulam stabilities for the integrodifferential equations

$$\left.\begin{aligned} &x'(t) = Ax(t) + F\left(t, x(t), \int_0^t g(t,s,x(s))ds\right), \quad t \in J,\\ &x(0) = x_0 \in X, \end{aligned}\right\} \tag{1}$$

where X is a Banach space, $J = [0,b]$ or $J = [0,\infty)$, $0 < b < \infty$. A is an infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, $F \in C(J \times X \times X, X)$ and $g \in C(J \times J \times X, X)$.

By using Pachpatte's inequality, they established sufficient conditions for Ulam-Hyers stability of the system (1), (Kucche & Shikhare, 2018).

The concept of controllability of systems governed by abstract impulsive integro-differential equations plays an important role in applied mathematics, physical phenomena, population dynamics and other technical sciences. The controllability problems which consist of a control function that drives the solutions of the system from an initial state to a final state have been explored by many authors, see for instance (Ntouyas & O'Regan 2009; Balachandran & Dauer, 2002; Li, Wang & Zhang, 2006; Kumar & Malik, 2019).

Motivated by the results of Kucche and Shikhare together with the concept of controllability, the author mainly investigates the question of Hyers-Ulam stability and controllability for a system described by the impulsive of integro-differential equations,

$$\left.\begin{aligned} &x'(t) = Ax(t) + B(u(t)) + F\left(t, x(t), \int_0^t g(t,s,x(s))ds\right),\ t \in J' = J - \{t_k\}_{k\in\mathbb{M}},\\ &\triangle x(t_k) = I_k(x(t_k^-)),\ \ k \in \mathbb{M},\\ &x(0) = x_0 \in X, \end{aligned}\right\} \tag{2}$$

where $J = [0, b]$ for some $b > 0$, $\mathbb{M} = \{1, 2, ..., K\}$. X is a Banach space and $A : \mathrm{dom}(A) \subset X \to X$ is an infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. The function $F : J \times X \times X \to X$, $g : J \times J \times X \to X$ are given function that will be specified later. $I_k : X \to X,\ k \in \mathbb{M}$, are impulsive functions. The sequence $\{t_k\}_{k\in\mathbb{M}}$ is a strictly increasing sequence, i.e., $0 < t_k < t_{k+1} \leq b$ for any $k \in \mathbb{M}$. $x(t_k^+) := \lim_{\varepsilon\to 0^+} x(t_k + \varepsilon)$ and $x(t_k^-) := \lim_{\varepsilon\to 0^-} x(t_k + \varepsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, and $\triangle x(t_k) := x(t_k^+) - x(t_k^-)$. U is a Banach space, $B : U \to X$ is a bounded linear operator and a control function $u(\cdot) \in L^2(J, U)$.

Traditionally, the general approaches used to investigate the Hyers-Ulam stability and controllability problems are Gronwall's lemmas and fixed point theorems. However, these methods cannot be adapted to complicated problems such as the system (2). To overcome this problems, the iterative methods are used to establish the Hyers-Ulam stability and controllability results for the system (2). By this approach, we simultaneously obtain the existence of mild solutions, Hyers-Ulam stability results and controllability results which is the main features in this paper.

A brief outline of this paper is given. In section 2, we provide some essential background including notations and the definitions of Hyers-Ulam stability and controllability for the system (2). A main result of the paper are presented in section 3. Moreover, we construct some examples to demonstrate a main theorem.

2. Hyers-Ulam Stability and Controllability

The aim of this section is to given some necessary notations, fundamental definitions and theorems which will be used in this paper. Let X be a Banach space and a corresponding norm on X is denoted by $\|\cdot\|_X$. A space of all bounded linear operators on X is denoted by $\mathcal{B}(X)$ equipped with the norm $\|\cdot\|_{\mathcal{B}(X)}$. We also introduce the space

$PC(J, X) := \Big\{x : J \to X \,\Big|\, x \in C((t_k, t_{k+1}], X), k \in \{0, 1, ..., K\}, t_0 = 0, t_{K+1} = b,\ K \in \mathbb{N}$ and there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k^-) = x(t_k)$ for any $k \in \{1, 2, ..., K\}\Big\}$. Evidently, the space $PC(J, X)$ is the Banach space with the norm $\|x\|_{PC} := \sup\{\|x(t)\|_X : t \in J\}$.

Definition 1.(Pazy, 2012) A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is said to be a **strongly continuous semigroup** (or a C_0−**semigroup**) of operators on X if it satisfies the following conditions

(i) $T(0) = I$, where I is an identity operator on X,

(ii) $T(t + s) = T(t) \circ T(s)$ for $t, s \geq 0$, (called the semigroup property), and

(iii) $\lim_{t\to 0^+} \|T(t)x - x\|_X = 0$ for all $x \in X$, (called the strong continuity at 0).

Definition 2. For a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, the operator $A : \mathrm{dom}(A) \subset X \longrightarrow X$ defined by

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad \text{for all } x \in \mathrm{dom}(A),$$

where $\mathrm{dom}(A) = \left\{x \in X \,\middle|\, \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists}\right\}$ is called an **infinitesimal generator** of $\{T(t)\}_{t\geq 0}$.

Theorem 1.(Mckibben, 2017) Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup. The mapping $t \mapsto \|T(t)\|_{\mathcal{B}(X)}$) is bounded on bounded subsets of $[0, \infty)$. In addition, there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\|_{\mathcal{B}(X)} \leq Me^{\omega t}, \quad \text{for all } t \in [0, \infty).$$

Definition 3. A function $x \in PC(J, X)$ is called a **mild solution** of the system (2) if it satisfies $x(0) = x_0, \triangle x(t_k) = I_k(x(t_k^-)),\ \ k \in \mathbb{M}$, and the following integral equation:

$$x(t) = T(t)x_0 + \int_0^t T(t-s)F\Big(s, x(s), \int_0^s g(s,r,x(r))dr\Big)ds + \int_0^t T(t-s)B(u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k)), \quad t \in J.$$

Definition 4. The system (2) is said to be **exactly controllable** on the interval J, if for the initial state $x_0 \in X$ and arbitrary final state $x_1 \in X$, there exists a control function $u \in L^2(J, U)$, such that the mild solution $x_u(t)$ of the system (2) satisfies

$$x_u(b) = x_1.$$

We introduce the concept of Hyers-Ulam stability for the system (2) which is inspired by Wang et al. (Wang, Feckan & Zhou, 2014). Let $\varepsilon > 0$. We examine the following inequalities

$$\left.\begin{aligned} \left\| y'(t) - Ay(t) - B(u(t)) - F\left(t, y(t), \int_0^t g(t,r,y(r))dr\right)\right\|_X \leq \varepsilon, \;\; t \in J', \\ \left\| \triangle y(t_k) - I_k(y(t_k^-))\right\|_X \leq \varepsilon, \;\; k \in \mathbb{M}, \end{aligned}\right\} \tag{3}$$

where $y \in \Omega := PC(J, X) \cap C(J', \mathrm{dom}(A)) \cap C^1(J', X)$ and $J' := J - \{t_k\}_{k\in\mathbb{M}}$.

Definition 5. The system (2) is called **Hyers-Ulam stable** if there exists a constant $C_{\mathbb{M},F} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in \Omega$ of the inequality (3), there exists a mild solution $x \in PC(J, X)$ of the system (2) that satisfies the following inequality

$$\left\| y(t) - x(t)\right\|_X \leq \varepsilon C_{\mathbb{M},F}, \;\; \text{for all} \;\; t \in J.$$

Remark 1. Consequently from the inequality (3), a function $y \in \Omega$ is a solution of the inequality (3) if and only if there exist a function $h \in C(J', X)$ and a sequence $h_k \in X, \; k \in \mathbb{M}$, such that:

(i) $\|h(t)\|_X \leq \varepsilon$ and $\|h_k\|_X \leq \varepsilon, \; t \in J', \; k \in \mathbb{M}$,

(ii) $y'(t) - Ay(t) - B(u(t)) - F\Big(t, y(t), \displaystyle\int_0^t g(t, s, y(s))ds\Big) = h(t), \;\; t \in J'$,

(iii) $\triangle y(t_k) - I_k\big(y(t_k^-)\big) = h_k, \;\; k \in \mathbb{M}$.

3. Mains Results

To discuss the stability and controllability results of the system (2), the following assumptions are presented:

(A1-i) Let $F \in C(J \times X \times X, X)$ and there exists a constant $L_F > 0$ satisfying

$$\left\| F(t, x_1, x_2) - F(t, y_1, y_2)\right\|_X \leq L_F\Big(\left\|x_1 - y_1\right\|_X + \left\|x_2 - y_2\right\|_X\Big),$$

for each $t \in J$, and $x_1, x_2, y_1, y_2 \in X$.

(A1-ii) Let $g \in C(J \times J \times X, X)$ and there exists a constant $G_g > 0$ satisfying

$$\left\| g(t, s, x_1) - g(t, s, x_2)\right\|_X \leq G_g\left\|x_1 - x_2\right\|_X,$$

for all $t, s \in J$ and $x_1, x_2 \in X$.

(A2) Let $I_k : X \to X$ and there exists a constant $\rho_k > 0$ satisfying

$$\left\| I_k(u) - I_k(v)\right\|_X \leq \rho_k\left\|u - v\right\|_X,$$

for all $u, v \in X, \; k \in \mathbb{M}$.

(A3) The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b T(b - s)B(u(s))ds,$$

induces a bounded inverse operator W^{-1} which takes values in $L^2(J, U)/\ker W$ and $\left\|W^{-1}\right\| \leq L_W$ and $\left\|B\right\| \leq L_B$. The construction of the bounded invertible operator W^{-1} see (Quinn & Carmichael, 1985).

(A4) Under assumptions **(A1-i)**,**(A1-ii)**,**(A2)** and **(A3)**, let

$$\Big(ML_Fb + ML_FG_gb + KM\rho_{\max}\Big)\Big(1 + L_WL_BMb\Big) < 1,$$

where $M = \sup\limits_{t\in J}\left\|T(t)\right\|_{\mathcal{B}(X)}$ and $\rho_{\max} = \max\left\{ \rho_k \;\middle|\; k \in \mathbb{M} \right\}$.

By **Remark 1**, if $y \in \Omega$ is a solution of inequality (3), then there exist $h \in C(J', X)$ and a sequence $h_k \in X,\ k \in \mathbb{M}$, such that $\|h(t)\|_X \le \varepsilon,\ \|h_k\|_X \le \varepsilon$ and

$$y'(t) - Ay(t) - F\Big(t, y(t), \int_0^t g(t,s,y(s))ds\Big) - B(u(t)) = h(t),\quad t \in J',$$
$$\triangle y(t_k) - I_k(y(t_k^-)) = h_k,\quad k \in \mathbb{M}.$$

Setting $y(0) = x_0$ and using properties of a strongly continuous semigroup, (Pazy, 2012), we establish

$$\begin{aligned} y(t) =& T(t)x_0 + \int_0^t T(t-s)F\Big(s, y(s), \int_0^s g(s,r,y(r))dr\Big)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)) \\ &+ \sum_{0<t_k<t} T(t-t_k)g_k + \int_0^t T(t-s)B(u_y(s))ds + \int_0^t T(t-s)g(s)ds. \end{aligned}$$

Let $t \in J$. Define a sequence $\{y_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ by

$$\left.\begin{aligned} y_0(t) =& y(t) = T(t)x_0 + \int_0^t T(t-s)F\Big(s, y(s), \int_0^s g(s,r,y(r))dr\Big)ds + \sum_{0<t_k<t} T(t-t_k)I_i(y(t_k^-)) \\ &+ \sum_{0<t_k<t} T(t-t_k)g_k + \int_0^t T(t-s)B(u_{y_0}(s))ds + \int_0^t T(t-s)g(s)ds \\ y_n(t) =& T(t)x_0 + \int_0^t T(t-s)F\Big(s, y_{n-1}(s), \int_0^s g(s,r,y_{n-1}(r))dr\Big)ds + \int_0^t T(t-s)B(u_{y_n}(s))ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_{n-1}(t_k^-)) + \sum_{0<t_k<t} T(t-t_k)\frac{g_k}{2^n} + \int_0^t T(t-s)\frac{g(s)}{2^n}ds, \end{aligned}\right\} \tag{4}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} u_{y_0}(t) =& W^{-1}\Big[x_1 - T(b)x_0 - \int_0^b T(b-s)F\Big(s, y(s), \int_0^s g(s,r,y(r))dr\Big)ds \\ &- \sum_{0<t_k<t} T(b-t_k)I_k(y(t_k^-)) - \sum_{0<t_k<t} T(b-t_k)g_k - \int_0^b T(b-s)g(s)ds\Big](t) \text{ and} \\ u_{y_n}(t) =& W^{-1}\Big[x_1 - T(b)x_0 - \int_0^b T(b-s)F\Big(s, y_{n-1}(s), \int_0^s g(s,r,y_{n-1}(r))dr\Big)ds \\ &- \sum_{0<t_k<t} T(b-t_k)I_k(y(t_k^-)) - \sum_{0<t_k<t} T(b-t_k)\frac{g_k}{2^n} - \int_0^b T(b-s)\frac{g(s)}{2^n}ds\Big](t), \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Lemma 1. For $n \in \mathbb{N}$. If the assumptions **(A1)-(A3)** are satisfied, then

$$\|u_{y_1}(t) - u_{y_0}(t)\|_U \le \frac{L_W M b\varepsilon}{2} + \frac{K L_W M\varepsilon}{2} \quad \text{and}$$
$$\|u_{y_n}(t) - u_{y_{n-1}}(t)\|_U \le \big(ML_WL_Fb + ML_FL_WG_gb + ML_W\rho_{\max}K\big)\|y_{n-1}(t) - y_{n-2}(t)\|_X + \frac{ML_WK\varepsilon}{2^n} + \frac{ML_Wb\varepsilon}{2^n}, \text{ for all } t \in [0,b].$$

Proof. Let $t \in J$. By direct calculation under the assumptions **(A1)-(A3)**, the following inequality are established

$$\begin{aligned} \|u_{y_1}(t) - u_{y_0}(t)\|_U \le& \|W^{-1}\|\Big[\int_0^t \|T(b-s)\|_{\mathcal{B}(X)}\Big\|\frac{g(s)}{2}\Big\|_X ds + \sum_{i=1}^K \|T(b-t_i)\|_{\mathcal{B}(X)}\Big\|\frac{g_i}{2} - g_i\Big\|_X\Big] \\ \le& \frac{L_WMb\varepsilon}{2} + \frac{KL_WM\varepsilon}{2}, \quad \text{and} \\ \|u_{y_n}(t) - u_{y_{n-1}}(t)\|_U \le& \|W^{-1}\|\Big(\Big\|\int_0^b T(b-s)F\Big(s, y_{n-1}(s), \int_0^s g(s,r,y_{n-1}(r))dr\Big)ds - \int_0^b T(b-s)F\Big(s, y_{n-2}(s), \int_0^s g(s,r,y_{n-2}(r))dr\Big)ds\Big\|_X \\ &+ \Big\|\sum_{i=1}^K T(b-t_i)I_i(y_{n-1}(t_i^-)) - \sum_{i=1}^K T(b-t_i)I_i(y_{n-2}(t_i^-))\Big\|_X + \sum_{i=1}^K \|T(b-t_i)\|_{\mathcal{B}(X)}\Big\|\frac{g_i}{2^n} - \frac{g_i}{2^{n-1}}\Big\|_X \\ &+ \Big\|\int_0^b T(b-s)\frac{g(s)}{2^n}ds - \int_0^b T(b-s)\frac{g(s)}{2^{n-1}}ds\Big\|_X\Big) \\ \le& \big(ML_WL_Fb + ML_FL_WG_gb + ML_W\rho_{\max}K\big)\|y_{n-1}(t) - y_{n-2}(t)\|_X + \frac{ML_WK\varepsilon}{2^n} + \frac{ML_Wb\varepsilon}{2^n}. \end{aligned}$$

Corollary 1. For any $t \in J$, if a sequence $\{y_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in $PC(J, X)$, then $\{u_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in $L^2(J, U)$.

At this moment, we are going to state the main theorem and prove it by using iterative methods under the assumptions **(A1)-(A4)**.

Theorem 2. Suppose that **(A1)-(A4)** are satisfied. Then the system (2) is Hyers-Ulam stable and exactly controllable on the interval J.

Proof. Let $\varepsilon > 0$, $x_1 \in X$ and $y \in \Omega$ be a solution of inequality (3). Setting $y(0) = x_0$ and using properties of strongly continuous semigroup, we establish

$$\begin{aligned} y(t) =& T(t)x_0 + \int_0^t T(t-s)F\Big(s, y(s), \int_0^s g\big(s, r, y_{n-1}(r)\big)dr\Big)ds + \sum_{0<t_k<t} T(t-t_k)I_k\big(y(t_k^-)\big) + \int_0^t T(t-s)B(u_y(s))ds \\ &+ \int_0^t T(t-s)g(s)ds, \qquad t \in J. \end{aligned}$$

Define a sequence $\{y_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ as in (4). We observe that

$$\begin{aligned} \big\|y_1(t)-y_0(t)\big\|_X \leq& \int_0^t \big\|T(t-s)\big\|_{\mathcal{B}(X)}\Big\|\frac{g(s)}{2} - g(s)\Big\|_X ds + \sum_{i=1}^K \|T(t-t_i)\|_{\mathcal{B}(X)}\Big\|\frac{g_i}{2} - g_i\Big\|_X + \int_0^t \big\|T(t-s)\big\|_{\mathcal{B}(X)}\big\|B\big\|\big\|u_{y_1}(s) - u_{y_0}(s)\big\|ds \\ \leq& \Big[\frac{1}{2} + \frac{L_B L_W M b}{2}\Big]Mb\varepsilon + \Big[\frac{1}{2} + \frac{L_W L_B M b}{2}\Big]MK\varepsilon, \\ \big\|y_2(t)-y_1(t)\big\|_X \leq& \int_0^t \big\|T(t-s)\big\|_{\mathcal{B}(X)}\Big\|F\Big(s, y_1(s), \int_0^s g(s,r,y_1(r))dr\Big) - F\Big(s, y_0(s), \int_0^s g(s,r,y_0(r))dr\Big)\Big\|_X ds \\ &+ \sum_{i=1}^K \big\|T(t-t_i)\big\|_{\mathcal{B}(X)}\Big\|I_i\big(y_1(t_i^-)\big) - I_i\big(y_0(t_i^-)\big)\Big\|_X + \int_0^t \big\|T(t-s)\big\|_{\mathcal{B}(X)}\Big\|\frac{g(s)}{2^2} - \frac{g(s)}{2}\Big\|_X ds \\ &+ \sum_{i=1}^K \|T(t-t_i)\|_{\mathcal{B}(X)}\Big\|\frac{g_i}{2^2} - \frac{g_i}{2}\Big\|_X + \int_0^t \big\|T(t-s)\big\|_{\mathcal{B}(X)}\big\|B\big\|\big\|u_{y_2}(s) - u_{y_1}(s)\big\|ds \\ \leq& \Big[\frac{SR^2}{2} + \frac{R}{2^2}\Big]Mb\varepsilon + \Big[\frac{SR^2}{2} + \frac{R}{2^2}\Big]MK\varepsilon, \end{aligned}$$

where $S = ML_F b + ML_F G_g b + KM\rho_{\max}$, $R = 1 + L_B L_W M b$. Moreover, we have

$$\begin{aligned} \big\|y_3(t)-y_2(t)\big\|_X \leq& \Big[\frac{S^2R^3}{2} + \frac{SR^2}{2^2} + \frac{R}{2^3}\Big]Mb\varepsilon + \Big[\frac{S^2R^3}{2} + \frac{SR^2}{2^2} + \frac{R}{2^3}\Big]MK\varepsilon \\ =& R\Big[\frac{(SR)^2}{2} + \frac{SR}{2^2} + \frac{1}{2^3}\Big]Mb\varepsilon + R\Big[\frac{(SR)^2}{2} + \frac{SR}{2^2} + \frac{1}{2^3}\Big]MK\varepsilon \end{aligned}$$

Similarly, we obtain

$$\big\|y_n(t)-y_{n-1}(t)\big\|_X \leq R\Big[\frac{(SR)^{n-1}}{2} + \frac{(SR)^{n-2}}{2^2} + \cdots + \frac{SR}{2^{n-1}} + \frac{1}{2^n}\Big]Mb\varepsilon + R\Big[\frac{(SR)^{n-1}}{2} + \frac{(SR)^{n-2}}{2^2} + \cdots + \frac{SR}{2^{n-1}} + \frac{1}{2^n}\Big]MK\varepsilon,$$

for all $n \in \mathbb{N}$. Let $w_n(t) = y_n(t) - y_{n-1}(t)$ for all $n \in \mathbb{N}, t \in J$. Consider

$$\begin{aligned} \sum_{i=1}^n \big\|w_i(t)\big\|_X =& \big\|y_1(t)-y_0(t)\big\|_X + \big\|y_2(t)-y_1(t)\big\|_X + \cdots + \big\|y_n(t)-y_{n-1}(t)\big\|_X \\ \leq& R\Big[\frac{1}{2}\Big]Mb\varepsilon + R\Big[\frac{1}{2}\Big]MK\varepsilon + R\Big[\frac{SR}{2} + \frac{1}{2^2}\Big]Mb\varepsilon + R\Big[\frac{SR}{2} + \frac{1}{2^2}\Big]MK\varepsilon \\ &+ \cdots + R\Big[\frac{(SR)^{n-1}}{2} + \frac{(SR)^{n-2}}{2^2} + \cdots + \frac{SR}{2^{n-1}} + \frac{1}{2^n}\Big]Mb\varepsilon + R\Big[\frac{(SR)^{n-1}}{2} + \frac{(SR)^{n-2}}{2^2} + \cdots + \frac{SR}{2^{n-1}} + \frac{1}{2^n}\Big]MK\varepsilon \\ \leq& R\Big[1 + SR + \cdots + (SR)^{n-1}\Big]Mb\varepsilon + R\Big[1 + SR + \cdots + (SR)^{n-1}\Big]MK\varepsilon \\ \leq& R\Big[1 + SR + (SR)^2 + \cdots\Big]Mb\varepsilon + R\Big[1 + SR + (SR)^2 + \cdots\Big]MK\varepsilon \\ =& \Big(\frac{RMb + RMK}{1 - SR}\Big)\varepsilon < \infty. \end{aligned}$$

Thus,

$$\sum_{i=1}^n \big\|w_i(t)\big\|_X \leq \frac{\Big(1 + MbL_B L_W\Big)\Big(Mb + MK\Big)}{1 - \Big(ML_F b + ML_F G_g b + KM\rho_{\max}\Big)\Big(1 + L_W L_B M b\Big)}\varepsilon. \tag{5}$$

Then, $\sum_{i=1}^{\infty}\left\|w_i(t)\right\|_X = \lim_{n\to\infty}\sum_{i=1}^{n}\left\|w_i(t)\right\|_X < \infty$. Since the series $\sum_{i=1}^{\infty}\left\|w_i(t)\right\|_X$ converges, the series $\sum_{i=1}^{\infty} w_i(t)$ also converges. Consequently, $\lim_{n\to\infty} w_n(t) = \lim_{n\to\infty}(y_n(t) - y_{n-1}(t)) = 0$, this implies $\{y_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in the space $PC(J, X)$, which is a Banach space. Thus, $\{y_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ is a convergent sequence for any $t \in J$. Define $x(t) = \lim_{n\to\infty} y_n(t)$. In addition, by using **Corollary 1,** we also establish that $\{u_{y_n}(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $L^2(J, U)$ is a Banach space, $\{u_{y_n}(t)\}_{n\in\mathbb{N}}$ is a convergent sequence for $t \in J$. Define $u(t) = \lim_{n\to\infty} u_{y_n}(t)$. Because W^{-1} is linear (W is linear) and bounded, this implies that W^{-1} is continuous. Subsequently,

$$\begin{aligned}
u(t) &= \lim_{n\to\infty} W^{-1}\Big[x_1 - T(b)x_0 - \int_0^b T(b-s)F\Big(s, y_{n-1}(s), \int_0^s g(s,r,y_{n-1}(r))dr\Big)ds - \sum_{0<t_k<t} T(t-t_k)I_k(y_{n-1}(t_k^-)) - \sum_{0<t_k<t} T(t-t_k)\frac{g_k}{2^n}\\
&\quad - \int_0^b T(b-s)\frac{g(s)}{2^n}ds\Big](t)\\
&= W^{-1}\Big[x_1 - T(b)x_0 - \int_0^b T(b-s)F\Big(s, x(s), \int_0^s g(s,r,x(r))dr\Big)ds - \sum_{0<t_k<t} T(t-t_k)I_k\big(x(t_k)\big)\Big](t),\\
x(t) &= \lim_{n\to\infty}\Big(T(t)x_0 + \int_0^t T(t-s)F\Big(s, y_{n-1}(s), \int_0^s g(s,r,y_{n-1}(r))dr\Big)ds\\
&\quad + \int_0^t T(t-s)B(u_{y_n}(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_{n-1}(t_k^-)) + \sum_{0<t_k<t} T(t-t_k)\frac{g_k}{2^n} + \int_0^t T(t-s)\frac{g(s)}{2^n}ds\Big)\\
&= T(t)x_0 + \int_0^t T(t-s)F\Big(s, x(s), \int_0^s g(s,r,x(r))dr\Big)ds + \int_0^t T(t-s)B(u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))
\end{aligned}$$

which is a mild solution of the system (2). The following estimate is followed directly form triangle inequality.

$$\left\|y_n(t) - y_0(t)\right\|_X \le \sum_{i=1}^{n}\left\|w_i(t)\right\|_X.$$

It follows that $\lim_{n\to\infty}\left\|y_n(t) - y_0(t)\right\|_X = \left\|x(t) - y(t)\right\|_X \le \lim_{n\to\infty}\sum_{i=1}^{n}\left\|w_i(t)\right\|_X$. Using (5), we obtain the estimate

$$\left\|x(t) - y(t)\right\|_X \le \frac{\Big(1 + MbL_BL_W\Big)\Big(Mb + MK\Big)}{1 - \Big(ML_Fb + ML_FG_gb + KM\rho_{\max}\Big)\Big(1 + L_WL_BMb\Big)}\varepsilon.$$

Putting $C_F = \dfrac{\Big(1 + MbL_BL_W\Big)\Big(Mb + MK\Big)}{1 - \Big(ML_Fb + ML_FG_gb + KM\rho_{\max}\Big)\Big(1 + L_WL_BMb\Big)}$, then the system (2) is Hyers-Ulam stable.
Moreover, we obtain

$$\begin{aligned}
x(0) &= T(0)x_0 = x_0\\
x(b) &= T(b)x_0 + \int_0^b T(b-s)F\Big(s, x(s), \int_0^s g(s,r,x(r))dr\Big)ds + \int_0^b T(b-s)B(u(s))ds + \sum_{0<t_k<t} T(b-t_k)I_k(x(t_k^-))\\
&= x_1.
\end{aligned}$$

Then, the system (2) is exactly controllable.

Remark 2. The control function $u(t)$ is estimated by the following inequality;

$$\begin{aligned}
\left\|u(t)\right\|_U &\le \left\|W^{-1}\right\|\Big[\ \Big\|x_1 - T(b)x_0 - \sum_{i=1}^{K} T(t-t_i)I_i\big(x(t_i)\big) - \int_0^b T(b-s)F\Big(s, x(s), \int_0^s g(s,r,x(r))dr\Big)ds\Big\|_X\ \Big]\\
&\quad + \int_0^b \left\|T(b-s)\right\|_{B(X)}\Big\|F\Big(t, x(t), \int_0^s g(s,r,x(r))dr\Big)\Big\|_X ds\ \Big]\\
&\le L_W\Big[\|x_1\|_X + M\|x_0\|_X + M\sum_{i=1}^{K}\Big\|I_i\big(x(t_i)\big)\Big\|_X + Mb\Big\|F\Big(t, x(t), \int_0^t g(t,s,x(s))ds\Big)\Big\|_X\ \Big]\\
&\le L_WL_{u(t)},
\end{aligned}$$

where

$$L_{u(t)} = \|x_1\|_X + M\|x_0\|_X + M\sum_{i=1}^{K}\Big\|I_i\big(x(t_i)\big)\Big\|_X + Mb\Big\|F\Big(t, x(t), \int_0^t g(t,s,x(s))ds\Big)\Big\|_X.$$

Moreover, we have

$$\frac{1}{L_W L_{u(t)}} \leq \frac{1}{\|u(t)\|_U}. \tag{6}$$

Example.

This section is designed to demonstrate our main results. Consider the following impulsive partial integro-differential equations

$$\left.\begin{aligned}
\frac{\partial}{\partial t}z(t,x) &= \frac{\partial^2}{\partial x^2}z(t,x) + \frac{4.5+\sin(z(t,x))}{2} + \int_0^t \frac{4.5+\cos(z(s,x))}{2}ds \\
&\quad + (0.5)^{\frac{1}{\mu}}u(t),\ \ 0<x<1,\ t\in[0,1]-\Big\{\frac{1}{2},\frac{1}{4}\Big\}, \\
\triangle z\Big(\frac{1}{2^i},x\Big) &= \frac{1}{2^{i+3}}z\Big(\frac{1}{2^i}^{-},x\Big),\ \ i=1,2, \\
z(0,t) &= z(1,t)=0, \\
z(0,x) &= z_0(x),
\end{aligned}\right\} \tag{7}$$

where $z_0(x)$ is continuous, $u\in L^2([0,1],U)$, $0.001<u(t)<1$ for all $t\in[0,1]$, U is a non-empty set and $U\subset[0,1]$. Moreover,

$$\mu := \inf\Big\{u(t)\ \Big|\ u\in L^2([0,1],U),\ 0.001<u(t)<1\Big\}.$$

First, we are attempting to transform the impulsive partial differential equations (7) into the abstract ordinary differential equation. We identify the terms $J=[0,1],\ J'=[0,1]-\{\frac{1}{2},\frac{1}{4}\},\ K=2,\ I_k(x)=\dfrac{x}{2^{k+3}}$ for $k=1,2$ and

$$F\Big(t,y(t),\int_0^t g(t,s,y(s))ds\Big) = \frac{4.5+\sin(y(t))}{2} + \int_0^t \frac{4.5+\cos(y(s))}{2}ds.$$

Let

$$X = L^2([0,1],\mathbb{R}) := \Big\{f\ \Big|\ f:[0,1]\to\mathbb{R},\ \int_0^1 |f(x)|^2dx<\infty\Big\},$$

and define a corresponding norm on X by $\|f\|_X = \Big(\int_0^1 |f(x)|^2dx\Big)^{\frac{1}{2}}$. Therefore, X is a Banach space. Assume that $z_0(\cdot)\in X$ and identify the solution and an initial condition as follows

$$\begin{aligned}
y(t)[x] &= z(t,x),\ 0<x<1,\ t\in[0,1], \\
y_0[x] &= z(0)[x] = z(0,x) = z_0(x).
\end{aligned}$$

Define the operator $A:\mathrm{dom}(A)\subseteq X\to X$ by $Af = \dfrac{\partial^2 f}{\partial x^2}$ for all $f\in\mathrm{dom}(A)$ with

$$\mathrm{dom}(A) = \Big\{f\in X\ \Big|\ \frac{\partial f}{\partial x},\frac{\partial^2 f}{\partial x^2}\in X,\ f(0)=f(1)=0\Big\}.$$

Let $B:U\to X$ defined by $B(u(t)) = 0.5^{\frac{1}{\mu}}(u(t))$, where $t\in[0,1]$. Evidently, B is the linear operator and

$$\Big|B(u(t))\Big| = \Big|0.5^{\frac{1}{\mu}}(u(t))\Big| \leq (0.5)^{\frac{1}{\mu}}\Big|u(t)\Big|,\ \ \text{for all}\ \ t\in[0,1].$$

Then, the system (7) can be transformed into the abstracted form

$$\left.\begin{aligned}
y'(t) &= Ay(t) + \frac{4.5+\sin(y(t))}{2} + \int_0^t \frac{4.5+\cos(y(s))}{2}ds + B(u(t)),\ t\in[0,1]-\Big\{\frac{1}{2},\frac{1}{4}\Big\}, \\
\triangle y\Big(\frac{1}{2^i}\Big) &= \frac{1}{2^{i+3}}y\Big(\frac{1}{2^i}^{-}\Big),\ \ i=1,2, \\
y(0) &= y_0.
\end{aligned}\right\} \tag{8}$$

We have a strongly continuous semigroup of the operator A (Mckibben, 2017), which is

$$T(t)y = \sum_{m=1}^{\infty} 2e^{-(m\pi)^2 t}\langle y_0(\cdot),\sin(m\pi)\rangle_{L^2([0,1],\mathbb{R})}\sin(m\pi y).$$

Observe that $g:[0,1]\times[0,1]\times X\to X$ defined by $g(t,s,y(s)) = \dfrac{4.5+\cos(y(s))}{2}$.

The following estimate follows form Mean value Theorem.

$$\Big\|g\Big(t,s,y_1(s)\Big) - g\Big(t,s,y_2(s)\Big)\Big\|_X \leq \frac{1}{2}\Big\|y_1(s)-y_2(s)\Big\|_X,$$

for all $s, t \in J,\ y \in X$. Therefore, it satisfies **(A1-ii)** and $G_g = \frac{1}{2}$.

Consider $F : [0,1] \times X \times X \to X$ defined by

$$F\Big(t, y(t), \int_0^t g(t,s,y(s))ds\Big) = \frac{4.5 + \sin(y(t))}{2} + \int_0^t \frac{4.5 + \cos(y(s))}{2} ds.$$

Observe that

$$\big\|F(t,x_1,x_2) - F(t,y_1,y_2)\big\|_X \le \frac{1}{2}\Big(\big\|x_1(t) - y_1(t)\big\|_X + \big\|x_2(s) - y_2(s)\big\|_X\Big),$$

for all $x_1, x_2, y_1, y_2 \in X,\ t \in J$. Thus, it satisfies **(A1-i)** with $L_F = \frac{1}{2}$.

For $k = 1,2$, we have, $I_k : X \to X$ defined by $I_k(y(t)) = \frac{y(t)}{2^{k+3}}$ and

$$\big\|I_k(y_1(t)) - I_k(y_2(t))\big\|_X \le \frac{1}{2^{k+3}}\big\|y_1(t) - y_2(t)\big\|_X,$$

for all $t \in J,\ y \in X$. Thus they satisfy **(A2)** and $\rho_{\max} = \frac{1}{2^4}$.

Take $W : L^2([0,1], U) \to X$ given by

$$Wu = \int_0^1 T(1-s)B(u(s))ds = \int_0^1 \sum_{m=1}^{\infty} 2e^{-(m\pi)^2(1-s)}\Big\langle y_0(\cdot),\ \sin(m\pi)\Big\rangle_{L^2([0,1],\mathbb{R})} \sin(m\pi y)\cdot 0.5^{\frac{1}{\mu}}(u(s))ds$$

and suppose that W^{-1} exists and takes values in $L^2([0,1],U)/\ker W$. Let L_W be a constant satisfying $\|W^{-1}\| \le L_W$. Now, we have $M = 1,\ K = 2,\ \rho_{\max} = \frac{1}{2^4},\ L_F = \frac{1}{2},\ G_g = \frac{1}{2},\ b = 1, L_B = 0.5^{\frac{1}{\mu}}$ and

$$\begin{aligned}
\Big(ML_Fb + ML_FG_gb + KM\rho_{\max}\Big)\Big(1 + L_WL_BMb\Big) &\le \frac{7}{8}\Big(1 + L_W(0.5)^{\frac{1}{|u(t)|}}\Big)\\
&\le \frac{7}{8}\Big(1 + L_W(0.5)^{\frac{1}{L_WL_u}}\Big)\\
&\le \frac{7}{8}\Big(1 + L_W0.5\Big)^{\frac{1}{L_WL_u}}\\
&\le \frac{7}{8}\Big(1 + (\frac{1}{L_WL_u})L_W(0.5)\Big) \text{ (By Bernoulli's inequality)}\\
&= \frac{7}{8} + \frac{3.5}{8L_u},
\end{aligned}$$

where

$$L_u = \|x_1\|_X + M\|x_0\|_X + M\sum_{i=1}^{K}\big\|I_i\big(x(t_i)\big)\big\|_X + Mb\Big\|F\Big(t, x(t), \int_0^t g(t,s,x(s))ds\Big)\Big\|_X$$

and $x_0 \ne x_1$. Observe that

$$\begin{aligned}
L_u &= \|x_1\|_X + M\|x_0\|_X + M\sum_{i=1}^{K}\big\|I_i\big(x(t_i)\big)\big\|_X + Mb\Big\|F\Big(t, x(t), \int_0^t g(t,s,x(s))ds\Big)\Big\|_X\\
&> Mb\Big\|F\Big(t, x(t), \int_0^t g(t,s,x(s))ds\Big)\Big\|_X\\
&= (1)(1)\Big|\frac{4.5 + \sin(y(t))}{2} + \int_0^1 \frac{4.5 + \sin(y(s))}{2} ds\Big|\\
&\ge 3.5\ .
\end{aligned}$$

Thus, $\frac{3.5}{8L_u} < \frac{1}{8}$. This establishes $\Big(ML_Fb + ML_FG_gb + KM\rho_{\max}\Big)\Big(1 + L_WL_BMb\Big) < 1$. Then it satisfies **(A4)**. Applying Theorem, the systems (7) is Hyers-Ulam stable and exactly controllable on the interval $[0,1]$.

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