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Optimal Control for a Degenerate Population Model in Divergence Form With Incomplete Data

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Abstract

In this paper, we study the control of a degenerate population dynamics system in divergence form with unknown information on the boundary. We use the no-regret control concept of J. L. Lions treated in (Lions, 1992) to investigate the problem. At first, we define notions of no-regret control. Using an appropriate Hilbert space, we show that the no-regret control is the limit near the origin of a series of low-regret controls defined by a quadratic perturbation previously used by (Nakoulima, Omrane, & Velin, 2000) corresponding to the disturbed system and for which we give a singular optimality system.

Keywords: population dynamics, degenerate equation, incomplete data, low-regret control, no-regret control

Mathematics subject classification: 35Q93; 49J20; 92D25; 93C05; 93C41

1. Introduction

We consider a degenerate population model in its divergence form with incomplete data:

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y & = & f + v\chi_\omega & \text{in} & Q \\ y(t,a,1) = y(t,a,0) & = & g & \text{on} & Q_{T,A} \\ y(0,a,x) & = & y^0(a,x) & \text{in} & Q_{A,1} \\ y(t,0,x) & = & \displaystyle\int_0^A \beta y da & \text{in} & Q_{T,1} \end{cases} \tag{1}$$

Here $Q = (0,T)\times(0,A)\times(0,1)$, $Q_{T,A} = (0,T)\times(0,A)$, $Q_{A,1} = (0,A)\times(0,1)$, $Q_{T,1} = (0,T)\times(0,1)$, and $Q_\omega = (0,T)\times(0,A)\times\omega$ where the subset $\omega \subset (0,1)$ is the region where a control v is acting. The control v can correspond to a supply of individuals or to a removal of individuals on the subdomain ω. In this model, $y(t,a,x)$ is the distribution of certain individuals at the point $x \in (0,1)$, at time $t \in (0,T)$, where T is fixed, and age $a \in (0,A)$, A being the life expectancy, β and μ denote the natural rates of fertility and mortality, respectively. The formula $\displaystyle\int_0^A \beta y da$ is the proportion of newborns at time t and at location x. In this model, χ_ω is the characteristic function of the control domain $\omega \subset (0,1)$;$y^0 = y^0(a,x) \in L^2(Q_{T,1})$ is the initial distribution of individuals; the data $f \in L^2(Q)$ matches to an external supply. The function g belongs to $G \subset L^2(Q_{T,A})$. We say that (1) is a system with incomplete data because the information on the boundary are missing. Then k is a function of the space variable x which designates the dispersion coefficient. We assume that it degenerates at the boundary of the domain.

In the follow, we define the following notions:

Definition 1.1. *We say that the function k is* ***weakly degenerate (W.D.)*** *if* $k \in W^{1,1}([0,1])$, $k > 0$ *in (0,1) and* $k(0) = k(1) = 0$, *for all* $x \in [0,1]$, *there exists two constants* $M_1, M_2 \in (0,1)$ *such that* $xk'(x) \le M_1 k(x)$ *and* $(x-1)k'(x) \le M_2 k(x)$.

Definition 1.2. *We say that the function k is* ***Strongly degenerate (S.D.)*** *if* $k \in W^{1,\infty}([0,1]), k > 0$ *in (0,1) and* $k(0) = k(1) = 0$, *for all* $x \in [0,1]$, *there exists two constants* $M_1, M_2 \in [1,2)$ *such that* $xk'(x) \le M_1 k(x)$ *and* $(x-1)k'(x) \le M_2 k(x)$.

In recent years, population dynamics models have been widely studied by several authors from many points of view.

The majority of them have investigated the null controllability of the system for example, (Boutaayamou & Echarroudi, 2017), (Fragnelli, 2018), (Fragnelli, 2019), (Fragnelli, 2020). In effect, y can designate the proportion of a pest insect population, for example (He & Ainseba, 2014). Thus it is important to control it. In (He & Ainseba, 2014), the system (1) models insect growth, and the control corresponds to the removal of individuals by using pesticides. Authors (Boutaayamou & Echarroudi, 2017), are concerned with the null controllability of a population dynamics system with an interior degenerate diffusion. To this end, they proved first a new Carleman estimate for the full adjoint system, and afterward, they deduce a suitable observability inequality which will be needed to establish the existence of control acting on a subset of the space which leads the population to extinction in a finite time. (Fragnelli, 2019) and (Fragnelli, 2020) deal with a degenerate system describing the dynamics of a population depending on time, age, and space in divergence form. He assumes that the degeneracy can occur at the boundary or in the interior of the space domain and he focuses on the null controllability problem. To this aim, he proves first Carleman estimates for the associated adjoint problem, then, via cut off functions, he proves the existence of a null control function localized in the interior of the space domain in both papers. In the second one, he considers two cases: either the control region contains the degeneracy point x_0, or it is a reunion union of two domains each located on one side of x_0. Whereas in (Fragnelli, 2018), the same previous research is done but on a degenerate population equation in non-divergence form.

According to the authors, the non-trivial solutions of the system (commonly named LotkaCMcKendrick systems) have exponentially rising or falling asymptotic behavior, depending on the size of a certain biological amount (the so called net reproduction rate), see (Anita, 2000) and also (Fragnelli, Martinez, & Vancostenoble, 2005) for related results concerning time-independent steady states. In (Ainseba & Langlais, 2000), authors consider the optimal control problem for a population dynamics system with age dependence, spatial structure, and a nonlocal birth processus appearing as a boundary condition. They examine the controllability at a given time T and prove that the approximate controllability is valid for any fixed finite time T. Accordingly, they established a new result of condition continuation which is unique.

As much as we know, the first null controllability work for an age population dynamics model is due to (Ainseba & Langlais, 2000), where the authors proved that a set of profiles is approximately reachable. Later, in (Ainseba & Anita, 2004), a local exact controllability was proved. In particular, in (Ainseba & Ianneli, 2003), the authors showed that, if the initial data is sufficiently small, it is possible to find a control that drives the population to extinction. In the last one, the null controllability is also studied for a non-linear model of population dynamics in the diffusive form whenever the fertility and the mortality rates respectively depend on the total population. In (Traore, 2006), the authors considered a nonlinear distribution of newborns of the form $F(\int_0^A \beta(t,a,x)y(t,a,x)da)$.

But, in all the above articles, the dispersion coefficient k is a scalar or a strictly positive function. To our best knowledge, (Ainseba & Ianneli, 2003) is the first paper where the dispersion coefficient, which depends on the space variable x, can degenerate. In particular, the authors assume that k degenerates at the edges (for example $k(x) = x^\alpha$, being $x \in (0,1)$ and $\alpha > 0$). The authors apply Carleman estimates on the adjoint problem and prove a zero controllability result for (1) under the condition $T \geq A$. But, this hypothesis is incorrect when A becomes large enough. To overcome this problem in (Echarroudi & Maniar, 2014), the authors employed Carleman estimates and the fixed point method of Leray-Schauder.

In (Birgit & Omrane, 2010), B.Jacob and A.Omrane are concerned with the optimal control for linear age-structured population dynamics system with incomplete data. More precisely, the initial population age distribution is supposed to be unknown. They used the notion of no-regret control of J.L.Lions in (Lions, 1992) to such singular population dynamics, following the method by Nakoulima et al. as in (Nakoulima, Omrane, & Velin, 2000). They prove that the problem they are considering has a unique no-regret control that they characterize by a singular optimality system.

In the present paper, we are interested with the no-regret control of a degenerate population dynamics system describing a single species in divergence form with unknown information on the boundary which to our knowledge has not been treated. We consider the minimization of the following cost functional:

$$J(v;g) = \|y(v;g) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)} \tag{2}$$

where $z_d \in L^2(Q)$ and $N > 0$ are given. We deal with solving the optimization problem above:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(Q_{T,A})} J(v;g)$$

But noticing that we could have obtained:

$$\sup_{g\in L^2(Q_{T,A})} J(v;g) = +\infty,$$

We consider the next problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(Q_{T,A})} (J(v;g) - J(0;g)) \tag{3}$$

Then we research the control that does not make things worse than a given control v_0 (or to than doing nothing, $v_0 = 0$ in our case), independently of the perturbations which may be of an infinite number. Lions used the notions of Pareto control (Lions, 1986) and equivalently the no-regret control (Lions, 1992) in application to the control of systems having missing data. The no-regret concept was previously used in statistics by Savage (Savage, 1972). The no-regret control over incomplete data problems is not easy to characterize directly. We will use an approximate control: the low-regret control. To achieve the no-regret control, we give the singular optimality system for the low-regret control for the incomplete data population dynamics (1)C(2), using a quadratic perturbation used by Nakoulima et al. in (Nakoulima, Omrane, & Velin, 2000) (see also (Nakoulima, Omrane, & Velin, 2003)). Next, we give a singular optimality system that characterizes the no-regret control that is the limit of a standard control problem.

The paper is organized as follows. In Section 2, we give well-posedness and some regularity results. We study the low-regret and no-regret control and their characterizations in sections 3 and 4 respectively.

2. Well-posedness Result and Preliminaries

In the sequel, we will assume that k satisfies the following hypotheses:

$$k \in C([0;1]) \cap C^1((0;1]) \; ; \; k > 0 \text{ in } (0;1], \; k(0) = 0;$$

there is a constant $M_1 \in [0,1)$ such that $xk'(x) \leq M_1 k(x)$ for all $x \in [0;1]$
In plus, we make the following assumptions about the functions μ and β defined in (1):

- $\beta \in C(\bar{Q}_{A,1})$ and $\beta \geq 0$ in $Q_{A,1}$,
- $\mu \in C(\bar{Q})$ and $\mu \geq 0$ in Q.

To show that the problem (1) is well posed, we need to introduce the following Sobolev spaces:

$$H^1_k = \{u \in L^2(0,1) | u \text{ absolutely continuous in } [0,1], \; \sqrt{k}u_x \in L^2(0,1) \textit{ and } u(1) = u(0) = 0\}$$

and

$$H^2_k = \{u \in H^1_k(0,1) |\; ku_x \in H^1(0,1)\}.$$

with their respective norms:

$$\|u\|^2_{H^1_k(0,1)} = \|u\|^2_{L^2(0,1)} + \|\sqrt{k}u_x\|^2_{L^2(0,1)} \quad \forall u \in H^1_k(0,1)$$

$$\|u\|^2_{H^2_k(0,1)} = \|u\|^2_{H^1_k(0,1)} + \|(ku_x)_x\|^2_{L^2(0,1)} \quad \forall u \in H^2_k(0,1)$$

Let the unbounded operator $A : D(A) = H^2_k(0,1) \to L^2(0,1)$ defined by $Au = (k(x)u_x)_x$, $u \in D(A)$, closed, symmetric, self-adjoint and negative operator and whose domain is dense in $L^2(0,1)$ (Cannarsa, Martinez, & Vancostenoble, 2005). In addition, it generates an analytical semi-group in space $L^2(0,1)$. By setting $L^2(Q) = L^2(0,T;L^2(Q_{A,1}))$ the following result on the existence and uniqueness of the solution of the model (1) holds:

Theorem 2.1. *Assume that k is weakly or strongly degenerated in 0 and/or in 1. For all $f \in L^2(Q)$ and $y_0 \in L^2(Q_{A,1})$, the system* (1) *admits a unique solution*

$$y \in \mathcal{E} = C([0,T];L^2(Q_{A,1})) \cap L^2(0,T;H^1(0,A;H^1_k(0,1))$$

and

$$\sup_{t\in[0,T]} \|y(t)\|^2_{L^2(Q_{A,1})} + \int_0^T \int_0^A \|\sqrt{k}y_x\|^2_{L^2(0,1)} \mathrm{d}a\mathrm{d}t \leq C\|y_0\|^2_{L^2(Q_{A,1})} + C\|f\|^2_{L^2(Q)} \tag{4}$$

where C is a positive constant independent of k, y_0 and f. In addition, if $f \equiv 0$, then $y \in C^1([0,T];L^2(Q_{A,1}))$.

The proof is similar to those in ((Engel & Nagel, 2000), (Fragnelli, 2020), (Lions & Magenes, 1972))

Lemma 1. *for any $y \in W(T,A) = \{y \in L^2((0,T)\times(0,A);H^1_k(0,1))$ such that $\dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} \in L^2((0,T)\times(0,A);H^{-1}_k(0,1)\}$, one can define the trace at $t = t_0$ in $L^2(Q_{A,1})$. One can define also the trace at $a = a_0$ in $L^2(Q_{T,1})$. The applications "trace" are continuous for weak and strong topologies.*

For more details on the latter lemma, see Oumar in [Sur un problme de dynamique de populations(2003)]

Remark 1. *1. The space $H^1_k(0,1)$ is compactly embedded in $L^2(0,1)$. See (Alabau-Boussouira, Cannarsa, & Fragnelli, 2006)*

2. $W(T,A) \subset C([0,T], L^2(Q_{A,1}))$ and $W(T,A) \subset C([0,A], L^2(Q_{T,1}))$. See (Langlais, 1979)

Proposition 2.1. *Let $y = y(v,.)$ be solution of system (1), then the application $v \mapsto y(v,.)$ is continuous from $L^2(Q_\omega)$ to $L^2((0,T)\times(0,A); H^1_k(0,1))$.*

Proof 1. *Let $v_0 \in L^2(Q_\omega)$. And let us show that $\lim_{v\to v_0} y(v,.) = y(v_0,.)$.*
We set $\bar{y} = y(v,.) - y(v_0,.)$, then $\bar{y}$ is solution of :

$$\left\{\begin{array}{lcll} \frac{\partial \bar{y}}{\partial t} + \frac{\partial \bar{y}}{\partial a} - (k(x)\bar{y}_x)_x + \mu\bar{y} & = & (v-v_0)\chi_\omega & \text{in} \quad Q \\ \bar{y}(t,a,1) = \bar{y}(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ \bar{y}(0,a,x) & = & 0 & \text{in} \quad Q_{A,1} \\ \bar{y}(t,0,x) & = & \displaystyle\int_0^A \beta\bar{y}da & \text{in} \quad Q_{T,1} \end{array}\right. \tag{5}$$

If we set $z = e^{-rt}\bar{y}$ with $r > 0$, we get that z is solution of:

$$\left\{\begin{array}{lcll} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - (k(x)z_x)_x + (\mu+r)z & = & (v-v_0)e^{-rt}\chi_\omega & \text{in} \quad Q \\ z(t,a,1) = z(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ z(0,a,x) & = & 0 & \text{in} \quad Q_{A,1} \\ z(t,0,x) & = & \displaystyle\int_0^A \beta z da & \text{in} \quad Q_{T,1} \end{array}\right. \tag{6}$$

Multiply the first equation of (6) by z then integrate by parts on Q, we get:

$$\frac{1}{2}\|z(T,.,.)\|^2_{L^2(Q_{A,1})} - \frac{1}{2}\|z(0,.,.)\|^2_{L^2(Q_{A,1})} + \frac{1}{2}\|z(.,A,.)\|^2_{L^2(Q_{T,1})} - \frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k}z_x\|^2_{L^2(Q)}$$
$$+\|\sqrt{r+\mu}z\|^2_{L^2(Q)} = \int_{Q_\omega} z(v-v_0)e^{-rt}\,\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

It then follows:

$$-\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k}z_x\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z\|^2_{L^2(Q)} \leq \int_{Q_\omega} z(v-v_0)e^{-rt}\,\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

By observing that:$\|z(.,0,.)\|^2_{L^2(Q_{T,1})} = \left\|\int_0^A \beta z da\right\|^2_{L^2(Q_{T,1})}$*, one can obtain obviously:*

$$\|\sqrt{k}z_x\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z\|^2_{L^2(Q)} - \frac{A\beta^2_\infty}{2}\|z\|^2_{L^2(Q)} - \frac{1}{2}\|z\|^2_{L^2(Q)} \leq \frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}$$

with $\beta_\infty = \|\beta\|_{L^\infty(0,A)}$ and $\mu_0 = r + \mu - \frac{1}{2} - \frac{A\beta^2_\infty}{2}$ with $r \geq \frac{1}{2} + \frac{A\beta^2_\infty}{2}$:

we obtain:

$$\|\sqrt{k}z_x\|^2_{L^2(Q)} + \mu_0\|z\|^2_{L^2(Q)} \leq \frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}$$

we can choose μ_0 such that:

$$\|z\|^2_{L^2((0,T)\times(0,A);H^1_k(0,1))} \leq \frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}$$

This means that the map $v \mapsto y(v,.)$ is a lipschitz function on $L^2(Q_\omega)$ onto $L^2((0,T)\times(0,A); H^1_k(0,1))$.

Proposition 2.2. *For all $v \in L^2(Q_\omega)$ the application $v \mapsto S(v,.)$ is continuous on $L^2(Q_\omega)$ onto $L^2(Q_\omega)$ to $L^2((0,T)\times(0,A);H^1_k(0,1))$. where $S = S(v,.)$ is solution of*

$$\begin{cases} -\dfrac{\partial S}{\partial t} - \dfrac{\partial S}{\partial a} - (k(x)S_x)_x + \mu S &= -(y(v,0) - y(0,0)) \quad \text{in} \quad Q \\ S(t,a,1) = S(t,a,0) &= 0 \quad \text{on} \quad Q_{T,A} \\ S(T,a,x) &= 0 \quad \text{in} \quad Q_{A,1} \\ S(t,A,x) &= 0 \quad \text{in} \quad Q_{T,1} \end{cases} \tag{7}$$

Proof 2. *Let be $v_1, v_2 \in L^2(Q_\omega)$ and let be $\bar{S} = S(v_1) - S(v_2)$. Then $\bar{S}$ satifies the system:*

$$\begin{cases} -\dfrac{\partial \bar{S}}{\partial t} - \dfrac{\partial \bar{S}}{\partial a} - (k(x)\bar{S}_x)_x + \mu \bar{S} &= -(y(v_1,.) - y(v_2,.)) \quad \text{in} \quad Q \\ \bar{S}(t,a,1) = \bar{S}(t,a,0) &= 0 \quad \text{on} \quad Q_{T,A} \\ \bar{S}(T,a,x) &= 0 \quad \text{in} \quad Q_{A,1} \\ \bar{S}(t,A,x) &= 0 \quad \text{in} \quad Q_{T,1} \end{cases} \tag{8}$$

If we set $z = e^{-rt}\bar{S}$ with $r > 0$, we get that z is solution of:

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \dfrac{\partial z}{\partial a} - (k(x)z_x)_x + (\mu + r)z &= -(y(v_1,.) - y(v_2,.))e^{-rt} \quad \text{in} \quad Q \\ z(t,a,1) = z(t,a,0) &= 0 \quad \text{on} \quad Q_{T,A} \\ z(0,a,x) &= 0 \quad \text{in} \quad Q_{A,1} \\ z(t,0,x) &= 0 \quad \text{in} \quad Q_{T,1} \end{cases} \tag{9}$$

Multiply the first equation of (9) by z then integrate by parts on Q:

$$\frac{1}{2}\|z(0,.,.)\|^2_{L^2(Q_{A,1})} + \frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k}z_x\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z\|^2_{L^2(Q)} = -\int_Q z(y(v_1,.) - y(v_2,.))e^{-rt}\,\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

Then,

$$\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k}z_x\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z\|^2_{L^2(Q)} \leq -\int_Q z(y(v_1,.) - y(v_2,.))e^{-rt}\,\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

Hence

$$\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k}z_x\|^2_{L^2(Q)} + (r+\mu)\|z\|^2_{L^2(Q)} \leq \frac{1}{2}\|z\|^2_{L^2(Q)} + \frac{1}{2}\|y(v_1,.) - y(v_2,.)\|^2_{L^2(Q)}$$

By setting $r_0 = r + \mu - \frac{1}{2}$

$$\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k}z_x\|^2_{L^2(Q)} + r_0\|z\|^2_{L^2(Q)} \leq \frac{1}{2}\|y(v_1,.) - y(v_2,.)\|^2_{L^2(Q)}$$

we can choose r_0 such that:

$$\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})} + \|z\|^2_{L^2((0,T)\times(0,A);H^1_k(\Omega))} \leq \frac{1}{2}\|y(v_1,.) - y(v_2,.)\|^2_{L^2(Q)}$$

Returning to $z = e^{-rt}\bar{S}$

$$\frac{1}{2}\|\bar{S}(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\bar{S}\|^2_{L^2((0,T)\times(0,A);H^1_k(\Omega))} \leq e^T\|y(v_1,.) - y(v_2,.)\|^2_{L^2(Q)}$$

and consequently

$$\|\bar{S}\|^2_{L^2((0,T)\times(0,A);H^1_k(\Omega))} \leq e^T\|y(v_1,.) - y(v_2,.)\|^2_{L^2(Q)}$$

Using Proposition 2.1, we get that $v \mapsto S(v,.)$ is continuous on $L^2(Q)$ onto $L^2((0,T)\times(0,A);H^1_k(0,1))$

Lemma 2. *For all $v \in L^2(Q)$ and $g \in L^2(Q_{T,A})$, we have:*

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu}d\sigma \tag{10}$$

Proof 3. *Let be $f \in L^2(Q)$, $v \in L^2(Q_w)$, $g \in L^2(Q_{T,A})$, $\mu \in L^\infty(Q)$ and $\beta \in L^\infty(Q_{T,1})$, $k \in W^{1,1}([0,1])$ and $y_0 \in L^2(Q_{A,1})$. Now let be $y(v,0)$, $y(0,g)$ and $y(0,0)$ the respective solutions of systems:*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y & = & f + v\chi_\omega & \text{in} & Q \\ y(t,a,1) = y(t,a,0) & = & 0 & \text{on} & Q_{T,A} \\ y(0,a,x) & = & y^0(a,x) & \text{in} & Q_{A,1} \\ y(t,0,x) & = & \displaystyle\int_0^A \beta y da & \text{in} & Q_{T,1} \end{cases} \tag{11}$$

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y & = & f & \text{in} & Q \\ y(t,a,1) = y(t,a,0) & = & g & \text{on} & Q_{T,A} \\ y(0,a,x) & = & y^0(a,x) & \text{in} & Q_{A,1} \\ y(t,0,x) & = & \displaystyle\int_0^A \beta y da & \text{in} & Q_{T,1} \end{cases} \tag{12}$$

and

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y & = & f & \text{in} & Q \\ y(t,a,1) = y(t,a,0) & = & 0 & \text{on} & Q_{T,A} \\ y(0,a,x) & = & y^0(a,x) & \text{in} & Q_{A,1} \\ y(t,0,x) & = & \displaystyle\int_0^A \beta y da & \text{in} & Q_{T,1} \end{cases} \tag{13}$$

Remember that according to (7) that the functional S is the solution of system:

$$\begin{cases} -\dfrac{\partial S}{\partial t} - \dfrac{\partial S}{\partial a} - (k(x)S_x)_x + \mu S & = & -(y(v,0) - y(0,0)) & \text{in} & Q \\ S(t,a,1) = S(t,a,0) & = & 0 & \text{on} & Q_{T,A} \\ S(T,a,x) & = & 0 & \text{in} & Q_{A,1} \\ S(t,A,x) & = & 0 & \text{in} & Q_{T,1} \end{cases}$$

As a result, as $y(v,0) - y(0,0) \in L^2(Q)$, the problem below admits a unique solution.
Also, noting that $S = S(v) \in L^2(Q_{T,A}, H^2_k(0,1))$, $\dfrac{\partial S}{\partial\nu}$ exists and belongs to $L^2(Q_{T,A})$. Therefore, there is a constant $C > 0$ such that

$$\|\frac{\partial S}{\partial\nu}\|_{L^2(Q_{T,A})} \le C\|y(v,0) - y(0,0)\|_{L^2(Q)}$$

Now multiply the first equation of (7) by $y(0,g) - y(0,0)$ then integrate by parts over Q. We get

$$\int_Q(-\frac{\partial S}{\partial t} - \frac{\partial S}{\partial a} - (k(x)S_x)_x + \mu S)(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t = -\int_Q (y(v,0) - y(0,0))(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

$$\int_Q(-\frac{\partial S}{\partial t} - \frac{\partial S}{\partial a} - (k(x)S_x)_x + \mu S)y(0,g)\mathrm{d}x\mathrm{d}a\mathrm{d}t + \int_Q(-\frac{\partial S}{\partial t} - \frac{\partial S}{\partial a} - (k(x)S_x)_x + \mu S)y(0,0)\mathrm{d}x\mathrm{d}a\mathrm{d}t$$
$$= -\int_Q (y(v,0) - y(0,0))(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

$$\int_Q (\frac{\partial y(0,g)}{\partial t} + \frac{\partial y(0,g)}{\partial a} - (k(x)y_x(0,g))_x + \mu y(0,g))S\,\mathrm{d}x\mathrm{d}a\mathrm{d}t + \int_Q (\frac{\partial y(0,0)}{\partial t} + \frac{\partial y(0,0)}{\partial a}$$
$$-(k(x)y_x(0,0))_x + \mu y(0,0))S\,\mathrm{d}x\mathrm{d}a\mathrm{d}t - \int_{Q_{A,1}} [y(0,g)(T)S(T) - y(0,g)(0)S(0)]\mathrm{d}x\mathrm{d}a$$
$$-\int_{Q_{T,1}} [y(0,g)(A)S(A) - y(0,g)(0)S(0)]\mathrm{d}x\mathrm{d}t - \int_{Q_{T,A}} \frac{\partial(k(x)S)}{\partial\nu} y(0,g)d\sigma$$
$$-\int_{Q_{A,1}} [y(0,0)(T)S(T) - y(0,0)(0)S(0)]\mathrm{d}x\mathrm{d}a - \int_{Q_{T,1}} [y(0,0)(A)S(A) - y(0,0)(0)S(0)]\mathrm{d}x\mathrm{d}t$$
$$-\int_{Q_{T,A}} \frac{\partial(k(x)S)}{\partial\nu} y(0,0)d\sigma = -\int_Q (y(v,0) - y(0,0))(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

Then

$$-\int_{Q_{T,A}} \frac{\partial(k(x)S)}{\partial\nu} y(0,g)d\sigma = -\int_Q (y(v,0) - y(0,0))(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t$$

Finally, we get:

$$\int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu} d\sigma = \int_Q (y(v,0) - y(0,0))(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t \tag{14}$$

On the other hand, noting that: $y(v,g) = y(v,0) + y(0,g) - y(0,0)$*, we get:*

$$\begin{aligned}
J(v,g) &= \|y(v,g) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)} \\
&= \|y(v,0) + y(0,g) - y(0,0) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)} \\
&= \|y(v,0) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)} + \|y(0,g) - y(0,0)\|^2_{L^2(Q)} \\
&+ 2\int_Q (y(v,0) - z_d)(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&= \|y(v,0) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)} + \|y(0,g) - z_d\|^2_{L^2(Q)} + \|y(0,0) - z_d\|^2_{L^2(Q)} \\
&- 2\int_Q (y(0,g) - z_d)(y(0,0) - z_d)\mathrm{d}x\mathrm{d}a\mathrm{d}t + 2\int_Q (y(v,0) - z_d)(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&= J(v,0) + J(0,g) + J(0,0) - 2\int_Q (y(0,g) - z_d)(y(0,0) - z_d)\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&+ 2\int_Q (y(v,0) - z_d)(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&= J(v,0) + J(0,g) + J(0,0) - 2\int_Q (y(0,g) - y(0,0))(y(0,0) - z_d)\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&- 2\int_Q (y(0,0) - z_d)(y(0,0) - z_d)\mathrm{d}x\mathrm{d}a\mathrm{d}t + 2\int_Q (y(v,0) - z_d)(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&= J(v,0) + J(0,g) + J(0,0) - 2J(0,0) - 2\int_Q (y(0,g) - y(0,0))(y(0,0) - z_d)\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&+ 2\int_Q (y(v,0) - z_d)(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t \\
&= J(v,0) + J(0,g) - J(0,0) + 2\int_Q (y(v,0) - y(0,0))(y(0,g) - y(0,0))\mathrm{d}x\mathrm{d}a\mathrm{d}t
\end{aligned}$$

and using equality (14)*, we get:*

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu} d\sigma \tag{15}$$

Remark 2. *The problem* (14) *has a meaning if the expression* $\int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu} d\sigma$ *is bounded in* $L^2(Q_{T,A})$*.*

Then, using (14), the expression (3) becomes :

$$\inf_{v\in\mathcal{U}} \left(J(v,0) - J(0,0) + 2\sup_{g\in L^2(Q_{T,A})} \int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu} d\sigma \right) \tag{16}$$

We can get:

$$\sup_{g\in L^2(Q_{T,A})}\int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu}d\sigma = +\infty \tag{17}$$

or

$$\sup_{g\in L^2(Q_{T,A})}\int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu}d\sigma = 0 \tag{18}$$

The problem (16) admits a solution only in the case (18). And the control v is choosen in suitable space $\mathcal{U}_{ad}$ subset convex close non empty of $L^2(Q_\omega)$ defined by:

$$\mathcal{U}_{ad} = \{v \in L^2(Q_\omega), \text{such that} \int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu}d\sigma = 0, \forall g \in L^2(Q_{T,A})\} \tag{19}$$

As such control is not easy to characterize, we consider the following low-regret control problem:

$$\inf_{v\in\mathcal{U}} \quad \left(J(v,0) - J(0,0) + 2\sup_{g\in L^2(Q_{T,A})}\int_{Q_{T,A}} g\frac{\partial(k(x)S)}{\partial\nu}d\sigma - \gamma\|g\|^2_{L^2(Q_{T,A})}\right), \quad \forall\gamma > 0 \tag{20}$$

Using the Legendre-Fenchel transformation, problem (20) is equivalent to solving:

$$\inf_{v\in\mathcal{U}} \qquad J^\gamma(v) \tag{21}$$

with $\quad J^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|\frac{\partial(k(x)S)}{\partial\nu}\|^2_{L^2(Q_{T,A})}$

3. Existence and Characterization of Low-regret Control

In this section, we propose an existence result for the family of low-regret controls. Then we give the singular optimality system allowing us to characterize it.

Proposition 3.1. *There exists a unique low-regret control u_γ solution of problem* (21).

Proof 4. *The proof uses Propositions 2.1 and 2.2 to show that the functional J^γ is continuous, on the one hand, and the strict convexity of J^γ, on the other hand, to show the uniqueness of the solution of the problem* (21). *Therefore, the sequence $y(v_n,.)$ weakly converges to $y(u_\gamma,.)$ in $L^2(Q)$. The sequence $S(v_n,.)$ weakly converges to $S(u_\gamma,.)$ in $L^2(Q)$. By continuity of the trace application, the sequence $\frac{\partial(k(x)S(v_n,.))}{\partial\nu}$ weakly converges toward $\frac{\partial(k(x)S(u_\gamma,.))}{\partial\nu}$ in $L^2(Q_{T,A})$. Therefore,*

$$J^\gamma(u_\gamma) \le \inf_{n\in\mathbf{N}} J^\gamma(v_n) = \inf_{n\in\mathbf{N}}(J(v,0) - J(0,0) + \frac{1}{\gamma}\|\frac{\partial(k(x)S(v_n,.))}{\partial\nu}\|^2_{L^2(Q_{T,A})}) = d_\gamma$$

Thus u_γ is a solution to the problem (21).

We now turn to the characterization of low-regret control u_γ.

Proposition 3.2. *Let $u_\gamma \in L^2(Q_\omega)$ be the solution of the problem* (21). *Then there exists $p_\gamma \in L^2((0,T)\times(0,A); H^1_k(0,1))$*

and $q_\gamma \in L^2((0,T)\times(0,A); H_k^1(0,1))$ *such as the quadruplet* $\{y_\gamma, \xi_\gamma, p_\gamma, q_\gamma\}$ *be solution of systems:*

$$\begin{cases} \dfrac{\partial y_\gamma}{\partial t} + \dfrac{\partial y_\gamma}{\partial a} - (k(x)y_{\gamma x})_x + \mu y_\gamma &= f + u_\gamma\chi_\omega & \text{in} & Q \\ y_\gamma(t,a,1) = y_\gamma(t,a,0) &= 0 & \text{on} & Q_{T,A} \\ y_\gamma(0,a,x) &= y^0(a,x) & \text{in} & Q_{A,1} \\ y_\gamma(t,0,x) &= 0 & \text{in} & Q_{T,1} \end{cases} \tag{22}$$

$$\begin{cases} -\dfrac{\partial \xi_\gamma}{\partial t} - \dfrac{\partial \xi_\gamma}{\partial a} - (k(x)\xi_{\gamma x})_x + \mu \xi_\gamma &= y_\gamma - z_d & \text{in} & Q \\ \xi_\gamma(t,a,1) = \xi_\gamma(t,a,0) &= 0 & \text{on} & Q_{T,A} \\ \xi_\gamma(0,a,x) &= 0 & \text{in} & Q_{A,1} \\ \xi_\gamma(t,0,x) &= 0 & \text{in} & Q_{T,1} \end{cases} \tag{23}$$

$$\begin{cases} \dfrac{\partial p_\gamma}{\partial t} + \dfrac{\partial p_\gamma}{\partial a} - (k(x)p_{\gamma x})_x + \mu p_\gamma &= 0 & \text{in} & Q \\ p_\gamma(t,a,1) = p_\gamma(t,a,0) &= \dfrac{1}{\sqrt{\gamma}}\dfrac{\partial(k(x)S(u_\gamma,.))}{\partial \nu} & \text{on} & Q_{T,A} \\ p_\gamma(0,a,x) &= 0 & \text{in} & Q_{A,1} \\ p_\gamma(t,0,x) &= 0 & \text{in} & Q_{T,1} \end{cases} \tag{24}$$

$$\begin{cases} -\dfrac{\partial q_\gamma}{\partial t} - \dfrac{\partial q_\gamma}{\partial a} - (k(x)q_{\gamma x})_x + \mu q_\gamma &= y_\gamma + \frac{1}{\sqrt{\gamma}}p_\gamma - z_d & \text{in} & Q \\ q_\gamma(t,a,1) = q_\gamma(t,a,0) &= 0 & \text{on} & Q_{T,A} \\ q_\gamma(0,a,x) &= 0 & \text{in} & Q_{A,1} \\ q_\gamma(t,0,x) &= 0 & \text{in} & Q_{T,1} \end{cases} \tag{25}$$

and

$$N u_\gamma + q_\gamma = 0 \quad dans \quad Q_\omega \tag{26}$$

Proof 5. *The optimality condition of Euler-Lagrange which characterizes the low-regret control* u_γ *is given by:*

$$\lim_{\lambda\to 0}\frac{J^\gamma(u_\gamma+\lambda v) - J^\gamma(u_\gamma)}{\lambda} = 0, \qquad \forall v \in L^2(Q_\omega)$$

After some calculations, we get the relation:

$$\int_Q (y(u_\gamma,0) - z_d)y(v,0) + \frac{1}{\gamma}\int_{Q_{T,A}} \frac{\partial(k(x)\xi(u_\gamma,.))}{\partial \nu}\frac{\partial(k(x)\xi(v,.))}{\partial \nu} + N\int_{Q_\omega} u_\gamma v = 0, \quad \forall v \in L^2(Q_\omega) \tag{27}$$

where $y(v,0) = y_v$ *and* $\xi(v,0) = \xi_v$ *are respective solutions of:*

$$\begin{cases} \dfrac{\partial y_v}{\partial t} + \dfrac{\partial y_v}{\partial a} - (k(x)y_{vx})_x + \mu y_v &= v\chi_\omega & \text{in} & Q \\ y_v(t,a,1) = y_v(t,a,0) &= 0 & \text{on} & Q_{T,A} \\ y_v(0,a,x) &= 0 & \text{in} & Q_{A,1} \\ y_v(t,0,x) &= 0 & \text{in} & Q_{T,1} \end{cases} \tag{28}$$

and

$$\begin{cases} -\dfrac{\partial \xi_v}{\partial t} - \dfrac{\partial \xi_v}{\partial a} - (k(x)\xi_{vx})_x + \mu \xi_v &= -y(v,0) & \text{in} & Q \\ \xi_v(t,a,1) = \xi_v(t,a,0) &= 0 & \text{in} & Q_{T,A} \\ \xi_v(0,a,x) &= 0 & \text{in} & Q_{A,1} \\ \xi_v(t,0,x) &= 0 & \text{in} & Q_{T,1} \end{cases} \tag{29}$$

Multiply the first equation of (28) *by* q_γ *solution of* (25) *then integrate by parts on* Q*. We get:*

$$\int_Q y_v\left(y_\gamma + \frac{1}{\sqrt{\gamma}}p_\gamma - z_d\right) = \int_{Q_\omega} q_\gamma v, \quad \forall v \in L^2(Q_\omega) \tag{30}$$

Multiply the first equation of (29) *by* $\frac{1}{\sqrt{\gamma}}p_\gamma$ *and integrate by parts on* Q*. We get:*

$$\int_{Q_{T,A}} \frac{\partial(k(x)\xi_{vx})}{\partial \nu}p_\gamma = \int_Q y_v p_\gamma \tag{31}$$

Reduce the relation (27) *by using relations* (30) *and* (31)*, we get:*

$$\int_{Q_\omega} q_\gamma v + N \int_{Q_\omega} u_\gamma v = 0, \quad \forall v \in L^2(Q_\omega) \tag{32}$$

And we find the relation (26)*:*

$$N u_\gamma + q_\gamma = 0 \quad in \quad Q_\omega \tag{33}$$

4. Existence and Characterization of the No-regret Control

In this section, we give an existence result for the no-regret control as a limit of the family of low-regret controls in the neighborhood of the origin. Then we establish the singular optimality system allowing us to characterize it.

Proposition 4.1. *The low-regret control u_γ converges toward the no-regret control u in $L^2(Q_\omega)$.*

Proof 6. *As u_γ is solution of* (21)*, then*

$$\begin{aligned} J^\gamma(u_\gamma) \leq J^\gamma(v) \Longleftrightarrow \quad & J(u_\gamma, 0) - J(0,0) + \frac{1}{\gamma} \| \frac{\partial(k(x)\xi(u_\gamma, .))}{\partial \nu} \|^2_{L^2(Q_{T,A})} \leq J(v,0) - J(0,0) \\ & + \frac{1}{\gamma} \| \frac{\partial(k(x)\xi(v, .))}{\partial \nu} \|^2_{L^2(Q_{T,A})} \end{aligned}$$

In particular, for $v = 0$, we have:

$$J(u_\gamma, 0) - J(0,0) + \frac{1}{\gamma} \| \frac{\partial(k(x)\xi(u_\gamma, .))}{\partial \nu} \|^2_{L^2(Q_{T,A})} \leq 0$$

The structure of $J(u_\gamma, 0)$ provides us:

$$\|y_\gamma - z_d\|^2_{L^2(Q)} + N\|u_\gamma\|^2_{L^2(Q_\omega)} + \frac{1}{\gamma} \| \frac{\partial(k(x)\xi(u_\gamma, .))}{\partial \nu} \|^2_{L^2(Q_{T,A})} \leq J(0,0)$$

Thus, we deduce the following estimates:

$$\|u_\gamma\|_{L^2(Q_\omega)} \leq C \tag{34}$$

$$\|y_\gamma - z_d\|_{L^2(Q)} \leq C \tag{35}$$

$$\frac{1}{\gamma} \| \frac{\partial(k(x)\xi(u_\gamma, .))}{\partial \nu} \|^2_{L^2(Q_{T,A})} \leq C \tag{36}$$

Using the relation (35) *and the fact that y_γ is solution of* (22)*, we deduce:*

$$\|y_\gamma\|_{L^2((0,T)\times(0,A);H^1_k(0,1))} \leq C \tag{37}$$

where C is a positive constant. From the relations (35) *and* (37)*, we deduce that there exists $u \in L^2(Q_\omega)$ and $y \in L^2((0,T)\times(0,A); H^1_k(0,1))$ such that:*

$$u_\gamma \rightharpoonup u \qquad weakly\ in \quad L^2(Q_\omega) \tag{38}$$

$$y_\gamma \rightharpoonup y \qquad weakly\ in \quad L^2((0,T)\times(0,A); H^1_k(0,1)) \tag{39}$$

For the rest of the demonstration, we proceed by steps:
Step 1:*Verify that (u, y) is solution of* (22)
Let be $\varphi \in C^\infty(Q)$ with $\varphi = 0$ on $Q_{T,A}$, $\varphi(T, ., .) = 0$ in $Q_{A,1}$ and $\varphi(., A, .) = 0$ in $Q_{T,1}$. Multiply the first equation of (22) *by φ then integrate by parts over Q:*

$$\int_Q \varphi \left(\frac{\partial y_\gamma}{\partial t} + \frac{\partial y_\gamma}{\partial a} - (k(x) y_{\gamma x})_x + \mu y_\gamma \right) = \int_Q \varphi (f + u_\gamma \chi_\omega)$$

We get:

$$\int_Q y_\gamma \left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu \varphi \right) = \int_{Q_\omega} \varphi (f + u_\gamma)$$

by passing to the limit when $\gamma \to 0$, we get:

$$\int_Q y\left(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) = \int_{Q_\omega} \varphi(f+u)$$

We integrate by parts the last relation on Q and we get:

$$\int_Q \varphi\left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y\right) = \int_{Q_\omega} \varphi(f+u)$$

Thus

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y = f + u\chi_\omega \quad a.e.\ sur\ Q \tag{40}$$

Due the fact that $y \in L^2((0,T)\times(0,A); H^1_k(0,1))$, then $y(t,a)\mid_{Q_{T,A}}$ exists and belongs to $L^2(Q_{T,A})$. In addition, using relation (40), we deduce that $\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} \in L^2((0,T)\times(0,A); H^{-1}_k(0,1))$. Hence $y \in W(T,A)$. Thanks to Lemma 1, the trace $y(0,.,.)$ and $y(.,0,.)$ exists and belongs respectively to $L^2(Q_{A,1})$ and $L^2(Q_{T,1})$.

Now, consider $\varphi \in C^\infty(Q)$ with $\varphi = 0$ on $Q_{T,A}$, $\varphi(T,.,.) = 0$ in $Q_{A,1}$ and $\varphi(.,A,.) = 0$ in $Q_{T,1}$. Multiply the first equation of (22) by φ then integrate by parts on Q:

$$\begin{aligned}
\int_Q \varphi(f + u_\gamma\chi_\omega) &= \int_Q \varphi\left(\frac{\partial y_\gamma}{\partial t} + \frac{\partial y_\gamma}{\partial a} - (k(x)y_{\gamma x})_x + \mu y_\gamma\right)\\
&= \int_Q y_\gamma\left(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) + \int_{Q_{T,1}} \left(\varphi(A)y_\gamma(A) - \varphi(0)y_\gamma(0)\right)\\
&+ \int_{Q_{A,1}} (\varphi(T)y_\gamma(T) - \varphi(0)y_\gamma(0)) + \int_{Q_{T,A}} \left(\frac{\partial(k(x)y_\gamma)}{\partial\nu}\right) - \int_{Q_{T,A}} \frac{\partial(k(x)\varphi)}{\partial\nu}
\end{aligned}$$

by taking into account the conditions at the boundary and/or limits, it comes:

$$\int_Q \varphi(f + u_\gamma\chi_\omega) = \int_Q y_\gamma\left(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) - \int_{Q_{A,1}} \varphi(0,.,.)y_\gamma(0,.,.)$$

passing to limits when $\gamma \to 0$, we obtain:

$$\int_Q \varphi(f + u\chi_\omega) = \int_Q y\left(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) - \int_{Q_{A,1}} \varphi(0,.,.)y_\gamma(0,.,.)$$

We integrate this last equality by parts on Q:

$$\begin{aligned}
\int_Q \varphi(f + u\chi_\omega) &= \int_Q \varphi\left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y\right) + \int_{Q_{T,1}} (\varphi(A)y(A) - \varphi(0)y(0))\\
&+ \int_{Q_{A,1}} (\varphi(T)y(T) - \varphi(0)y(0)) + \int_{Q_{T,A}} \left(\frac{\partial(k(x)y)}{\partial\nu}\right) - \int_{Q_{T,A}} \frac{\partial(k(x)\varphi)}{\partial\nu} - \int_{Q_{A,1}} y^0\varphi(0,.,.)\\
&\forall\varphi \in C^\infty(Q) \text{ such that } \varphi = 0 \text{ on } Q_{T,A},\ \varphi(T,.,.) = 0 \text{ in } Q_{A,1} \text{ and } \varphi(.,A,.) = 0 \text{ in } Q_{T,1}.
\end{aligned}$$

by taking into account the conditions at the boundary and/or limits, it comes:

$$\begin{aligned}
\int_Q \varphi(f + u\chi_\omega) &= \int_Q \varphi\left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y\right) + \int_{Q_{T,1}} \varphi(.,0,.)y(.,0,.) + \int_{Q_{A,1}} \varphi(0,.,.)y(0,.,.)\\
&- \int_{Q_{A,1}} y^0\varphi(0,.,.) - \int_{Q_{T,A}} y\frac{\partial(k(x)\varphi)}{\partial\nu}\\
&= \int_Q \varphi\left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y\right) + \int_{Q_{T,1}} \varphi(.,0,.)y(.,0,.) - \int_{Q_{T,A}} y\frac{\partial(k(x)\varphi)}{\partial\nu}\\
&- \int_{Q_{A,1}} (y^0 - y(0,.,.))\varphi(0,.,.)\\
0 &= \int_{Q_{T,1}} \varphi(.,0,.)y(.,0,.) - \int_{Q_{A,1}} (y^0 - y(0,.,.))\varphi(0,.,.) - \int_{Q_{T,A}} y\frac{\partial(k(x)\varphi)}{\partial\nu}\\
&\forall\varphi \in C^\infty(Q) \text{ such that } \varphi = 0 \text{ on } Q_{T,A},\ \varphi(T,.,.) = 0 \text{ in } Q_{A,1} \text{ and } \varphi(.,A,.) = 0 \text{ in } Q_{T,1}.
\end{aligned}$$

Then

$$y(0, ., .) = y^0 \text{ in } Q_{A,1} \tag{41}$$
$$y(., 0, .) = 0 \text{ in } Q_{T,1} \tag{42}$$
$$y\mid_{Q_{T,A}} = 0 \tag{43}$$

By combining (40), (41), (42) *and* (43), *we deduce that* $y = y(u, 0)$ *is solution of*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y &= f + u\chi_\omega & \text{in} & Q \\ y(t, a, 1) = y(t, a, 0) &= 0 & \text{on} & Q_{T,A} \\ y(0, a, x) &= y^0(a, x) & \text{in} & Q_{A,1} \\ y(t, 0, x) &= 0 & \text{in} & Q_{T,1} \end{cases} \tag{44}$$

***Step 2** : show that* $\xi = \xi(u, .)$ *is solution of* (23)

Let $\psi \in C^\infty(Q)$ *such that* $\psi = 0$ *on* $Q_{T,A}$ *and* $\psi(0, ., .) = 0$ *in* $Q_{A,1}$. *Multiply the first equation of* (23) *by* ψ *then integrate by parts on Q:*

$$\int_Q \psi\left(-\frac{\partial \xi_\gamma}{\partial t} - \frac{\partial \xi_\gamma}{\partial a} - (k(x)\xi_{\gamma x})_x + \mu\xi_\gamma\right) = \int_Q \psi(y_\gamma - z_d)$$

it comes:

$$\int_Q \xi_\gamma\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right) = \int_Q \psi(y_\gamma - z_d)$$

passing to the limit when $\gamma \to 0$, *we obtain:*

$$\int_Q \xi\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right) = \int_Q \psi(y - z_d)$$

We integrate this last equality by parts on Q:

$$\int_Q \psi\left(-\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial a} - (k(x)\xi_x)_x + \mu\xi\right) = \int_Q \psi(y - z_d)$$

we deduce that:

$$-\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial a} - (k(x)\xi_x)_x + \mu\xi = y - z_d \tag{45}$$

As $\xi \in L^2((0, T) \times (0, A); H^1_k(0, 1))$, *then* $\psi(t, a)\mid_{Q_{T,A}}$ *exists and belongs to* $L^2(Q_{T,A})$ *a.e.*$(t, a) \in (0, T) \times (0, A)$.*On the other hand, using* (45) *in more,* $\dfrac{\partial \psi}{\partial t} + \dfrac{\partial \psi}{\partial a} \in L^2((0, T) \times (0, A); H^{-1}_k(0, 1))$. *Hence* $\psi \in W(T, A)$. *It follows from Lemma 1 that the trace* $\psi(T, ., .)$ *exists and belongs to* $L^2(Q_{A,1})$ *and the traces* $\psi(., A, .)$, $\psi(., 0, .)$ *exist and belong to* $L^2(Q_{T,1})$.

Let be $\psi \in C^\infty(Q)$ *knowing that* $\psi = 0$ *on* $Q_{T,A}$ *and* $\psi(0, ., .) = 0$ *in* $Q_{A,1}$. *Multiply the first equation of* (23) *by* ψ *then integrate by parts on Q:*

$$\begin{aligned} \int_Q \psi(y_\gamma - z_d) &= \int_Q \psi\left(-\frac{\partial \xi_\gamma}{\partial t} - \frac{\partial \xi_\gamma}{\partial a} - (k(x)\xi_{\gamma x})_x + \mu\xi_\gamma\right) \\ &= \int_Q \xi_\gamma\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right) - \int_{Q_{T,1}} \Big(\psi(A)\xi_\gamma(A) - \psi(0)\xi_\gamma(0)\Big) \\ &\quad - \int_{Q_{A,1}} (\psi(T)\xi_\gamma(T) - \psi(0)\xi_\gamma(0)) + \int_{Q_{T,A}} \psi\frac{\partial(k(x)\xi_\gamma)}{\partial \nu} - \int_{Q_{T,A}} \xi_\gamma\frac{\partial(k(x)\psi)}{\partial \nu} \end{aligned}$$

which gives, by taking into account the boundary and/or limits conditions:

$$\int_Q \psi(y_\gamma - z_d) = \int_Q \xi_\gamma\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right) + \int_{Q_{T,1}} \psi(., 0, .)\xi_\gamma(., 0, .)$$

passing to the limit when $\gamma \to 0$, we have:

$$\int_Q \psi(y - z_d) = \int_Q \xi\left(\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right) + \int_{Q_{T,1}} \tau\psi(.,0,.)$$

We integrate this last equality by parts on Q:

$$\int_Q \psi(y - z_d) = \int_Q \psi\left(-\frac{\partial\xi}{\partial t} - \frac{\partial\xi}{\partial a} - (k(x)\xi_x)_x + \mu\xi\right) + \int_{Q_{T,1}} (\psi(A)\xi(A) - \psi(0)\xi(0))$$
$$+ \int_{Q_{A,1}} (\psi(T)\xi(T) - \psi(0)\xi(0)) + \int_{Q_{T,A}} \psi\frac{\partial(k(x)\xi)}{\partial\nu} - \int_{Q_{T,A}} \xi\frac{\partial(k(x)\psi)}{\partial\nu} + \int_{Q_{T,1}} \tau\psi(.,0,.).$$

$\forall\psi \in C^\infty(Q)$ *such that* $\psi = 0$ *on* $Q_{T,A}$ *and* $\psi(0,.,.) = 0$ *in* $Q_{A,1}$

.

by taking into account the boundary and/or limits conditions, it comes:

$$\int_Q \psi(y - z_d) = \int_Q \psi\left(-\frac{\partial\xi}{\partial t} - \frac{\partial\xi}{\partial a} - (k(x)\xi_x)_x + \mu\xi\right) + \int_{Q_{T,1}} \psi(.,A,.)\xi(.,A,.) + \int_{Q_{A,1}} \psi(T,.,.)\xi(T,.,.)$$
$$+ \int_{Q_{T,1}} \tau\psi(.,0,.) - \int_{Q_{T,1}} \psi(.,0,.)\xi(.,0,.) - \int_{Q_{T,A}} \xi\frac{\partial(k(x)\psi)}{\partial\nu}$$
$$0 = \int_{Q_{T,1}} \psi(.,A,.)\xi(.,A,.) + \int_{Q_{A,1}} \psi(T,.,.)\xi(T,.,.) - \int_{Q_{T,A}} \xi\frac{\partial(k(x)\psi)}{\partial\nu}$$
$$- \int_{Q_{T,1}} (\xi(.,0,.) - \tau)\psi(.,0,.)$$

$\forall\psi \in C^\infty(Q)$ *such that* $\psi = 0$ *on* $Q_{T,A}$ *and* $\psi(0,.,.) = 0$ *in* $Q_{A,1}$

.

We obtain:

$$\xi(.,0,.) = \tau \text{ in } Q_{T,1} \tag{46}$$
$$\xi\,|_{Q_{T,A}} = 0 \tag{47}$$
$$\xi(.,A,.) = 0 \text{ in } Q_{T,1} \tag{48}$$
$$\xi(T,.,.) = 0 \text{ in } Q_{A,1} \tag{49}$$

By combining (45), (47), (48) *and* (49), *we find that* $\xi = \xi(u,0)$ *is solution of*

$$\begin{cases} -\frac{\partial\xi}{\partial t} - \frac{\partial\xi}{\partial a} - (k(x)\xi_x)_x + \mu\xi & = y - z_d & \text{in } Q \\ \xi(t,a,1) = \xi(t,a,0) & = 0 & \text{on } Q_{T,A} \\ \xi(T,a,x) & = 0 & \text{in } Q_{A,1} \\ \xi(t,A,x) & = 0 & \text{in } Q_{T,1} \end{cases} \tag{50}$$

Now, using relation (38) *and proposition (2.2), we deduce that:*

$$\xi(u_\gamma,.) \rightharpoonup \xi(u,.) \quad weakly\ in \quad L^2(Q_{T,A}) \tag{51}$$

If in addition, we exploit the relation (36), *we get that:*

$$\xi(u_\gamma,0) \to 0 \quad strongly\ in \quad L^2(Q_{T,A}) \tag{52}$$

From the uniqueness of the limit, we conclude that:

$$\xi(u_\gamma,.) \to \xi(u,.) \quad weakly\ in \quad L^2(Q_{T,A}) \tag{53}$$

Using the continuity of the trace application:

$$\frac{\partial(k(x)\xi(u_\gamma,.))}{\partial\nu} \to \frac{\partial(k(x)\xi(u,.))}{\partial\nu} = 0 \tag{54}$$

Consequently,

$$\int_{Q_{T,A}} g\frac{\partial(k(x)\xi(u_\gamma,.))}{\partial\nu}\mathrm{d}\sigma \to \int_{Q_{T,A}} g\frac{\partial(k(x)\xi(u,.))}{\partial\nu}\mathrm{d}\sigma = 0 \tag{55}$$

Thus $u \in \mathcal{U}_{ad}$. *The strict convexity of* J^γ *allows to deduce that* u *is the unique control, solution of* (21).

We conclude that the low-regret control u_γ *converge in* $L^2(Q_\omega)$ *toward the unique no-regret control* u

In what follows, we will try to characterize the unique no-regret control u.

Proposition 4.2. *The no-regret control* u *solution of problem* (3) *is characterized by the quadruplet* $\{y, \xi, p, q\}$ *solution of optimality system :*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - (k(x)y_x)_x + \mu y = f + u\chi_\omega & \text{in } Q \\ y(t,a,1) = y(t,a,0) = 0 & \text{on } Q_{T,A} \\ y(0,a,x) = y^0(a,x) & \text{in } Q_{A,1} \\ y(t,0,x) = 0 & \text{in } Q_{T,1} \end{cases} \tag{56}$$

$$\begin{cases} -\dfrac{\partial \xi}{\partial t} - \dfrac{\partial \xi}{\partial a} - (k(x)\xi_x)_x + \mu \xi = y - z_d & \text{in } Q \\ \xi(t,a,1) = \xi(t,a,0) = 0 & \text{on } Q_{T,A} \\ \xi(T,a,x) = 0 & \text{in } Q_{A,1} \\ \xi(t,A,x) = 0 & \text{in } Q_{T,1} \end{cases} \tag{57}$$

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - (k(x)p_x)_x + \mu p = 0 & \text{in } Q \\ p(t,a,1) = p(t,a,0) = k_1 & \text{on } Q_{T,A} \\ p(0,a,x) = 0 & \text{in } Q_{A,1} \\ p(t,0,x) = 0 & \text{in } Q_{T,1} \end{cases} \tag{58}$$

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - (k(x)q_x)_x + \mu q = y - z_d + k_2 & \text{in } Q \\ q(t,a,1) = q(t,a,0) = 0 & \text{on } Q_{T,A} \\ q(T,a,x) = 0 & \text{in } Q_{A,1} \\ q(t,A,x) = 0 & \text{in } Q_{T,1} \end{cases} \tag{59}$$

and

$$Nu + q = 0 \quad in \quad Q_\omega \tag{60}$$

where $k_1 = \lim_{\gamma\to 0} \dfrac{1}{\sqrt{\gamma}}\dfrac{\partial(k(x)\xi(u,.))}{\partial\nu}$ *and* $k_2 = \lim_{\gamma\to 0}\dfrac{1}{\sqrt{\gamma}}p_\gamma$

Proof 7. *The results* (56) *and* (57) *have been demonstrated during the proof of the previous proposition. It remains to be demonstrated* (58)-(60). *For this, we will proceed by steps:*

Step 1: *show that* p *is solution of* (58).

Using relation (36) *on the one hand, we deduce that there exists a constant* $C > 0$ *such that:*

$$\left\|\frac{1}{\sqrt{\gamma}}\frac{\partial(k(x)\xi(u_\gamma,.))}{\partial\nu}\right\|_{L^2(Q_{T,A})} \le C \tag{61}$$

On the other hand, noting that $p_\gamma = p(u_\gamma,.)$ *satisfies system* (24), *we deduce that there exists an other constant* $C > 0$ *such that:*

$$\|p_\gamma\|_{L^2(Q)} \le C \tag{62}$$

With regard to estimates (61) *and* (62), *we deduce that there exist* $k_1 \in L^2(Q_{T,A})$ *and* $p \in L^2(Q)$ *such that:*

$$\frac{1}{\sqrt{\gamma}}\frac{\partial(k(x)\xi(u_\gamma,.))}{\partial\nu} \rightharpoonup k_1 \qquad weakly\ in \quad L^2(Q_{T,A}) \tag{63}$$

$$p_\gamma \rightharpoonup p \qquad weakly\ in \quad L^2(Q) \tag{64}$$

Now multiply the first equation of (24) *by a test function* $\varphi \in C^{\infty}(Q)$ *such that* $\varphi = 0$ *on* $Q_{T,A}$, $\varphi(T, ., .) = 0$ *in* $Q_{A,1}$ *and* $\varphi(., A, .) = 0$ *in* $Q_{T,1}$. *Then we integrate by parts on Q:*

$$\begin{aligned} 0 &= \int_Q \varphi\left(\frac{\partial p_\gamma}{\partial t} + \frac{\partial p_\gamma}{\partial a} - (k(x)p_{\gamma x})_x + \mu p_\gamma\right) \\ &= \int_Q p_\gamma\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) \end{aligned}$$

passing to the limit when $\gamma \to 0$ *in the last equation and using* (64)*, we obtain:*

$$0 = \int_Q p\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right)$$

We integrate this last equality by parts on Q:

$$0 = \int_Q \varphi\left(\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - (k(x)p_x)_x + \mu p\right)$$

So we deduce that:

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - (k(x)p_x)_x + \mu p = 0 \qquad a.e. \text{ in } \quad Q \tag{65}$$

Then, $p \in L^2((0, T) \times (0, A); H^1_k(0, 1))$ *implies* $p(t, a)\,|_{Q_{T,A}}$ *exists* $a.e.(t, a) \in (0, T) \times (0, A)$ *and belongs to* $L^2(Q_{T,A})$. *Using* (65) *on the other hand, we deduce that* $p \in L^2((0, T) \times (0, A); H^1_k(0, 1))$ *involves* $\dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} \in L^2((0, T) \times (0, A); H^{-1}_k(0, 1))$. *Which implies* $p \in W(T, A)$. *Thus using Lemma 1, the traces* $p(., 0, .)$ *and* $p(0, ., .)$ *exist and belong respectively to* $L^2(Q_{T,1})$ *and to* $L^2(Q_{A,1})$.

Consider now $\varphi \in C^{\infty}(Q)$ *such that* $\varphi = 0$ *on* $Q_{T,A}$, $\varphi(T, ., .) = 0$ *in* $Q_{A,1}$ *and* $\varphi(., A, .) = 0$ *in* $Q_{T,1}$. *Multiply the first equation of* (24) *by* φ *then integrate by parts on Q:*

$$\begin{aligned} 0 &= \int_Q \varphi\left(\frac{\partial p_\gamma}{\partial t} + \frac{\partial p_\gamma}{\partial a} - (k(x)p_{\gamma x})_x + \mu p_\gamma\right) \\ &= \int_Q p_\gamma\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) + \int_{Q_{T,1}} \left(\varphi(A)p_\gamma(A) - \varphi(0)p_\gamma(0)\right) \\ &+ \int_{Q_{A,1}} (\varphi(T)p_\gamma(T) - \varphi(0)p_\gamma(0)) + \int_{Q_{T,A}} \varphi\frac{\partial(k(x)p_\gamma)}{\partial \nu} - \int_{Q_{T,A}} p_\gamma\frac{\partial(k(x)\varphi)}{\partial \nu} \end{aligned}$$

which gives, by taking into account the boundary and/or limits conditions:

$$0 = \int_Q p_\gamma\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) - \int_{Q_{T,A}} \frac{1}{\sqrt{\gamma}}\frac{\partial(k(x)\xi(u_\gamma, .))}{\partial \nu}\frac{\partial(k(x)\varphi)}{\partial \nu}$$

passing to the limit in this last equality when $\gamma \to 0$ *then using* (63) *and* (64)*, it comes:*

$$0 = \int_Q p\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right) - \int_{Q_{T,A}} k_1\frac{\partial(k(x)\varphi)}{\partial \nu}$$

We integrate this last equality by parts on Q:

$$\begin{aligned} 0 &= \int_Q \varphi\left(\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - (k(x)p_x)_x + \mu p\right) - \int_{Q_{T,1}} (\varphi(A)p(A) - \varphi(0)p(0)) \\ &- \int_{Q_{A,1}} (\varphi(T)p_\gamma(T) - \varphi(0)p_\gamma(0)) + \int_{Q_{T,A}} p\frac{\partial(k(x)\varphi)}{\partial \nu} - \int_{Q_{T,A}} \varphi\frac{\partial(k(x)p)}{\partial \nu} - \int_{Q_{T,A}} k_1\frac{\partial(k(x)\varphi)}{\partial \nu} \end{aligned}$$

by taking into account the boundary and/or limits conditions:

$$0 = \int_{Q_{T,1}} \varphi(.,0,.)p(.,0,.) + \int_{Q_{A,1}} \varphi(0,.,.)p(0,.,.) + \int_{Q_{T,A}} (p-k_1)\frac{\partial(k(x)\varphi)}{\partial\nu}$$

$\forall\varphi \in C^\infty(Q)$ *such that* $\varphi = 0$ *on* $Q_{T,A}$, $\varphi(T,.,.) = 0$ *in* $Q_{A,1}$ *and* $\varphi(.,A,.) = 0$ *in* $Q_{T,1}$.

We obtain:

$$p(.,0,.) = 0 \text{ in } Q_{T,1} \tag{66}$$

$$p(0,.,.) = 0 \text{ in } Q_{A,1} \tag{67}$$

$$p = k_1 \text{ on } Q_{T,A} \tag{68}$$

By combining (65), (66), (67) *and* (68), *we deduce that p is solution of system*

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - (k(x)p_x)_x + \mu p & = & 0 & \text{in} & Q \\ p(t,a,1) = p(t,a,0) & = & k_1 & \text{on} & Q_{T,A} \\ p(0,a,x) & = & 0 & \text{in} & Q_{A,1} \\ p(t,0,x) & = & 0 & \text{in} & Q_{T,1} \end{cases} \tag{69}$$

Step 2: *show that* q_γ *converge toward q solution of* (59)
Let q_1^γ *and* q_2^γ *be respective solutions of:*

$$\begin{cases} -\dfrac{\partial q_1^\gamma}{\partial t} - \dfrac{\partial q_1^\gamma}{\partial a} - (k(x)q_{1x}^\gamma)_x + \mu q_1^\gamma & = & y_\gamma - z_d & \text{in} & Q \\ q_1^\gamma(t,a,1) = q_1^\gamma(t,a,0) & = & 0 & \text{on} & Q_{T,A} \\ q_1^\gamma(T,a,x) & = & 0 & \text{in} & Q_{A,1} \\ q_1^\gamma(t,A,x) & = & 0 & \text{in} & Q_{T,1} \end{cases} \tag{70}$$

and

$$\begin{cases} -\dfrac{\partial q_2^\gamma}{\partial t} - \dfrac{\partial q_2^\gamma}{\partial a} - (k(x)q_{2x}^\gamma)_x + \mu q_2^\gamma & = & \frac{1}{\sqrt{\gamma}}p_\gamma & \text{in} & Q \\ q_2^\gamma(t,a,1) = q_2^\gamma(t,a,0) & = & k_1 & \text{on} & Q_{T,A} \\ q_2^\gamma(T,a,x) & = & 0 & \text{in} & Q_{A,1} \\ q_2^\gamma(t,A,x) & = & 0 & \text{in} & Q_{T,1} \end{cases} \tag{71}$$

Then $q^\gamma = q_1^\gamma + q_2^\gamma$ *is solution of* (25).
According to the estimate (36), *there exists constant* $C > 0$ *such that:*

$$\|q_1^\gamma\|_{L^2((0,T)\times(0,A);H_k^1(0,1))} \le C \tag{72}$$

So, there exists $q_1 \in L^2((0,T)\times(0,A);H_k^1(0,1))$ *such that:*

$$q_1^\gamma \rightharpoonup q_1 \quad \textit{weakly in} \quad L^2((0,T)\times(0,A);H_k^1(0,1)) \tag{73}$$

Now let $\varphi_1 \in C^\infty(Q)$ *such that* $\varphi_1 = 0$ *on* $Q_{T,A}$ *and* $\varphi_1(0,.,.) = 0$ *in* $Q_{A,1}$. *Multiply the first equation of* (70) *by* φ_1 *then integrate by parts on Q:*

$$\int_Q \varphi_1\left(-\frac{\partial q_1^\gamma}{\partial t} - \frac{\partial q_1^\gamma}{\partial a} - (k(x)q_{1x}^\gamma)_x + \mu q_1^\gamma\right) = \int_Q \varphi_1(y_\gamma - z_d)$$

we obtain:

$$\int_Q q_1^\gamma\left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial a} - (k(x)\varphi_{1x})_x + \mu\varphi_1\right) = \int_Q \varphi_1(y_\gamma - z_d)$$

passing to the limit when $\gamma \to 0$ *in the last equality and using, we get:*

$$\int_Q q_1\left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial a} - (k(x)\varphi_{1x})_x + \mu\varphi_1\right) = \int_Q \varphi_1(y - z_d)$$

We integrate this last equality by parts on Q:

$$\int_Q \varphi_1\left(-\frac{\partial q_1}{\partial t}-\frac{\partial q_1}{\partial a}-(k(x)q_{1x})_x+\mu q_1\right)=\int_Q \varphi_1(y-z_d)$$

we deduce that:

$$-\frac{\partial q_1}{\partial t}-\frac{\partial q_1}{\partial a}-(k(x)q_{1x})_x+\mu q_1=y-z_d \tag{74}$$

As $q_1\in L^2((0,T)\times(0,A);H^1_k(0,1))$*, then* $q_1(t,a)\mid_{Q_{T,A}}$ *exists and belongs to* $L^2(Q_{T,A})$ *a.e.*$(t,a)\in(0,T)\times(0,A)$*. On the other hand, relation* (74) *and the fact that* $q_1\in L^2((0,T)\times(0,A);H^1_k(0,1))$ *implie that* $-\frac{\partial q_1}{\partial t}-\frac{\partial q_1}{\partial a}\in L^2((0,T)\times(0,A);H^{-1}_k(0,1))$*. Whence* $q_1\in W(T,A)$*. Then according Lemma 1, the traces* $q_1(T,.,.)$ *and* $q_1(.,A,.)$ *exist and belong respectively to* $L^2(Q_{A,1})$*; and to* $L^2(Q_{T,1})$*. Consider* $\varphi_1\in C^\infty(Q)$ *knowing that* $\varphi_1=0$ *on* $Q_{T,A}$ *and* $\varphi_1(0,.,.)=0$ *on* $Q_{A,1}$*. Multiply the first equation of* (70) *by* φ_1 *then integrate by parts on Q:*

$$\begin{aligned}\int_Q \varphi_1(y_\gamma-z_d)&=\int_Q \varphi_1\left(-\frac{\partial q_1^\gamma}{\partial t}-\frac{\partial q_1^\gamma}{\partial a}-(k(x)q_{1x}^\gamma)_x+\mu q_1^\gamma\right)\\&=\int_Q q_1^\gamma\left(\frac{\partial \varphi_1}{\partial t}+\frac{\partial \varphi_1}{\partial a}-(k(x)\varphi_{1x})_x+\mu\varphi_1\right)-\int_{Q_{T,1}}\left(\varphi_1(A)q_1^\gamma(A)-\varphi_1(0)q_1^\gamma(0)\right)\\&-\int_{Q_{A,1}}(\varphi_1(T)q_1^\gamma(T)-\varphi_1(0)q_1^\gamma(0))+\int_{Q_{T,A}}\varphi_1\frac{\partial(k(x)q_1^\gamma)}{\partial\nu}-\int_{Q_{T,A}}q_1^\gamma\frac{\partial(k(x)\varphi_1)}{\partial\nu}\end{aligned}$$

by taking into account the boundary and/or limits conditions:

$$\int_Q \varphi_1(y_\gamma-z_d)=\int_Q q_1^\gamma\left(\frac{\partial \varphi_1}{\partial t}+\frac{\partial \varphi_1}{\partial a}-(k(x)\varphi_{1x})_x+\mu\varphi_1\right)+\int_{Q_{T,1}}\varphi_1(.,0,.)q_1^\gamma(.,0,.)$$

passing to the limit when $\gamma\to0$ *in the last equality and taking into account* (73) *and* (39)*, we get:*

$$\int_Q \varphi_1(y-z_d)=\int_Q q_1\left(\frac{\partial \varphi_1}{\partial t}+\frac{\partial \varphi_1}{\partial a}-(k(x)\varphi_{1x})_x+\mu\varphi_1\right)+\int_{Q_{T,1}}\tau\varphi_1(.,0,.)$$

If we integrate this last equality by parts over Q, we get:

$$\begin{aligned}\int_Q \varphi_1(y-z_d)&=\int_Q \varphi_1\left(-\frac{\partial q_1}{\partial t}-\frac{\partial q_1}{\partial a}-(k(x)q_{1x})_x+\mu q_1\right)+\int_{Q_{T,1}}(\varphi_1(A)q_1(A)-\varphi_1(0)q_1(0))\\&+\int_{Q_{A,1}}(\varphi_1(T)q_1(T)-\varphi_1(0)q_1(0))+\int_{Q_{T,A}}\varphi_1\frac{\partial(k(x)q_1)}{\partial\nu}-\int_{Q_{T,A}}q_1\frac{\partial(k(x)\varphi_1)}{\partial\nu}\\&+\int_{Q_{T,1}}\tau\varphi_1(.,0,.)\\&\forall\varphi_1\in C^\infty(Q)\text{ such that }\varphi_1=0\text{ on }Q_{T,A}\text{ and }\varphi_1(0,.,.)=0\text{ in }Q_{A,1}.\end{aligned}$$

by taking into account the boundary and/or limits conditions:

$$\begin{aligned}\int_Q \varphi_1(y-z_d)&=\int_Q \varphi_1\left(-\frac{\partial q_1}{\partial t}-\frac{\partial q_1}{\partial a}-(k(x)q_{1x})_x+\mu q_1\right)+\int_{Q_{T,1}}(\tau-q_1(.,0,.))\varphi_1(.,0,.)\\&+\int_{Q_{T,1}}q_1(.,A,.)\varphi_1(.,A,.)+\int_{Q_{A,1}}q_1(T,.,.)\varphi_1(T,.,.)-\int_{Q_{T,A}}q_1\frac{\partial(k(x)\varphi_1)}{\partial\nu}\\0&=\int_{Q_{T,1}}(\tau-q_1(.,0,.))\varphi_1(.,0,.)+\int_{Q_{T,1}}q_1(.,A,.)\varphi_1(.,A,.)+\int_{Q_{A,1}}q_1(T,.,.)\varphi_1(T,.,.)\\&-\int_{Q_{T,A}}q_1\frac{\partial(k(x)\varphi_1)}{\partial\nu}\\&\forall\varphi_1\in C^\infty(Q)\text{ such that }\varphi_1=0\text{ on }Q_{T,A}\text{ and }\varphi_1(0,.,.)=0\text{ in }Q_{A,1}.\end{aligned}$$

we get:

$$q_1(.,0,.) = \tau \ in \ Q_{T,1} \tag{75}$$

$$q_1|_{Q_{T,A}} = 0 \tag{76}$$

$$q_1(.,A,.) = 0 \ in \ Q_{T,1} \tag{77}$$

$$q_1(T,.,.) = 0 \ in \ Q_{A,1} \tag{78}$$

By combining (74)*,* (76)*,* (77) *and* (78)*, we conclude that* q_1 *satisfies system:*

$$\begin{cases} -\dfrac{\partial q_1}{\partial t} - \dfrac{\partial q_1}{\partial a} - (k(x)q_{1x})_x + \mu q_1 & = y - z_d & \text{in } Q \\ q_1(t,a,1) = q_1(t,a,0) & = 0 & \text{on } Q_{T,A} \\ q_1(T,a,x) & = 0 & \text{in } Q_{A,1} \\ q_1(t,A,x) & = 0 & \text{in } Q_{T,1} \end{cases} \tag{79}$$

Analogously with which above, it is obviously to show that q_2 *satisfies system:*

$$\begin{cases} -\dfrac{\partial q_2}{\partial t} - \dfrac{\partial q_2}{\partial a} - (k(x)q_{2x})_x + \mu q_2 & = k_2 & \text{in } Q \\ q_2(t,a,1) = q_2(t,a,0) & = 0 & \text{on } Q_{T,A} \\ q_2(T,a,x) & = 0 & \text{in } Q_{A,1} \\ q_2(t,A,x) & = 0 & \text{in } Q_{T,1} \end{cases} \tag{80}$$

Back to expression (27)*, which means:*

$$\int_Q (y(u_\gamma,0) - z_d)y(\omega,0) + \frac{1}{\gamma}\int_{Q_{T,A}} \frac{\partial(k(x)\xi(u_\gamma,.))}{\partial\nu}\frac{\partial(k(x)\xi(\omega,.))}{\partial\nu} + N\int_{Q_\omega} u_\gamma\omega = 0, \quad \forall\omega \in L^2(Q_\omega) \tag{81}$$

By substituting the expression (31) *in* (27)*, we obtain:*

$$\int_Q (y(u_\gamma,0) - z_d)y(\omega,0) + \frac{1}{\sqrt{\gamma}}\int_{Q_{T,A}} p_\gamma y(\omega,0) + N\int_{Q_\omega} u_\gamma\omega = 0, \quad \forall\omega \in L^2(Q_\omega) \tag{82}$$

We set $O = \{y(\omega,0), \omega \in L^2.(Q_\omega)\}$*. Then* $O \subset L^2(Q)$*. We define on* $O \times O$ *the inner product*

$$\langle y(v,0), y(\omega,0)\rangle_O = \int_{Q_\omega} v\omega + \int_Q y(v,0)y(\omega,0), \quad \forall(y(v,0)y(\omega,0)) \in O \times O$$

The set O *endowed with norm*

$$\|y(\omega,0)\|_O^2 = \|\omega\|^2_{L^2(Q_\omega)} + \|y(\omega,0)\|^2_{L^2(Q)}$$

is a Hilbert space.

By setting $T_\gamma(u_\gamma) = \dfrac{1}{\sqrt{\gamma}}p_\gamma$ *in* (82)*, we have:*

$$\int_Q T_\gamma(u_\gamma)y(\omega,0) = -\int_Q (y(u_\gamma,0) - z_d)y(\omega,0) - N\int_{Q_\omega} u_\gamma\omega, \quad \forall\omega \in L^2(Q_\omega)$$

On the one hand, we have:

$$\left|\int_Q T_\gamma(u_\gamma)y(\omega,0)\right| \leq \left(\|y(u_\gamma,0) - z_d\|_{L^2(Q)} + N\|u_\gamma\|_{L^2(Q_\omega)}\right)\|y(\omega,0)\|_O, \quad \forall\omega \in L^2(Q_\omega)$$

and on the other hand, we have:

$$\left|\int_Q T_\gamma(u_\gamma)y(\omega,0)\right| \leq \|T_\gamma(u_\gamma)\|_{L^2(Q)}\|y(\omega,0)\|_O, \quad \forall\omega \in L^2(Q_\omega)$$

We deduce that:

$$\|T_\gamma(u_\gamma)\|_{L^2(Q)} \leq C$$

$$\Rightarrow \|\frac{1}{\sqrt{\gamma}}p_\gamma\|_{L^2(Q)} \leq C \tag{83}$$

where $C = C(N, \|u_\gamma\|_{L^2(Q_\omega)}, \|z_d\|_{L^2(Q)}, \|y(u_\gamma, 0)\|_{L^2(Q)}, \|f\|_{L^2(Q)})$.
Using (71) *and* (83)*, there exists* $C > 0$ *such that*

$$\|q_2^\gamma\|_{L^2((0,T)\times(0,A);H_k^1(0,1))} \leq C \tag{84}$$

According to estimates (83) *and* (84)*, there exists* $k_2 \in L^2(Q)$ *and* $q_2 \in L^2((0,T)\times(0,A); H_k^1(0,1))$ *such that:*

$$\frac{1}{\sqrt{\gamma}} p_\gamma \rightharpoonup \quad k_2 \quad \text{weakly in} \quad L^2(Q) \tag{85}$$

$$q_2^\gamma \rightharpoonup \quad q_2 \quad \text{weakly in} \quad L^2((0,T)\times(0,A); H_k^1(0,1)) \tag{86}$$

For the rest, using the same reasoning as for q_1^γ*, we prove by using* (85) *and* (86) *that* q_2 *satisfies:*

$$\begin{cases} -\dfrac{\partial q_2}{\partial t} - \dfrac{\partial q_2}{\partial a} - (k(x)q_{2x})_x + \mu q_2 & = k_2 \quad \text{in } Q \\ q_2(t,a,1) = q_2(t,a,0) & = 0 \quad \text{on } Q_{T,A} \\ q_2(T,a,x) & = 0 \quad \text{in } Q_{A,1} \\ q_2(t,A,x) & = 0 \quad \text{in } Q_{T,1} \end{cases}$$

Now back to equality $q^\gamma = q_1^\gamma + q_2^\gamma$*. Using* (72) *and* (84)*, we deduce that*

$$\|q^\gamma\|_{L^2((0,T)\times(0,A);H_k^1(0,1))} \leq C \tag{87}$$

Then, there exists $q \in L^2((0,T)\times(0,A); H_k^1(0,1))$ *such that:*

$$q^\gamma \rightharpoonup q \quad \text{weakly in} \quad L^2(Q) \tag{88}$$

By proceeding in the same way as for q_1^γ *and* q_2^γ *and using* (88)*, we show that* q *is the solution of* (59).
Finally, passing to the limit in (33) *and using* (38) *and* (88) *, we deduce* (60).

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