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Optimal Investment, Consumption and Life Insurance Problem with Stochastic Environments

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Abstract

Optimal investment, consumption and life insurance problem with stochastic environments for a CRRA wage-earner is solved in this study. The wage-earner invests in the financial market with one risk-free security, one risky security, receives labor income and has a life insurance policy in the insurance market. A life insurance policy is purchased to hedge the financial wealth for the beneficiary in case of wage earner premature death. The interest rate and the volatility are stochastic. The stochastic interest rate dynamics of risk-free security follow a Ho-Lee model and the risky security follow Hestons model with stochastic volatility parameter dynamics following a Cox-Ingersoll-Ross (CIR) model. The objective of the wage-earner is to allocate wealth between risky security and risk-free security but also buy a life insurance policy during the investment period to maximize the expected discounted utilities derived from consumption, legacy and terminal wealth over an uncertain lifetime. By applying Bellman's optimality principle, the associated HJB PDE for the value function is established. The power utility function is employed for our analysis to obtain the value function and optimal policies. Finally, numerical examples and simulations are provided.

Keywords: Investment-consumption-life insurance problem, Ho-Lee model, Heston model, Cox-Ingersoll-Ross (CIR) model, Value function, optimal policies

1. Introduction

The stochastic optimal control problem is key in practice. Such problems are a major concern to individual and institutional investors who seek to allocate the wealth among various assets over an uncertain lifetime. So far, different researchers have explicitly solved stochastic optimal control problems via methods, such as the dynamic programming principle, the maximum principle (Yong & Zhou, 1999), and the convex duality martingale method. Our research work builds on the celebrated work of Merton (1969) and Merton (1971) who solved the stochastic optimal control problem for an agent who invests in one risk-free asset and one risky asset but under constant interest rate and volatility rate. In reality, interest rates and volatility rates are stochastic due to uncertain events such as Covid19, climate change, wars, inflation, natural disasters, fiscal policy and financial policy adjustments e.t.c. Merton derived closed-form solutions by applying the dynamic programming principle. In our study, we solve optimal investment, consumption and life insurance problem for a wage-earner who earns labour income and pay a life insurance premium with constant relative risk aversion (CRRA) case. The wage-earner invests in life insurance to hedge against the risk of premature death. The wage-earner who earns labour income can decide to invest it in money market account or bond, stock or stock index, life insurance policy, and can make consumption decisions. The stochastic interest rate for a money market account evolves as a Ho-Lee model. The stock price of the risky stock evolves as Hestons model with volatility following a CIR model. We extend Merton's work by choosing unique financial market models and life insurance models for a wage earner with labour income and has life insurance contract. The objective of the wage-earner is to choose an optimal investment-consumption-insurance strategy that maximizes the expected, discounted utilities derived from intermediate consumption, legacy and terminal wealth over an uncertain horizon. Our major contribution in this study is considering stochastic interest rate, stochastic volatility and life insurance product simultaneously in modeling wealth.

2. Links to the Literature

Mertons work has attracted a number of extensions. For instance, Richard (1975) was the first to extend Mertons work in a study titled optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous-time model. He added life insurance to Mertons work. Thus, combining the financial market and the insurance market. (Pliska & Ye, 2007) extended Mertons work by adding life insurance but with constant labor income in the study titled

Optimal life insurance purchase and consumption/investment under uncertain lifetime. In their study, the agent has an initial wealth but also receives an income continuously which can be terminated upon premature death. They used the dynamic programming principle to solve for explicit solutions for the CRRA utility case. (Shen & Wei, 2014) combined the HamiltonCJacobiCBellman (HJB) equation and backward stochastic differential (BSD)equation to obtained closed-form solutions in the study titled optimal investment-consumption-insurance with random parameters. They assumed that some stochastic parameters are adapted to the filtration generated by Brownian motion. Ye (2006) explored the problem of optimal life insurance purchase, consumption and portfolio investment strategies for a wage earner under an uncertain horizon as a Ph.D. dissertation. Two methods adopted in this research include the dynamic programming principle method and the martingale method to obtain explicit solutions for CRRA utility functions. Ye (2007) analyzed optimal Life Insurance Purchase, Consumption and Portfolio under Uncertainty by applying the Martingale method to obtain closed-form solutions for CRRA cases. Duarte *et al.* (2017) investigated a model for optimal insurance purchase, consumption and investment for a wage earner with an uncertain lifetime. They considered an underlying financial market consisting of one risk-free security and a fixed number of risky securities with diffusive terms evolving as a multidimensional Brownian motion. They applied the dynamic programming principle technique to obtain explicit solutions for constant relative risk aversion utility case. The study by (Bruhn & Steffensen, 2011) analyzed household consumption, investment and life insurance. This paper developed optimizers of future utility from consumption by controlling consumption, investments and purchase of life insurance for each person in the household. Huang *et al.* (2008) considered portfolio choice and life insurance for the CRRA utility case. They considered the correlation between the dynamics of human capital and financial capital and modeled the utility of the family as opposed to separating consumption and bequest. The Hamilton-Jacobi-Bellman equation was determined and used the reduction technique to obtain a numerical solution. Results showed that life insurance hedges human capital. Kwak *et al.* (2011) examined on optimal investment and consumption decision of a family with life insurance. They applied the Martingale method to obtain analytic solutions for the value function and the optimal controls. The study by (Liang & Guo, (2016) treated an optimal insurance-consumption-investment problem for a wage earner in an incomplete market when the stock price has a mean-reverting drift. They applied the Martingale method to determine explicit solutions for power and logarithmic utilities. The study by Pirvu *et al.* (2012) solved the problem of optimal investment, consumption and life insurance for a wage earner who has CRRA preferences. They considered a complete market model is complete with uncertainty driven by Brownian motion and the stock price has a mean-reverting drift. They derived the HJB equation by applying the dynamic programming principle and found closed-form solutions. (Guambe & Kufakunesu, 2018) explored optimal investment, consumption and insurance problems. They considered a market with a real zero-coupon bond, the inflation-linked real money account and a risky share following a jump-diffusion process. They applied backward stochastic differential equation (BSDE) with jumps to derive the explicit solutions.

In our paper, we solve optimal investment, consumption and life insurance problem with stochastic interest and volatility rates for an agent with a CRRA case. We let the wage-earner receive a stream of labour income $i(t)$ and can decide to consume, invest in a stock, bond and buy a life insurance policy. We let T be retirement time. A life insurance policy is purchased in order to hedge the financial wealth. The risk-free interest rate evolves as a Ho-Lee model. In addition, the risky stock evolves as Hestons model with volatility following a CIR model. We extend Mertons work in a unique way by considering stochastic interest rate, stochastic volatility and life insurance policy simultaneously.

The outline of this paper is as follows. Section 1, introduction. Section 2, literature review. Section 3, description of the financial market model. Section 4, description of the insurance market model. In Section 5, we determine the wealth model using models identified in sections 3 and 4. In Section 6, we describe the optimization criterion. In Section 7, we show results and discussion. In Section 8, numerical examples and simulations are provided, and in Section 9, we conclude and suggest possible future research work.

3. Financial Market Model

Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space with filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions such as $(\mathcal{F}_t)_{0\leq t\leq T}$ being right continuous complete filtration and $\mathbb{P}$-complete. Let all stochastic processes be well-defined and adapted in the filtered complete probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Let T be a finite time horizon.

Consider a financial market of a single wage-earner with a portfolio consisting of one risk-free security (e.g. a money market account or bond) $\mathcal{B}(t)$, one risky security (e.g. a stock or stock index) $\mathcal{S}(t)$ and life insurance policy to protect the beneficiary in case of wage-earner premature death. We start by describing the financial market, followed by the description of the insurance market and finally, define the wealth model for the wage earner.

Let the price dynamics of the risk-free security $\mathcal{B}(t)$ evolve as follows:

$$\begin{cases} d\mathcal{B}(t) = r(t)\mathcal{B}(t)dt, \\ \mathcal{B}(0) = 1, \end{cases} \tag{1}$$

with stochastic interest rate $r(t)$ following a Ho-Lee model given by:

$$\begin{cases} dr(t) = \theta_0(t)dt + \sigma_0 dW^r(t), \\ r(0) = r_0 > 0, \end{cases} \tag{2}$$

where $\theta_0(t)$ is the expected instantaneous change in the interest rate, $\sigma_0 > 0$ is a constant volatility factor and $W^r(t)$ is a one-dimensional wiener process on a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Assumed that $\theta_0(t)$ can be written as $\theta_0(t) = \gamma[\beta - r(t)]$, where γ and β are constants.

Let the price dynamics of the risky security $\mathcal{S}(t)$, follow a Heston's model given by:

$$\begin{cases} d\mathcal{S}(t) = \mathcal{S}(t)[r(t) + k\eta(t)]dt + \sigma_1 \sqrt{\eta(t)} S(t) dW^{\mathcal{S}}(t), \\ \mathcal{S}(0) = \mathcal{S}, \end{cases} \tag{3}$$

where $k\eta(t)$ is the appreciation factor, $\sigma_1 \sqrt{\eta(t)}$ is the volatility of the risky price and $W^S(t)$ is a Wiener process on a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Note that $S(t)$ is risky stock price, $r(t)$ is risk-free interest rate, $k > 0$ is the expected returns parameter of risky asset and σ_1 is the volatility of the volatility $\sqrt{\eta(t)}$ of risky asset.

In addition, let $\eta(t)$ follow a Cox-Ingersoll-Ross (CIR) model given by:

$$\begin{cases} d\eta(t) = [\theta_2 - b\eta(t)]dt + \sigma_2 \sqrt{\eta(t)} dW^{\eta}(t), \\ \eta(0) = \eta_0 > 0, \end{cases} \tag{4}$$

where $\theta_2 > 0$, $b > 0$, and $\sigma_2 > 0$ are constants. Also note that $\eta(t) > 0$ for all $t \geqslant 0$. W^{η} is a wiener process on a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$.

4. Life Insurance Market Model

Suppose the wage-earner has nonnegative lifetime denoted by τ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is alive at time $t = 0$. Assume that the lifetime τ is independent of the filtration $\mathcal{F}_t$, $\forall t \in [0, T]$. Let the instantaneous death rate for the wage-earner denoted by $\lambda(t)$ be defined by has a distribution function given by:

$$\lambda(t) := \lim_{\Delta t \to 0} \frac{\mathbb{P}(t \leq \tau < t + \Delta t | \tau \geq t)}{\Delta t}. \tag{5}$$

Implying the conditional probability survival function denoted by $F(t)$ is defined as:

$$F(s, t) = \mathbb{P}(\tau > s \mid \tau > t). \tag{6}$$

Alternatively, the conditional probability survival function $F(t)$ can be defined in terms of $\lambda(t)$ as follows:

$$F(t, s) := \exp\Big[- \int_t^s \lambda(u) du \Big]. \tag{7}$$

The conditional probability density for death denoted by $\overline{F}(t)$ of the wage-earner at time s conditional upon the wage-earner being alive at time $t \leq s$ is related to the hazard rate and defined as follows:

$$\overline{F}(t) = \lambda(s) exp\Big[- \int_t^s \lambda(u) du \Big]. \tag{8}$$

Going back to the insurance market, we make the following assumptions about the market. Let the wage-earner receive deterministic and non-negative income streams $i(t)$ up to time $\tau \wedge T$, which means the income will be terminated by the death or retirement of the wage-earner.

Let the wage-earner pay premiums at the rate $p(t)$ at time t for the life insurance contract and the insurance company will pay $\frac{p(\tau)}{\psi(\tau)}$ to the beneficiary upon his death, where $\psi(\tau)$ is the insurance premium-payout ratio. The policy end when the wage-earner dies or retires. When the wage-earner dies, the total legacy to his beneficiary is given by

$$\mathcal{Z}(\tau) = \mathcal{X}(\tau) + \frac{p(\tau)}{\psi(\tau)}, \tag{9}$$

where $\mathcal{X}(\tau)$ is the net wealth process of the wage-earner at time τ from the financial market and $\frac{p(\tau)}{\psi(\tau)}$ is the insurance benefit paid by the insurance company to the beneficiary from the insurance market if death occurs at time τ. That is, for $p(\tau) = P20,000$, $\psi(\tau) = 0.4$, then the benefit paid by the insurance is $\frac{p(\tau)}{\psi(\tau)} = P50,000$. In general, $\psi(\tau) > \lambda(\tau)$ in order for the insurance to also benefit, but in this study, we assume $\psi(\tau) = \lambda(\tau)$.

5. The Wealth Model

Consider a wage-earner with initial amount of money $\mathcal{X}(0) = x_0 > 0$. Let $C(t)$ denote the rate of continuous consumption. Let $\pi(t)$ denote the wealth invested in the risky security, $\mathcal{S}(t)$, then the amount invested in the risk-free security is given by $\mathcal{X}(t) - \pi(t)$. Let $(\pi(t), \mathcal{X}(t) - \pi(t))$ be the vector (portfolio process) of the financial market amounts the wage earner invest in the money market account $\mathcal{B}(t)$ and the risky security account $\mathcal{S}(t)$, respectively. Let the wage-earner pay premiums at the rate $p(t)$ at time t for the life insurance policy. Let the wage-earner receive deterministic and non-negative income stream $i(t)$ up to time $\tau \wedge T$. Note that the pair $(C(t), \pi(t), p(t))$ is a trading strategy.

The wealth evolve as a stochastic differential equation (SDE) given as follows:

$$\begin{cases} dX^i(t) = [\mathcal{X}(t)r(t) - C(t) - p(t) + i(t) + \pi(t)k\eta(t)]dt + \pi(t)\sigma_1\sqrt{\eta(t)}dW^S(t), \\ \mathcal{X}(0) = x_0 > 0. \end{cases} \tag{10}$$

We follow the idea of (Munk & Sorensen, 2010) to discount the future labour income stream (human capital) $i(t)$ and change our problem to a problem without labour income. Let the wage-earner sell the remaining labour income stream. This will result in converting our problem to one without labour income at time t. The net wealth 10 changes to the stochastic differential equation (SDE) without labour income stream $i(t)$ given as follows:

$$\begin{cases} d\mathcal{X}(t) = [\mathcal{X}(t)r(t) - C(t) - p(t) + \pi(t)k\eta(t)]dt + \pi(t)\sigma_1\sqrt{\eta(t)}dW^S(t), \\ \mathcal{X}(0) = x_0 > 0. \end{cases} \tag{11}$$

In case of wage-earner's premature death, the total legacy $Z(\tau)$ is calculated as wealth plus the insurance company amount as follows:

$$\mathcal{Z}(\tau) = \mathcal{X}(\tau) + \frac{p(\tau)}{\psi(\tau)}. \tag{12}$$

Remark 1. *For the sake of explicit solution we assume:*

- *The correlation coefficient $\rho \in \{-1, 1\}$ of $dW^r dW^S$, $dW^\eta dW^S$ and $dW^r dW^\eta$.*
- *Interest rate for risk-free security and risky security are equal.*
- *A premature death occurs on the set $\{\omega \in \Omega \mid \tau(\omega) < T\}$.*

Taking $\rho \in \{-1, 1\}$ is not realistic as it implies that the risks of stock and the risks of volatility are the same. But such an assumption will enable us to determine closed-form solutions.

6. The Optimization Criterion

Suppose the set of all admissible strategies is denoted by $\mathcal{A}$.

Definition 1

An investment, consumption and life insurance triple strategy $\mathcal{A} = (\pi(t), C(t), p(t))$ is said to be admissible if the following conditions are satisfied.

- The triple $(\pi(t), C(t), p(t))$ is progressively $\mathcal{F}_t - measurable$ and $\int_0^T \pi(t)^2 dt < \infty$, $\int_0^T C(t)dt < \infty$, $\int_0^T \mid p(t) \mid dt < \infty$.
- $\mathbb{E}[\int_0^T (\pi(t)\sigma_1\sqrt{\eta(t)})^2 dt] < \infty$.
- For all pair $(\pi(t), C(t), p(t))$, the wealth process 11 with $\mathcal{X}(0) = x_0 \geq 0$ has a path wise unique solution.

Remark 1

The wage-earners problem is to find strategies $(\pi(t), C(t), p(t)) \in \mathcal{A}$ that maximize the expected discounted utility obtained from

- intermediate consumption during $[0, \tau \wedge T]$,

- Legacy if premature death occur before time T and
- Terminal wealth at retirement date T if still alive.

Mathematically, for $t \in [0, \tau \wedge T]$, remark can be formulated as follows:

$$\max_{(\pi(t),C(t),p(t))\in\mathcal{A}} \mathbb{E}\Big[\int_0^{T\wedge\tau} e^{-\gamma s}\mathcal{U}_1(C(s), s)ds + \lambda e^{-\gamma\tau}\mathcal{U}_2(\mathcal{Z}(\tau), \tau)1_{\tau\leq T} + e^{-\gamma T}\mathcal{U}_3 X((T))1_{\tau>T} \mid \tau > t\Big], \tag{13}$$

where $\mathbb{E}$ is conditional expectation given the following initial values:

$$\begin{cases} X(0) = x_0, \\ t(0) = t_0, \\ r(0) = r_0, \\ \eta(0) = \eta_0. \end{cases} \tag{14}$$

and λ is the weight on the wage-earners legacy in case of premature death. The positive constant γ denotes time preference.

Definition 2

For any admissible strategies $(\pi(t), C(t), p(t))$, the value function is defined as:

$$V(t, r, \eta, x) = \sup_{(\pi(t),C(t),p(t))\in\mathcal{A}} \mathbb{E}\Big[\int_0^{T\wedge\tau} e^{-\gamma s}\mathcal{U}_1(C(s), s)ds + \lambda e^{-\gamma\tau}\mathcal{U}_2(\mathcal{Z}(\tau), \tau)1_{\tau\leq T} + e^{-\gamma T}\mathcal{U}_3 X((T))1_{\tau>T} \mid \tau > t\Big], \tag{15}$$

where $\mathbb{E}$ is conditional expectation given the following initial values:

$$\begin{cases} X(0) = x_0, \\ t(0) = t_0, \\ r(0) = r_0, \\ \eta(0) = \eta_0, \end{cases} \tag{16}$$

and $\mathcal{U}_1(C, .)$ is the utility function of consumption in the time interval $[0, \tau \wedge T]$ and is assumed to be strictly concave in C. $\mathcal{U}_2(\mathcal{Z}, .)$ is the wage earners legacy in case of premature death and is assumed to be strictly concave in $\mathcal{Z}$. $\mathcal{U}_3(X)$ is the terminal wealth at time T in the case the wage earner survives and is assumed to be strictly concave in X. $1_{\mathcal{A}}$ is the indicator function of event $\mathcal{A}$.

7. Results and Discussions

7.1 The Hamilton-Jacobi-Bellman Equation

We redefine the stochastic control problem in 13 into dynamic programming form to include fixed planning horizon in order to apply the Dynamic Programming Principle (DPP) and obtain the HJB PDE. The value function and closed-form expression of optimal strategies are then derived for the power utility function. In this case, the stochastic optimal control problem is equivalent to the problem of finding a solution to the HJB PDE.

7.2 Dynamic Programming Principle (DPP)

Redefining 13 into dynamic programming form by fixed planning horizon imply that the wage-earner facing premature death acts as if he will live until retirement time T and

$$\sup_{(\pi(t),C(t),p(t))\in\mathcal{A}} \mathbb{E}\Big[\int_0^{T\wedge\tau} e^{-\gamma(s-t)}\mathcal{U}_1(C(s), s)ds + \lambda e^{-\gamma(\tau-t)}\mathcal{U}_2(\mathcal{Z}(\tau), \tau)1_{\tau\leq T} + e^{-\gamma(T-t)}\mathcal{U}_3 X((T))1_{\tau>T} \mid \tau > t\Big], \tag{17}$$

where $\mathbb{E}$ is conditional expectation at time t given the following initial values:

$$\begin{cases} X(t) = x, \\ r(t) = r, \\ \eta(t) = \eta. \end{cases} \tag{18}$$

Note that using 7 and 8, the value function for 17 can be rewritten as a recursive relationship for the maximum expected discounted utility as a function of the wage-earners age and his wealth as follows:

$$\begin{aligned} V(t, r, \eta, x) = \sup_{(\pi(t), C(t), p(t)) \in \mathcal{A}} \mathbb{E}\Big[\int_t^T & F(s, t) e^{-\gamma(s-t)} \mathcal{U}_1(C(s), s) ds \\ & + \int_t^T \overline{F}(s, t) e^{-\gamma(\tau - t)} \mathcal{U}_2(\mathcal{Z}(\tau), \tau) 1_{\tau \leq T} \\ & + F(T, t) e^{-\gamma(T-t)} \mathcal{U}_3 X((T)) 1_{\tau > T} \mid \tau > t \Big]. \end{aligned} \tag{19}$$

where $\overline{F}(s, t)$ is the conditional probability density function for the wage-earners death to occur at time s conditional upon the wage earner being alive at time $t \leq s$, $F(s, t)$ is the conditional probability survival function for the wage-earner to be alive at time s conditional upon the wage-earner being alive at time $t \leq s$ and the nonnegative constant γ stands for the time preference. For proof see Ye (2007).

For any strategy $(\pi(t), C(t), p(t)) \in \mathcal{A}$, the corresponding HJB PDE for 17 given 19 is stated as follows:

$$\begin{aligned} & \tilde{V}_t(t, r, \eta, x) - (\gamma + \psi(t)) \tilde{V}(t, r, \eta, x) + \Big[X(t) r(t) - C(t) - p(t) + \pi(t) k \eta(t) \Big] \tilde{V}_x \\ & + \frac{1}{2} \pi^2 \sigma_1^2 \eta \tilde{V}_{xx} + \theta_0 \tilde{V}_r + \frac{1}{2} \sigma_0^2 \tilde{V}_{rr} + (\theta_2 - b\eta) \tilde{V}_\eta + \frac{1}{2} \sigma_2^2 \eta \tilde{V}_{\eta\eta} \\ & + \pi \sigma_1 \sigma_2 \eta \rho \tilde{V}_{x\eta} + U_1(C) + \lambda(t) U_2 \left(x + \frac{p(t)}{\psi(t)} \right) \\ & = 0. \end{aligned} \tag{20}$$

Next, we proceed to solving HJB PDE for 17 given 19 as follows.

Assume the value function is given as follows:

$$V(t, r, \eta, x) = e^{\int_0^t (\gamma(u) + \psi(u)) du} \tilde{V}(t, r, \eta, x). \tag{21}$$

Implying, we obtain the following partial of $\tilde{V}(t, r, \eta, x)$:

$$\begin{cases} \tilde{V}_t(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_t(t, r, \eta, x) - [\gamma(t) + \psi(t)] \tilde{V}(t, r, \eta, x), \\ \tilde{V}_r(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_r(t, r, \eta, x), \\ \tilde{V}_{rr}(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_{rr}(t, r, \eta, x), \\ \tilde{V}_x(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_x(t, r, \eta, x), \\ \tilde{V}_{xx}(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_{xx}(t, r, \eta, x), \\ \tilde{V}_\eta(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_\eta(t, r, \eta, x), \\ \tilde{V}_{\eta\eta}(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_{\eta\eta}(t, r, \eta, x), \\ \tilde{V}_{x\eta}(t, r, \eta, x) = e^{-\int_0^t (\gamma(u) + \psi(u)) du} V_{x\eta}(t, r, \eta, x). \end{cases} \tag{22}$$

The associated HJB PDE upon applying Ito's differentiation rule to $\tilde{V}(t, r, \eta, x)$ is then given in terms of $V(t, r, \eta, x)$ as follows:

$$\begin{aligned}
&V_t(t,r,\eta,x) - (\gamma + \psi(t))V(t,r,\eta,x) + \Big[\mathcal{X}(t)r(t) - C(t) - p(t) + \pi(t)k\eta(t)\Big]V_x \\
&+ \frac{1}{2}\pi^2\sigma_1^2\eta V_{xx} + \theta_0 V_r + \frac{1}{2}\sigma_0^2 V_{rr} + (\theta_2 - b\eta)V_\eta + \frac{1}{2}\sigma_2^2\eta V_{\eta\eta} \\
&+ \pi\sigma_1\sigma_2\eta\rho V_{x\eta} + U_1(C) + \lambda(t)U_2\left(x + \frac{p(t)}{\psi(t)}\right) \\
&= 0. \qquad (23)
\end{aligned}$$

Next, we determine the candidate policies:

Applying the first-order maximizing conditions to 23 result to the following candidate optimal policies (optimizers):

$$\pi^*(t) = -\frac{kV_x}{\sigma_1^2 V_{xx}} - \frac{\sigma_2\rho V_{x\eta}}{\sigma_1 V_{xx}}. \qquad (24)$$

$$C^*(t) = [V_x]^{-\frac{1}{\delta}} \qquad (25)$$

and

$$p^*(t) = \psi(t)\left\{\left[\frac{\psi(t)}{\lambda(t)}V_x\right]^{-\frac{1}{\delta}} - x\right\}. \qquad (26)$$

Definition 3

Let $\mathcal{X}(0) > 0$ be the initial wealth, the wage-earners optimal investment, consumption and life insurance problem is to maximize the expected utility over the pair of all admissible strategies $\mathcal{A} = (\pi(t), C(t), p(t))$ such that for all $(\pi^\star(t), C^\star(t), p^\star(t)) \in \mathcal{A}$,

$$V(\pi^\star(t)) = \sup_{(\pi(t),C(t),p(t))\in\mathcal{A}} V(\pi), \qquad (27)$$

$$V(C^\star) = \sup_{(\pi(t),C(t),p(t))\in\mathcal{A}} V(C) \qquad (28)$$

and

$$V(p^\star(t)) = \sup_{(\pi(t),C(t),p(t))\in\mathcal{A}} V(p). \qquad (29)$$

Substituting the candidate optimal policies 24, 25 and 26 into the HJB PDE 23, we obtain the following PDE:

$$\begin{aligned}
&V_t - [\gamma + \psi(t)]V + rxV_x + x\psi(t)V_x - \frac{k^2\eta V_x^2}{2\sigma_1^2 V_{xx}} - \frac{\sigma_2^2\rho^2\eta V_{x\eta}^2}{2V_{xx}} + \theta_0 V_r \\
&+ \frac{\sigma_0^2 V_{rr}}{2} + (\theta_2 - b\eta)V_\eta + \frac{\sigma_2^2\eta V_{\eta\eta}}{2} - \frac{k\sigma_2\rho\eta}{\sigma_1}\frac{V_x V_{x\eta}}{V_{xx}} \\
&+ \frac{\delta}{1-\delta}(V_x)^{\frac{\delta-1}{\delta}}\left[\lambda(t)\left(\frac{\psi(t)}{\lambda(t)}\right)^{\frac{\delta-1}{\delta}} + 1\right] = 0, \qquad (30)
\end{aligned}$$

with boundary conditions

$$V(T,r,\eta,x) = \frac{x^{1-\delta}}{1-\delta}. \qquad (31)$$

Let

$$Q(t) = \left[\lambda(t)\left(\frac{\psi(t)}{\lambda(t)}\right)^{\frac{\delta-1}{\delta}} + 1\right], \tag{32}$$

implying 30 becomes:

$$\begin{aligned} &V_t - [\gamma + \psi(t)]V + rxV_x + x\psi(t)V_x - \frac{k^2\eta V_x^2}{2\sigma_1^2 V_{xx}} - \frac{\sigma_2^2\rho^2\eta V_{x\eta}^2}{2V_{xx}} + \theta_0 V_r \\ &+ \frac{\sigma_0^2 V_{rr}}{2} + (\theta_2 - b\eta)V_\eta + \frac{\sigma_2^2\eta V_{\eta\eta}}{2} - \frac{k\sigma_2\rho\eta}{\sigma_1}\frac{V_x V_{x\eta}}{V_{xx}} \\ &+ \frac{\delta}{1-\delta}(V_x)^{\frac{\delta-1}{\delta}} Q(t) = 0, \end{aligned} \tag{33}$$

with boundary conditions:

$$V(T, r, \eta, x) = \frac{x^{1-\delta}}{1-\delta}. \tag{34}$$

7.3 The Value Function and Optimal Policies

In this study, we assume that the wage-earners preference towards risk is given by a power utility function defined as follows:

Definition 4

Power utility function is defined as:

$$\begin{cases} U_1(x) = U_2(x) = U_3(x) = \frac{x^{1-\delta}}{1-\delta}, \\ \delta > 0, \\ \delta \neq 1, \end{cases} \tag{35}$$

where δ is the risk aversion factor.

This is a case where the wage-earner has the same power utility functions for consumption, the legacy and his terminal wealth.

Assume the solution V for 33 take the form:

$$\begin{cases} V(t, r, \eta, x) = \frac{x^{1-\delta}}{1-\delta} g(t, r, \eta), \\ g(T, r, \eta) = 1. \end{cases} \tag{36}$$

Implying:

$$\begin{cases} V_t = \frac{x^{1-\delta}}{1-\delta} g_t, \\ V_x = x^{-\delta} g, \\ V_{xx} = -\delta x^{-\delta-1} g, \\ V_r = \frac{x^{1-\delta}}{1-\delta} g_r, \\ V_{rr} = \frac{x^{1-\delta}}{1-\delta} g_{rr}, \\ V_\eta = \frac{x^{1-\delta}}{1-\delta} g_\eta, \\ V_{\eta\eta} = \frac{x^{1-\delta}}{1-\delta} g_{\eta\eta}, \\ V_{x\eta} = x^{-\delta} g_\eta. \end{cases} \tag{37}$$

Substituting 36 and 37 into 33, gives:

$$\begin{aligned} &g_t + \left\{(1-\delta)\left[r + \psi - \lambda - \gamma + \frac{1}{2}\frac{k^2\eta}{\sigma_1^2\delta}\right]\right\} g - \frac{1}{2}\frac{\sigma_2^2\rho^2\eta(1-\delta)}{(\delta)}\frac{g_\eta g_\eta}{g} + \theta_0 g_r + \frac{1}{2}\sigma_0^2 g_{rr} \\ &+ \frac{1}{2}\sigma_2^2\eta g_{\eta\eta} + \left(\theta_2 - b\eta - \frac{k\eta\sigma_2\rho}{\sigma_1}\frac{1-\delta}{\delta}\right) g_\eta + \delta g^{\frac{\delta-1}{\delta}} Q(t) = 0. \end{aligned} \tag{38}$$

Assume the solution of 38 is of the form:

$$\begin{cases} g(t,r,\eta) = f(t,r,\eta)^{\delta}, \\ f(T,r,\eta) = 1. \end{cases} \tag{39}$$

The partial derivatives for 39 are given as follows:

$$\begin{cases} g_t = \delta f^{\delta-1} f_t, \\ g_r = \delta f^{\delta-1} f_r, \\ g_{rr} = \delta(\delta-1) f^{\delta-2} f_r^2 + \delta f^{\delta-1} f_{rr} \\ g_{\eta} = \delta f^{\delta-1} f_{\eta}, \\ g_{\eta\eta} = (\delta)(\delta-1) f^{\delta-2} f_{\eta}^2 + \delta f^{\delta-1} f_{\eta\eta}. \end{cases} \tag{40}$$

Substituting 40 into 38, we obtain:

$$\begin{aligned} & f_t + \left\{ \frac{1-\delta}{\delta} \left[r + \psi - \lambda - \gamma + \frac{1}{2} \frac{k^2 \eta}{\sigma_1^2 \delta} \right] \right\} f + \frac{1}{2} \sigma_2^2 \eta (\delta - 1)[\rho^2 - 1] f^{-1} f_{\eta}^2 \\ & + \theta_0 f_r + \frac{1}{2} \sigma_0^2 f_{rr} - \frac{1}{2} \sigma_0^2 (\delta - 1) f^{-1} f_r^2 + \frac{1}{2} \sigma_2^2 \eta f_{\eta\eta} + \left(\theta_2 - b\eta - \frac{k\eta\sigma_2\rho}{\sigma_1} \frac{\delta}{\delta - 1} \right) f_{\eta} \\ & + Q(t) = 0. \end{aligned} \tag{41}$$

The equation 41 is still nonlinear second-order PDE which is difficult to solve. Inspired by [Liu (2007), (Guan & Liang, 2014), (Guan & Liang, 2014)], we assume the following:

Lemma 1

Assume f given by:

$$f(t,r,\eta) = \int_t^T \hat{f}(u,r,\eta) du + \hat{f}(t,r,\eta), \tag{42}$$

is the solution to 41.

Then $\hat{f}(t,r,\eta)$ can be written as:

$$\begin{aligned} & \hat{f}_t + \left\{ \frac{1-\delta}{\delta} \left[r + \psi - \lambda - \gamma + \frac{1}{2} \frac{k^2 \eta}{\sigma_1^2 \delta} \right] \right\} \hat{f} + \frac{1}{2} \sigma_2^2 \eta (\delta - 1)[\rho^2 - 1] \hat{f}^{-1} \hat{f}_{\eta}^2 + \theta_0 \hat{f}_r + \frac{1}{2} \sigma_0^2 \hat{f}_{rr} \\ & - \frac{1}{2} \sigma_0^2 (\delta - 1) \hat{f}^{-1} \hat{f}_r^2 + \frac{1}{2} \sigma_2^2 \eta \hat{f}_{\eta\eta} + \left(\theta_2 - b\eta - \frac{k\eta\sigma_2\rho}{\sigma_1} \frac{\delta}{\delta - 1} \right) \hat{f}_{\eta} = 0, \end{aligned} \tag{43}$$

with the boundary condition $\hat{f}(T,r,\eta) = 1$.

Proof.

In lemma , we seek to convert PDE 41 to PDE 43. Define the differential operator Δ on any function $f(t,r,\eta)$ as:

$$\begin{aligned} \Delta f = & \left(\frac{1-\delta}{\delta} \left[r + \psi - \lambda - \gamma + \frac{1}{2} \frac{k^2 \eta}{\sigma_1^2 \delta} \right] \right) f + \frac{1}{2} \sigma_2^2 \eta (\delta - 1)[\rho^2 - 1] f^{-1} f_{\eta}^2 \\ & + \theta_0 f_r + \frac{1}{2} \sigma_0^2 f_{rr} - \frac{1}{2} \sigma_0^2 (\delta - 1) f^{-1} f_r^2 + \frac{1}{2} \sigma_2^2 \eta f_{\eta\eta} + \left(\theta_2 - b\eta - \frac{k\eta\sigma_2\rho}{\sigma_1} \frac{\delta}{\delta - 1} \right) f_{\eta}. \end{aligned} \tag{44}$$

Then equation 41 can also be written as:

$$\frac{\partial f}{\partial t} + \nabla f + Q(t) = 0, \tag{45}$$

where

$$f(T,r,\eta) = 1. \tag{46}$$

Notice that, on the other hand, we find:

$$\begin{aligned}\frac{\partial f}{\partial t}+\nabla f &= \frac{\partial}{\partial t}\left(\int_t^T \hat{f}(u,r,\eta)\right)+\nabla(\int_t^T \hat{f}(u,r,\eta)du)+\left(\frac{\partial}{\partial t}\hat{f}(t,r,\eta)+\nabla\hat{f}(t,r,\eta)\right)\\ &= -\hat{f}(t,r,\eta))+\left(\int_t^T \nabla\hat{f}(u,r,\eta)du\right)+\left(\frac{\partial}{\partial t}\hat{f}(t,r,\eta)+\nabla\hat{f}(t,r,\eta)\right)\\ &= -Q(t). \end{aligned} \tag{47}$$

Note also that

$$\begin{cases}\hat{f}(t,r,\eta)+\int_t^T \nabla\hat{f}(u,r,\eta)du = 1\\ \frac{\partial}{\partial t}\hat{f}(t,r,\eta)+\nabla\hat{f}(t,r,\eta)=0\end{cases}. \tag{48}$$

Therefore,

$$\frac{\partial\hat{f}}{\partial t}+\nabla\hat{f}=0, \tag{49}$$

where

$$\hat{f}(T,r,\eta)=1. \tag{50}$$

Thus, 41 can be converted to the following:

$$\begin{aligned}&\hat{f}_t+\left[\frac{1-\delta}{\delta}\left(r+\psi-\lambda-\gamma+\frac{1}{2}\frac{k^2\eta}{\sigma_1^2\delta}\right)\right]\hat{f}+\frac{1}{2}\sigma_2^2\eta(\delta-1)[\rho^2-1]\hat{f}^{-1}\hat{f}_\eta^2+\theta_0\hat{f}_r+\frac{1}{2}\sigma_0^2\hat{f}_{rr}\\ &-\frac{1}{2}\sigma_0^2(\delta-1)\hat{f}^{-1}\hat{f}_r^2+\frac{1}{2}\sigma_2^2\eta\hat{f}_{\eta\eta}+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho}{\sigma_1}\frac{\delta}{\delta-1}\right)\hat{f}_\eta\\ &=0,\end{aligned} \tag{51}$$

with the boundary condition $\hat{f}(T,r,\eta)=1$.

Thus, solving 33 or 41 is equivalent to solving 43. The nature of PDE 43 has well defined solutions.

Theorem 1 Suppose $V(t,r,\eta,x)$ is continuously differentiable and twice continuously differentiable for all $t\in[0,T]$ and $(r,x,\eta)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, then the solution of the PDE 33 is given by

$$V(t,r,\eta,x)=\frac{x^{1-\delta}}{1-\delta}[f(t,r,\eta)]^\delta, \tag{52}$$

where

$$\begin{aligned}f(t,r,\eta)=&\int_t^T \exp\{\mathcal{H}(u)\eta+\mathcal{L}(u)r+\mathcal{M}(u)\}du\\ &+\exp\{\mathcal{H}(t)\eta+\mathcal{L}(t)r+\mathcal{K}(t)\}.\end{aligned} \tag{53}$$

and $\mathcal{H}(t)$, $\mathcal{L}(t)$ and $\mathcal{M}(t)$ are determined in 71, 72 and 73 respectively.

In addition, the pair $(\pi^*,C^*,p^*(t))\in\mathcal{A}$ given by:

$$\pi^*(t)=\frac{k}{\sigma_1^2\delta}\mathcal{X}(t)+\frac{\sigma_2\rho f_\eta}{\sigma_1 f}\mathcal{X}(t), \tag{54}$$

$$C^*(t)=\mathcal{X}(t)f^{-1} \tag{55}$$

and

$$p^*(t)=\psi(t)\left[\frac{\psi(t)}{\lambda(t)}\right]^{-\frac{1}{\delta}}X(t)f^{-1}-\psi(t)X(t), \tag{56}$$

are the optimal policies.

Proof.

In theorem , assume the solution $\hat{f}(t,r,\eta)$ is given by

$$\hat{f}(t,r,\eta) = \exp\{\mathcal{H}(t)\eta + \mathcal{L}(t)r + \mathcal{M}(t)\}, \tag{57}$$

with the boundary condition $\mathcal{H}(T) = \mathcal{L}(T) = \mathcal{M}(T) = 0$.

From equation 57, we can obtain the following partial derivatives:

$$\begin{cases} \hat{f}_t = [\mathcal{H}'(t)\eta + \mathcal{L}'(t)r + \mathcal{M}'(t)]\hat{f}(t,r,\eta), \\ \hat{f}_r = \mathcal{L}(t)\hat{f}(t,r,\eta), \\ \hat{f}_{rr} = \mathcal{L}^2(t)\hat{f}(t,r,\eta) \\ \hat{f}_\eta = \mathcal{H}(t)\hat{f}(t,r,\eta), \\ \hat{f}_{\eta\eta} = \mathcal{H}^2(t)\hat{f}(t,r,\eta). \end{cases} \tag{58}$$

Substituting of 58 into 43, we obtain:

$$\begin{aligned} &(\mathcal{H}'(t)\eta + \mathcal{L}'(t)r + \mathcal{M}'(t))\hat{f}(t,r,\eta) + \left[\frac{1-\delta}{\delta}\left(r + \psi - \lambda - \gamma + \frac{1}{2}\frac{k^2\eta}{\sigma_1^2\delta}\right)\right]\hat{f}(t,r,\eta) \\ &+ \frac{1}{2}\sigma_2^2\eta(\delta-1)[\rho^2-1]\mathcal{H}^2\hat{f}(t,r,\eta) + \theta_0\mathcal{L}(t)\hat{f}(t,r,\eta) + \frac{1}{2}\sigma_0^2\mathcal{L}^2(t)\hat{f}(t,r,\eta) - \frac{1}{2}\sigma_0^2(\delta-1)\mathcal{L}^2(t)\hat{f}(t,r,\eta) \\ &+ \frac{1}{2}\sigma_2^2\eta\mathcal{H}^2(t)\hat{f}(t,r,\eta) + \left(\theta_2 - \eta - \frac{k\sigma_2\rho\eta}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t)\hat{f}(t,r,\eta) \\ &= 0. \end{aligned} \tag{59}$$

Canceling the term $\hat{f}(t,r,\eta)$ on both sides of 59 gives:

$$\begin{aligned} &(\mathcal{H}'(t)\eta + \mathcal{L}'(t)r + \mathcal{M}'(t)) + \left[\frac{1-\delta}{\delta}\left(r + \psi - \lambda - \gamma + \frac{1}{2}\frac{k^2\eta}{\sigma_1^2\delta}\right)\right] \\ &+ \frac{1}{2}\sigma_2^2\eta(\delta-1)[\rho^2-1]\mathcal{H}^2 + \theta_0\mathcal{L}(t) + \frac{1}{2}\sigma_0^2\mathcal{L}^2(t) \\ &- \frac{1}{2}\sigma_0^2(\delta-1)\mathcal{L}^2(t) + \frac{1}{2}\sigma_2^2\eta\mathcal{H}^2(t) + \left(\theta_2 - b\eta - \frac{k\sigma_2\rho\eta}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) \\ &= 0. \end{aligned} \tag{60}$$

Rewrite equation 60 to collect like terms in r and η gives:

$$\begin{aligned} &\eta\Big[\mathcal{H}'(t) + \frac{1}{2}\sigma_2^2\Big[(\delta-1)(\rho^2-1)+1\Big]\mathcal{H}^2(t) - \left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) + \frac{1}{2}\frac{k^2}{\sigma_1^2}\frac{(1-\delta)}{\delta^2}\Big] \\ &+ r\Big[\mathcal{L}'(t) + \frac{1-\delta}{\delta}\Big] \\ &+ \Big[\mathcal{M}'(t) - \frac{1}{2}\sigma_0^2\delta\mathcal{L}^2(t) + \theta_0\mathcal{L}(t) + \theta_2\mathcal{H}(t) - (-\psi+\lambda+\gamma)\frac{1-\delta}{\delta}\Big] \\ &= 0. \end{aligned} \tag{61}$$

Eliminating η and r, we split equation 61 into three ODE's as follows:

$$\begin{cases} \mathcal{H}'(t) = -\frac{1}{2}\sigma_2^2\Big[(\delta-1)(\rho^2-1)+1\Big]\mathcal{H}^2(t) + \left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) - \frac{1}{2}\frac{k^2}{\sigma_1^2}\frac{(1-\delta)}{\delta^2}, \\ \mathcal{H}(T) = 0. \end{cases} \tag{62}$$

$$\begin{cases} \mathcal{L}'(t) = -\frac{1-\delta}{\delta}, \\ \mathcal{L}(T) = 0. \end{cases} \tag{63}$$

and

$$\begin{cases} \mathcal{M}'(t) = \frac{1}{2}\sigma_0^2\delta\mathcal{L}^2(t) - \theta_0\mathcal{L}(t) - \theta_2\mathcal{H}(t) + (\psi + \lambda + \gamma)\frac{1-\delta}{\delta}, \\ \mathcal{M}(T) = 0. \end{cases} \tag{64}$$

Rewriting equation 62, we obtain:

$$\mathcal{H}'(t) = -\frac{1}{2}\sigma_2^2[(\delta-1)(\rho^2-1)+1]\Big[\mathcal{H}^2(t) - \frac{2}{\sigma_2^2[(\delta-1)(\rho^2-1)+1]}\left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) + \frac{k^2}{\sigma_1^2}\frac{(1-\delta)}{\delta^2\sigma_2^2[(\delta-1)(\rho^2-1)+1]}\Big]. \tag{65}$$

Let $\Delta_{\mathcal{H}}$ denote the discriminant of the quadratic equation

$$\mathcal{H}^2(t) - \frac{2}{\sigma_2^2[(\delta-1)(\rho^2-1)+1]}\left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) + \frac{k^2}{\sigma_1^2}\frac{(1-\delta)}{\delta^2\sigma_2^2[(\delta-1)(\rho^2-1)+1]} \tag{66}$$

Implying

$$\begin{aligned}\Delta_{\mathcal{H}} &= \frac{4}{\sigma_2^4\sigma_1^2((\delta-1)(\rho^2-1)+1)^2}\left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)^2 - \frac{4k^2}{\sigma_1^2}\frac{\delta}{\delta^2\sigma_2^2((\delta-1)(\rho^2-1)+1)} \\ &= \frac{4}{\sigma_2^4\sigma_1^2((\delta-1)(\rho^2-1)+1)^2}\Big[\frac{-k^2}{\delta-1} + \frac{\delta}{\delta-1}[(k\rho+k\rho)^2 + b^2(1-\rho^2)]\Big].\end{aligned} \tag{67}$$

Let the discriminant $\Delta_{\mathcal{H}}$ have distinct real solutions, that is $\Delta_{\mathcal{H}} > 0$, then we obtain the following condition for δ necessary for numerical analysis.

$$\delta < \frac{k^2}{(k\rho+k\rho)^2 + b^2(1-\rho^2)} < 1. \tag{68}$$

Considering condition 68, if we integrate both sides of 65 with respect to t, we obtain

$$\frac{1}{\xi_1-\xi_2}\int_t^T\Big[\frac{1}{\mathcal{H}-\xi_1} - \frac{1}{\mathcal{H}-\xi_2}\Big]d\mathcal{H}(t) = \frac{-1}{2}\sigma_2^2\sigma_1((\delta-1)(\rho^2-1)+1)(T-t), \tag{69}$$

where ξ_1 and ξ_2 are two distinct real solutions for 65 given by:

$$\begin{aligned}\xi_{1,2} = &\frac{1}{\sigma_2^2\sigma_1((\delta-1)(\rho^2-1)+1)^2}\Big[b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\Big] \\ &\pm\sqrt{\frac{1}{\sigma_2^4\sigma_1^2((\delta-1)(\rho^2-1)+1)^2}\Big[\frac{-k^2}{\delta-1} + \frac{\delta}{\delta-1}[(k\rho+k\rho)^2 + b^2(1-\rho^2)]\Big]}.\end{aligned} \tag{70}$$

Solving 69 with terminal conditions $\mathcal{H}(T) = 0$, we obtain:

$$\mathcal{H}(t) = \frac{\xi_1\xi_2\Big[1 - exp\Big[-\frac{\sigma_2^2\sigma_1}{2}[(\delta-1)(\rho^2-1)+1](\xi_1-\xi_2)(T-t)\Big]\Big]}{(\xi_1-\xi_2)exp\Big[-\frac{\sigma_2^2\sigma_1}{2}[(\delta-1)(\rho^2-1)+1](\xi_1-\xi_2)(T-t)\Big]}. \tag{71}$$

The solutions to the equations 63 and 64 is obtained directly as follows:

$$\mathcal{L}(t) = \frac{1-\delta}{\delta}(T-t) \tag{72}$$

and

$$\mathcal{M}(t) = \int_t^T\left[\frac{1}{2}\sigma_0^2\delta\mathcal{L}^2(t) - \theta_0\mathcal{L}(t) - \theta_2\mathcal{H}(t) + (-\psi + \lambda + \gamma)\frac{1-\delta}{\delta}\right]ds. \tag{73}$$

We characterize the value function as a classical solution of the HJB equation 33. Therefore, the value function V is represented as follows:

$$V(t,r,\eta,x) = \frac{x^{1-\delta}}{1-\delta}\Big[f(t,r,\eta)\Big]^{\delta}, \tag{74}$$

where

$$f(t,r,\eta) = \int_t^T \exp\{\mathcal{H}(u)\eta + \mathcal{L}(u)r + \mathcal{M}(u)\}du \\ + \exp\{\mathcal{H}(t)\eta + \mathcal{L}(t)r + \mathcal{K}(t)\}, \tag{75}$$

and $\mathcal{H}(t)$, $\mathcal{L}(t)$ and $\mathcal{M}(t)$ are determined in 71, 72 and 73 respectively.

In addition, the pair $(\pi^*, C^*, p^*(t)) \in \mathcal{A}$ given by

$$\pi^*(t) = \frac{k}{\sigma_1^2\delta}\mathcal{X}(t) + \frac{\sigma_2\rho f_\eta}{\sigma_1 f}\mathcal{X}(t), \tag{76}$$

$$C^*(t) = \mathcal{X}(t)f^{-1} \tag{77}$$

and

$$p^*(t) = \psi(t)\Big[\frac{\psi(t)}{\lambda(t)}\Big]^{-\frac{1}{\delta}}X(t)f^{-1} - \psi(t)X(t), \tag{78}$$

are the optimal policies.

8. Numerical Examples and Simulations

In this section, we show numerical examples, simulations and determine the relationship between optimal policies with different parameters.

When $\theta_0(t) = \sigma_0 = \sigma_2 = 0$, then the risk-free interest rate $r(t)$ and volatility term $\eta(t)$ of the stock are constants implying our problem becomes an investment, consumption and life insurance problem with constant interest rate and constant volatility rate with the following optimal policies

$$\pi^*(t) = \frac{k}{\sigma_1^2}\cdot\frac{1}{1-\delta}\mathcal{X}(t), \tag{79}$$

$$C^*(t) = \mathcal{X}(t)f^{-1}. \tag{80}$$

and

$$p^*(t) = \psi(t)\Big[\frac{\psi(t)}{\lambda(t)}\Big]^{-\frac{1}{\delta}}X(t)f^{-1} - \psi(t)X(t). \tag{81}$$

When $\rho = 1$, then the problem becomes an investment, consumption and life insurance problem with stochastic interest rate and stochastic volatility having the following optimal policies.

$$\pi^*(t) = \frac{k}{\sigma_1^2}\cdot\frac{1}{1-\delta}\mathcal{X}(t) + \frac{\sigma_2}{\sigma_1}\cdot\frac{f_\eta}{f}\mathcal{X}(t), \tag{82}$$

$$C^*(t) = \mathcal{X}(t)f^{-1}. \tag{83}$$

and

$$p^*(t) = \psi(t)\Big[\frac{\psi(t)}{\lambda(t)}\Big]^{-\frac{1}{\delta}}X(t)f^{-1} - \psi(t)X(t). \tag{84}$$

When $p(t) = 0$, the problem becomes an optimal investment and consumption without life insurance but with stochastic interest rate and stochastic volatility. The optimal policies are given as follows:

$$\pi^*(t) = \frac{k}{\sigma_1^2} \cdot \frac{1}{1-\delta}\mathcal{X}(t) + \frac{\sigma_2}{\sigma_1} \cdot \frac{f_\eta}{f}\mathcal{X}(t) \tag{85}$$

and

$$C^*(t) = \phi^{\frac{1}{1-\delta}}\mathcal{X}(t)f^{-1}. \tag{86}$$

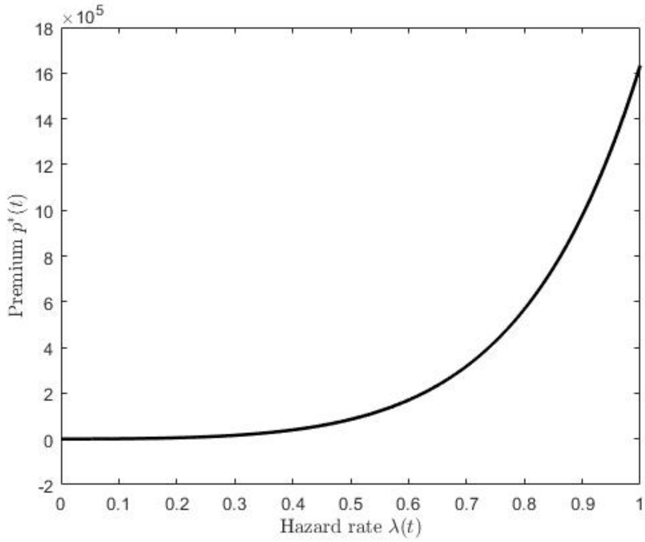

Figure 1. The effects of hazard rate $\lambda(t)$ on optimal premiums $p^*(t)$ when $t = 0 : 0.001 : 1$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\delta = 0.5$; $\lambda = 0 : 0.001 : 1$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\gamma = 1$; $\psi = 0.00511$; $\mathcal{X} = 2000$ and $\sigma_2 = 1.1$

In figure 1, the hazard rate $\lambda(t)$ has positive effect on optimal premium $p^*(t)$. In reality, mortality risk increases as the hazard rate increases. Thus, high mortality risk led to higher premiums. This agrees with practical investments and our intuition.

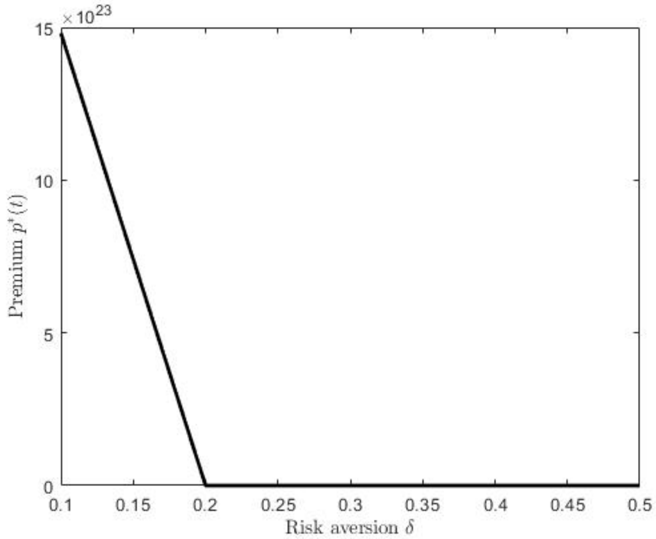

Figure 2. The effects of risk aversion δ on optimal premiums $p^*(t)$ when $t = 0 : 0.1 : 0.5$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\delta = 0 : 0.1 : 0.5$; $\lambda = 1$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\gamma = 1$; $\psi = 0.005$; $\mathcal{X} = 2000$ and $\sigma_2 = 1.1$

In figure 2, the risk-aversion δ has negative effect on optimal premium $p^*(t)$. The wage-earner's relative risk aversion is $1 - \delta$, so the wage-earner with high risk aversion purchases more life insurance.

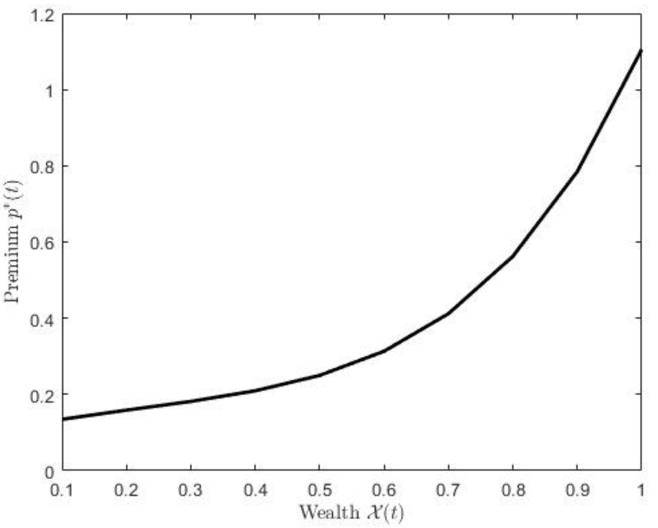

Figure 3. The effects of wealth $\mathcal{X}(t)$ on optimal premiums $p^*(t)$ when $t = 0 : 0.1 : 1$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\delta = 0.5$; $\lambda = 1$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\gamma = 1$; $\psi = 0.005$; $\mathcal{X} = 0 : 0.1 : 1$ and $\sigma_2 = 1.1$

In figure 3, accumulation of wealth has a positive effect on optimal premium $p^*(t)$. Note that the larger the wealth the more the wage-earner purchases life insurance to hedge against the risk in the financial market. Hence the wage-earner tends to invest more in the insurance market with the accumulation of wealth.

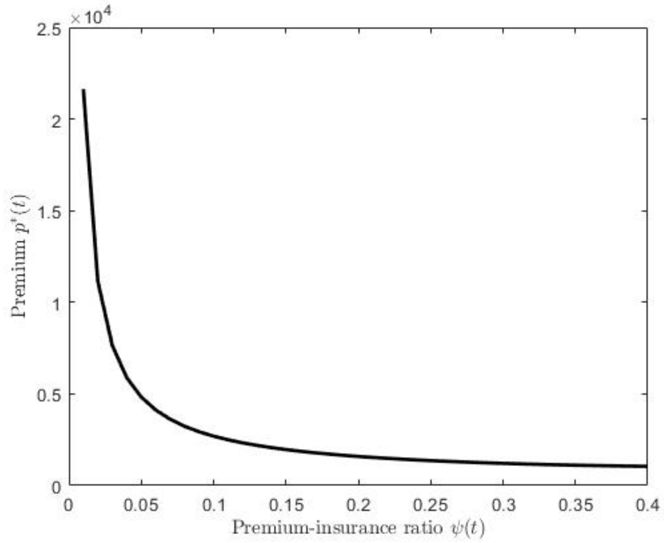

Figure 4. The effects of premium-insurance ratio $\psi(t)$ on optimal premiums $p^*(t)$ when $t = 0 : 0.01 : 0.4$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\delta = 0.5$; $\lambda = 1$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\gamma = 1$; $\psi = 0 : 0.01 : 0.4$; $\mathcal{X} = 2000$ and $\sigma_2 = 1.1$

In figure 4, the premium- insurance ratio $\psi(t)$ has negative effect on premium $p^*(t)$. Note that $\frac{p(t)}{\psi(t)}$ determines how big the benefit to be given to the beneficiary upon the wage-earners death at time $\tau < T$. For higher premium, the premium-insurance ratio, must be low in order for the policy to

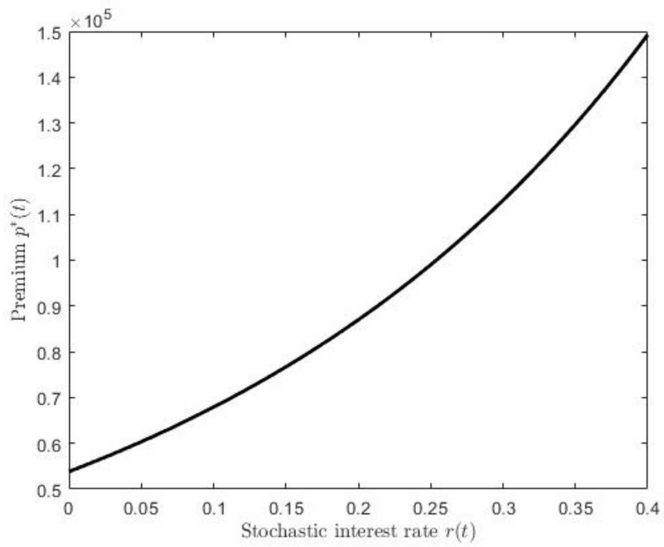

Figure 5. The effects of stochastic interest rate $r(t)$ on optimal premiums $p^*(t)$ when $t = 0 : 0.01 : 0.4$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0.6$; $r = 0 : 0.01 : 0.4$; $\delta = 0.5$; $\lambda = 1$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\gamma = 1$; $\psi = 0.005$; $\mathcal{X} = 2000$ and $\sigma_2 = 1.1$

In figure 5, the stochastic interest rate $r(t)$ has positive effect on optimal premium $p^*(t)$. In reality, when the stochastic interest rate increases the wage-earner invests less in the financial market. Thus, the wage-earner invests more in the insurance market when he/she faces higher interest rates in the financial market. This agrees with practical investments and our intuition.

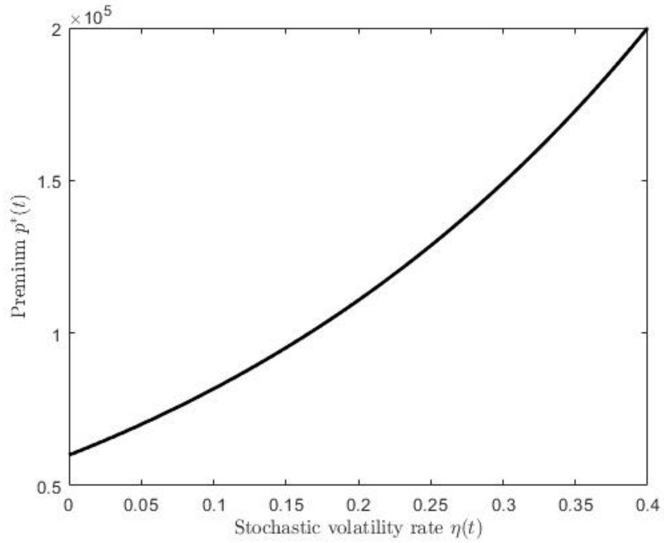

Figure 6. The effects of stochastic volatility rate $\eta(t)$ on optimal premiums $p^*(t)$ when $t = 0 : 0.01 : 0.4$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0 : 0.01 : 0.4$; $r = 0.4$; $\delta = 0.5$; $\lambda = 1$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\gamma = 1$; $\psi = 0.005$; $\mathcal{X} = 2000$ and $\sigma_2 = 1.1$

In figure 6, the stochastic volatility rate $\eta(t)$ has positive effect on optimal premium $p^*(t)$. When the stochastic volatility rate is high, the wage-earner invests less in the financial market. Instead, he/she invest more in the insurance market. This agrees with practical investments and our intuition.

9. Conclusion

We investigated an optimal investment, consumption and life insurance problem with stochastic interest rate and stochastic volatility. We considered a single wage-earner with a portfolio consisting of one risk-free security and one risky security and life insurance policy. The wage-earner invests in life insurance to hedge against the risk of unexpected death. By applying Bellman's optimality principle, we obtain the HJB PDE for the value function. Closed-form solutions when the risky preference of an investor satisfies a power utility are also established. Results showed that the risk aversion parameter δ and the premium-insurance ratio have negative effects on optimal premium $p^*(t)$. However, the Hazard rate $\lambda(t)$, wealth $\mathcal{X}(t)$, stochastic interest rate $r(t)$ and stochastic volatility $\eta(t)$ had positive effects on optimal premium. Our study can be extended in so many directions. For instance, we can introduce other utility functions. We can also consider other life insurance products on the market. Furthermore, we can also introduce multiple risky securities which leads to more sophisticated nonlinear second-order partial differential equations.

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