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Analytical Solutions of Classical and Fractional Navier-Stokes Equations by the SBA Method

Kamate Adama[1], Bationo Jeremie Yiyureboula[1], Djibet Mbaiguesse[2], Youssouf Pare[1]

[1] University Joseph KI-ZERBO, Burkina Faso

[2] University of N'Djamena, Tchad

Correspondence: Youssouf Pare, University Joseph KI-ZERBO, Burkina Faso

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Abstract

In this paper, we use the SBA method (a combination of the method of successive approximations, the Adomian decompositional method and Picard's principle) to obtain the analytical solutions of the systems of classical and fractional Navier-Stokes equations in Cartesian coordinates. The fractional derivative involved in the equations is the Caputo derivative. Then, we compare the solution of the classical system with the solution of the fractional system when the fractional derivative order α tends to 1.

Keywords: SBA method, Navier-Stokes equations, fractional integral, Caputo fractional derivative

Classification AMS 2020: 34G20, 47J25, 65J15, 97N40, 76D05

1. Introduction

The mathematical description of viscous fluid flows is given by the Navier-Stokes equations (Foias, C., Manley, O., Rosa, R., & Temam, R, 2001; Quartapelle, L., 1993; Teman, R., 1983; Heywood, J. G., 1990), a system of partial differential equations that result from the conservation laws of mass, momentum and energy. For an incompressible fluid, these equations are given by the following system:

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial t} - \nu\Delta \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla p = f & \forall\,(\mathrm{x},t) \in \Omega\times[0,T[\\ \mathrm{divu} = 0 & \forall\,(\mathrm{x},t) \in \Omega\times[0,T[\end{cases} \tag{1}$$

where $u = u(\mathrm{x}, t)$ is the velocity of the fluid at point $\mathrm{x} \in \Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ at time t, $p = p(\mathrm{x}, t)$ is the pressure, $f = f(\mathrm{x}, t)$ is the external volume force, and ν is the kinematic viscosity.

To this system, we add the initial condition

$$u(\mathrm{x}, 0) = u_0(\mathrm{x}) \tag{2}$$

In dimension two of space ($d = 2$), we use the SBA (SOME Blaise-Abbo) (Pare,Y., Bassono, F., & Some, B., 2012; Pare, Y., 2010; Zongo, G., So, O., & Pare, Y., 2016) method first to find the analytical solution of the classical Navier-Stokes system (1) in Cartesian coordinates with the initial condition (2). Then, we use it to find the analytical solution of the fractional Navier-Stokes system in time:

$$\begin{cases} \dfrac{\partial^\alpha \mathrm{u}}{\partial t^\alpha} - \nu\Delta \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla p = f & \forall\,(\mathrm{x},t) \in \Omega\times[0,T[\\ \mathrm{divu} = 0 & \forall\,(\mathrm{x},t) \in \Omega\times[0,T[\end{cases} \tag{3}$$

with the initial condition (2); where $\dfrac{\partial^\alpha}{\partial t^\alpha}$ is the fractional derivative in the Caputo sense of order α with $0 < \alpha < 1$.

The SBA method is a combination of the successive approximations method (Goursat, E., 1956; Picard, E.,1896; Picard, E.,1893), the Adomian decompositional method (Abbaoui, K., & Cherruault, Y., 1999; Abbaoui, 1995; Zongo et al, 2016; Adomian, 1994) and the Picard principle. It was proposed by Somé Blaise in 2006, and allows to bypass the difficulties linked to the computation of Adomian polynomials. It was used by Abbo Bakary in (Abbo, B., 2007) to solve ODEs and nonlinear PDEs in dimension 1 space. It has also been successfully used in (Pare, Y., 2010) for solving nonlinear evolutionary partial differential equations of the Cauchy and Cauchy-Von Neumann type in finite-dimensional space, integral equations and integro-differential equations.

2. Definitions and Basic Properties

2.1 Some Basic Functions of Fractional Calculus

Gamma function

The Euler Gamma function (Podlubny, I., & Kenneth, V. T., 1999; Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J., 2006) is defined on the half-plane $P = \{z \in \mathbb{C} / \ Re(z) > 0\}$ by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} dt \tag{4}$$

For any natural number $n : \Gamma(n+1) = n!$

Beta function

The Beta function (Hilfer, R., 2000; Kilbas et al., 2006) is defined for all complex numbers u and v such that $Re(u) > 0$ and $Re(v) > 0$ by

$$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt. \tag{5}$$

It is related to the gamma function by the relation (Hilfer, R., 2000; Samko, S. G., Kilbas, A. A., & Marichev, O. I., 1993; Kilbas et al., 2006)

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \tag{6}$$

Mittag-Leffler function

For any complex number z, we define the one-parameter Mittag-Leffler function (Hilfer, R., 2000; Kilbas et al., 2006) by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}; \ \alpha \in \mathbb{C}, \ Re(\alpha) > 0. \tag{7}$$

In particular, when $\alpha = 1$, this function coincides with the exponential function:

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \mathrm{e}^z \tag{8}$$

2.2 Fractional Riemman-Liouville Integral

Definition 1 Let $[a, b]$ be a finite interval of R and $f \in L^1([a, b])$. The fractional Riemann-Liouville left-handed integral of order $\alpha > 0$ of the function f is defined by (Hilfer, R., 2000)

$$I_{a,x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \tag{9}$$

Lemma 1 Let $f(x)$ be the power function $f(x) = (x-a)^\gamma$ Then, we have (Almeida, R., Tavares, D., Torres, D. F. M., 2019)

$$I_{a,x}^{\alpha} f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (x-a)^{\gamma+\alpha}, \ \gamma > -1 \tag{10}$$

2.3 Caputo Fractional Derivative

Definition 2 The fractional Caputo left derivative of order $\alpha > 0$ of the function $f(x)$, $x \in [a, b]$ is defined by (Almeida et al., 2019)

$$\begin{aligned} {}_C D_{a,x}^{\alpha} f(x) &= I_{a,x}^{m-\alpha} \left(f^{(m)}(x) \right) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt \end{aligned} \tag{11}$$

Where $m = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $m = \alpha$ if $\alpha \in \mathbb{N}$.

Lemma 2 Let $f(x)$ be the power function $f(x) = (x-a)^{\gamma}$ Then, we have (Almeida et al., 2019)

$$ {}_{C}D^{\alpha}_{a,x} f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (x-a)^{\gamma-\alpha}, \ \gamma > m-1. \tag{12} $$

Where $m = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $m = \alpha$ if $\alpha \in \mathbb{N}$.

Theorem 1 Let $\alpha > 0$. $f \in C^m([a,b];\mathbb{R})$, then we have (Almeida et al., 2019)

$$ I^{\alpha}_{a,x} {}_{C}D^{\alpha}_{a,x} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(k+1)} (x-a)^k . \tag{13} $$

with $m = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $m = \alpha$ if $\alpha \in \mathbb{N}$.

3. Principle of the SBA Method

Consider for example the functional equation:

$$ \frac{\partial u}{\partial t} = Ru + Nu + f \tag{14} $$

where R is a linear operator and N a nonlinear operator defined in a suitably chosen space V; $f \in V$ and u the unknown function.

Let $L_t(.) = \dfrac{\partial(.)}{\partial t}$, Then the equation (14) is written

$$ L_t u = Ru + Nu + f \tag{15} $$

Applying L_t^{-1} to (15), we obtain the Adomian canonical form:

$$ u = \theta + L_t^{-1} f + L_t^{-1} Ru + L_t^{-1} Nu \tag{16} $$

where θ verifies $L_t\theta = 0$ and L_t^{-1} is the "inverse" of L_t.

Apply the method of successive approximations to (16), we get:

$$ u^k = \theta + L_t^{-1} f + L_t^{-1} Ru^k + L_t^{-1} Nu^{k-1}, \ k \geq 1 \tag{17} $$

The Adomian agorithm associated with (17) is as follows:

$$ \begin{cases} u_0^k = \theta + L_t^{-1} f + L_t^{-1} Nu^{k-1}, \ k \geq 1 \\ u_{n+1}^k = L_t^{-1} Ru^k, \ n \geq 0 \end{cases} \tag{18} $$

Let us apply Picard's principle to (18): we choose $u^0 \in V$ any root of the equation $Nu = 0$.

- Step 1

For $k = 1$, we compute u^1 using the following algorithm

$$ \begin{cases} u_0^1 = \theta + L_t^{-1} f \\ u_{n+1}^1 = L_t^{-1} Ru_n^1, \ n \geq 0 \end{cases} \tag{19} $$

If the series $\sum_{n\geq 0} u_n^1$ is convergent, then we get:

$$ u^1 = \sum_{n\geq 0} u_n^1 \tag{20} $$

approximate solution of (14) in step 1.

- Step 2

For $k = 2$, we compute u^2 using the following algorithm:

$$\begin{cases} u_0^2 = \theta + L_t^{-1} f + L^{-1} N u^1 \\ u_{n+1}^2 = L_t^{-1} R u_n^2, \ n \geq 0 \end{cases} \tag{21}$$

If the series $\sum_{n\geq 0} u_n^2$ is convergent, then we get:

$$u^2 = \sum_{n\geq 0} u_n^2 \tag{22}$$

approximate solution of (14) in step 2.

- **Step k**

Recursively, if the series $\sum_{n\geq 0} u_n^k$ is convergent for $k \geq 1$, then we get

$$u^k = \sum_{n\geq 0} u_n^k \tag{23}$$

approximate solution of (14) in step k.

The solution of (14) is then:

$$u = \lim_{k\to\infty} u^k \tag{24}$$

Proposition 1.

Let be the functional equation:

$$\frac{\partial u}{\partial t} = Ru + Nu + f \tag{25}$$

where R is a linear operator and N a nonlinear operator defined in a suitably chosen space V; $f \in V$ *and u the unknown function*

Let the SBA agorithm associated with (25):

$$\begin{cases} u_0^k = \theta + L_t^{-1} f + L_t^{-1} N u^{k-1}, \ k \geq 1 \\ u_{n+1}^k = L_t^{-1} R u_n^k, \ n \geq 0 \end{cases} \tag{26}$$

By Picard's principle, we choose $u^0 \in V$ *such that* $Nu^0 = 0$.

If $Nu^1 = 0$, *then the problem (25) admits a unique solution.*

Proof

Existence: Let u^1 be the approximate solution in step 1. Assume that $Nu^1 = 0$, so the scheme in step 2 is written:

$$\begin{cases} u_0^2 = \theta + L_t^{-1} f \\ u_{n+1}^2 = L_t^{-1} R u_n^2, \ n \geq 0 \end{cases} \tag{27}$$

This scheme is identical to the scheme in step 1. So the approximate solution in step 2 is $u^2 = u^1$.

We have $Nu^2 = Nu^1 = 0$; therefore the scheme at step 3 is also identical to the scheme at step 2. Therefore, the solution at step 3 is $u^3 = u^2 = u^1$.

Recursively, the approximate solution at step $k(k \geq 2)$ is $u^k = u^{k-1} = ... = u^1$.

The solution of the problem (25) is

$$u = \lim_{k\to\infty} u^k = u^1$$

Uniqueness: suppose that the problem (25)admits by the SBA method two distinct solutions u and v.Let $\varphi = u - v$. Then we have:

$$\begin{cases} u_0^k = \theta + L^{-1}f + L_t^{-1}Nu^{k-1}, \ k \geq 1 \\ u_{n+1}^k = L_t^{-1}Ru_n^k, \ n \geq 0 \end{cases} \quad (28)$$

and

$$\begin{cases} v_0^k = \theta + L^{-1}f + L_t^{-1}Nv^{k-1}, \ k \geq 1 \\ v_{n+1}^k = L_t^{-1}Rv_n^k, \ n \geq 0 \end{cases} \quad (29)$$

Making the difference (28)-(29), we get

$$\begin{cases} \varphi_0^k = L_t^{-1}Nu^{k-1} - L_t^{-1}Nv^{k-1}, \ k \geq 1 \\ \varphi_{n+1}^k = L_t^{-1}R(\varphi_n^k), \ n \geq 0 \end{cases} \quad (30)$$

where $\varphi_n^k = u_n^k - v_n^k$

Step 1: for $k = 1$, we have

$$\begin{cases} \varphi_0^1 = 0, \ k \geq 1 \\ \varphi_{n+1}^1 = L_t^{-1}R(\varphi_n^1), \ n \geq 0 \end{cases} \quad (31)$$

- for $n = 0$, we have

$$\varphi_1^1 = L_t^{-1}R(\varphi_0^1) = 0$$

- for $n = 1$, we have

$$\varphi_2^1 = L_t^{-1}R(\varphi_1^1) = 0$$

- We find that for all $n \geq 0$, $\varphi_n^1 = 0$. Therefore, we have:

$$\varphi^1 = \sum_{n\geq 0} \varphi_n^1 = 0$$

Therefore, we obtain $u^1 = v^1$.

Step 2: for $k = 2$, we have

$$\begin{cases} \varphi_0^2 = L_t^{-1}Nu^1 - L_t^{-1}Nv^1, \ k \geq 1 \\ \varphi_{n+1}^2 = L_t^{-1}R(\varphi_n^2), \ n \geq 0 \end{cases} \quad (32)$$

If $Nu^1 = 0$ and $Nv^1 = 0$, then

$$\begin{cases} \varphi_0^2 = 0, \ k \geq 1 \\ \varphi_{n+1}^2 = L_t^{-1}R(\varphi_n^2), \ n \geq 0 \end{cases} \quad (33)$$

This pattern is identical to the pattern in step1; thus for all $n \geq 0$, $\varphi_n^2 = 0$.Hence

$$\varphi^2 = \sum_{n\geq 0} \varphi_n^2 = 0$$

Therefore, we get$u^2 = v^2$.

Recursively, for all $k \geq 1$, $u^k = v^k$. Therefore$u = v$; which is absurd. So the problem (25) admits a unique solution.

4. Application of the SBA Method

4.1 Resolution of the Classical Navier-Stokes System in Dimension 2

In Cartesian coordinates (x, y), the velocity is represented by its components (u, v). The Navier-Stokes system is written:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + \dfrac{\partial p}{\partial x} = f_1 \\ \dfrac{\partial v}{\partial t} - \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + \dfrac{\partial p}{\partial y} = f_2 \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases} \tag{34}$$

For simplicity, we assume that $f_1 = f_2 = 0$; $\dfrac{\partial p}{\partial x} = -g$ and $\dfrac{\partial p}{\partial y} = g$, where g is a constant. The system (34) becomes:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = g \\ \dfrac{\partial v}{\partial t} - \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -g \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases} \tag{35}$$

Let's add to the system (35) the initial condition

$$(u(x, y, 0), v(x, y, 0)) = (-\cos(x + y), \cos(x + y)) \tag{36}$$

We will now solve the problem (35)-(36) by the SBA method.

Let $L_t(.) = \dfrac{\partial(.)}{\partial t}$, $R(.) = \dfrac{\partial^2(.)}{\partial x^2} + \dfrac{\partial^2(.)}{\partial y^2}$ and $N(.) = u\dfrac{\partial(.)}{\partial x} + v\dfrac{\partial(.)}{\partial y}$. Then, the first two equations of the system (35) yield the following system

$$\begin{cases} L_t u = \nu Ru - Nu + g \\ L_t v = \nu Rv - Nv - g \end{cases} \tag{37}$$

Applying L_t^{-1} to (37) and using the initial condition (36), we obtain the following Adomian canonical form:

$$\begin{cases} u(x, y, t) &= -\cos(x + y) + \nu L_t^{-1} Ru - L_t^{-1} Nu + L_t^{-1} g \\ v(x, y, t) &= \cos(x + y) + \nu L_t^{-1} Rv - L_t^{-1} Nv - L_t^{-1} g \end{cases} \tag{38}$$

where $L_t^{-1}(.) = \int_0^t (.)ds$.

By applying the method of successive approximation, we obtain

$$\begin{cases} u^k(x, y, t) = -\cos(x + y) + \nu L_t^{-1} Ru^k - L_t^{-1} Nu^{k-1} + L_t^{-1} g, \ \ k \geq 1 \\ v^k(x, y, t) = \cos(x + y) + \nu L_t^{-1} Rv^k - L_t^{-1} Nv^{k-1} - L_t^{-1} g, \ \ k \geq 1 \end{cases} \tag{39}$$

Let us apply to (39) the Adomian algorithm

$$\begin{cases} u_0^k(x,y,t) = -\cos(x+y) - L_t^{-1} N u^{k-1} + L_t^{-1} g, \ k \geq 1 \\ v_0^k(x,y,t) = \cos(x+y) - L_t^{-1} N v^{k-1} - L_t^{-1} g, \ k \geq 1 \\ u_n^k(x,y,t) = \nu L_t^{-1} R u_{n-1}^k, \ n \geq 1 \\ v_n^k(x,y,t) = \nu L_t^{-1} R v_{n-1}^k, \ n \geq 1 \end{cases} \tag{40}$$

Then we apply to (40) Picard's principle: we take $u^0(x,y,t) = 0$ a root of the equation $Nu = 0$.

Step 1

- For $k = 1$; we compute $u^1(x,y,t)$ using the following algorithm:

$$\begin{cases} u_0^1(x,y,t) &= -\cos(x+y) + gt \\ v_0^1(x,y,t) &= \cos(x+y) - gt \\ u_n^1(x,y,t) &= \nu L_t^{-1} R u_{n-1}^1, \ n \geq 1 \\ v_n^1(x,y,t) &= \nu L_t^{-1} R v_{n-1}^1, \ n \geq 1 \end{cases} \tag{41}$$

- For $n = 1$, we have:

$$\begin{cases} u_1^1(x,y,t) = \nu L_t^{-1} R u_0^1 = \nu L_t^{-1} \left(2\cos(x+y)\right) = 2\nu t \cos(x+y) \\ v_1^1(x,y,t) = \nu L_t^{-1} R v_0^1 = \nu L_t^{-1} \left(-2\cos(x+y)\right) = -2\nu t \cos(x+y) \end{cases} \tag{42}$$

- For $n = 2$, we have:

$$\begin{cases} u_2^1(x,y,t) &= \nu L_t^{-1} R u_1^1 = \nu L_t^{-1} \left(4\nu t \cos(x+y)\right) = -(2\nu)^2 \dfrac{t^2}{2} \cos(x+y) \\ v_2^1(x,y,t) &= \nu L_t^{-1} R v_1^1 = \nu L_t^{-1} \left(4\nu t \cos(x+y)\right) = (2\nu)^2 \dfrac{t^2}{2} \cos(x+y) \end{cases} \tag{43}$$

- For $n = 3$, we have:

$$\begin{cases} u_3^1(x,y,t) &= \nu L_t^{-1} R u_2^1 = \nu L_t^{-1} \left(2(2\nu)^2 \dfrac{t^2}{2} \cos(x+y)\right) = (2\nu)^3 \dfrac{t^3}{3!} \cos(x+y) \\ v_3^1(x,y,t) &= \nu L_t^{-1} R v_2^1 = \nu L_t^{-1} \left(-2(2\nu)^2 \dfrac{t^2}{2} \cos(x+y)\right) = (-2\nu)^3 \dfrac{t^3}{3!} \cos(x+y) \end{cases} \tag{44}$$

We note that for any integer $n \geq 1$:

$$\begin{cases} u_n^1(x,y,t) &= \dfrac{(-1)^{n-1}(2\nu t)^n}{n!} \cos(x+y) = -\dfrac{(-2\nu t)^n}{n!} \cos(x+y) \\ v_n^1(x,y,t) &= \dfrac{(-2\nu t)^n}{n!} \cos(x+y) \end{cases} \tag{45}$$

We have:

$$\begin{aligned} u^1(x,y,t) = \sum_{n=0}^{+\infty} u_n^1(x,y,t) &= u_0^1(x,y,t) + \sum_{n=1}^{+\infty} u_n^1(x,y,t) \\ &= -\cos(x+y) + gt - \cos(x+y) \sum_{n=1}^{+\infty} \frac{(-2\nu t)^n}{n!} \\ &= gt - \cos(x+y) \sum_{n=0}^{+\infty} \frac{(-2\nu t)^n}{n!} \\ &= gt - \cos(x+y) e^{-2\nu t} \end{aligned} \tag{46}$$

$$\begin{aligned} v^1(x,y,t) &= \sum_{n=0}^{+\infty} v_n^1(x,y,t) = v_0^1(x,y,t) + \sum_{n=1}^{+\infty} v_n^1(x,y,t) \\ &= \cos(x+y) - gt + \cos(x+y)\sum_{n=1}^{+\infty}\frac{(-2\nu t)^n}{n!} \\ &= -gt + \cos(x+y)\sum_{n=0}^{+\infty}\frac{(-2\nu t)^n}{n!} \\ &= -gt + \cos(x+y)\mathrm{e}^{-2\nu t} \end{aligned} \tag{47}$$

The approximate solution of the problem (37)-(36) in step1 is thus

$$\left(u^1, v^1\right) = \left(gt - \cos(x+y)\mathrm{e}^{-2\nu t}, -gt + \cos(x+y)\mathrm{e}^{-2\nu t}\right) \tag{48}$$

Step 2

We have

$$N(u^1) = u^1\frac{\partial u^1}{\partial x} + v^1\frac{\partial u^1}{\partial y} \tag{49}$$

$$\begin{aligned} &= \left(gt - \cos(x+y)\mathrm{e}^{-2\nu t}\right)\left(\sin(x+y)\mathrm{e}^{-2\nu t}\right) \\ &+ \left(-gt + \cos(x+y)\mathrm{e}^{-2\nu t}\right)\left(\sin(x+y)\mathrm{e}^{-2\nu t}\right) = 0 \end{aligned} \tag{50}$$

and

$$N(v^1) = u^1\frac{\partial v^1}{\partial x} + v^1\frac{\partial v^1}{\partial y} \tag{51}$$

$$\begin{aligned} &= \left(gt - \cos(x+y)\mathrm{e}^{-2\nu t}\right)\left(-\sin(x+y)\mathrm{e}^{-2\nu t}\right) \\ &+ \left(-gt + \cos(x+y)\mathrm{e}^{-2\nu t}\right)\left(-\sin(x+y)\mathrm{e}^{-2\nu t}\right) = 0 \end{aligned} \tag{52}$$

Therefore, according to Proposition1, the problem (37)-(36) admits a unique solution

$$(u(x,y,t), v(x,y,t)) = (u^1, v^1) = \left(gt - \cos(x+y)\mathrm{e}^{-2\nu t}, -gt + \cos(x+y)\mathrm{e}^{-2\nu t}\right) \tag{53}$$

The pair (53) checks the incompressibility condition. Therefore, the exact solution of the problem (35)-(36) is

$$(u(x,y,t), v(x,y,t)) = \left(gt - \cos(x+y)\mathrm{e}^{-2\nu t}, -gt + \cos(x+y)\mathrm{e}^{-2\nu t}\right) \tag{54}$$

4.2 Solving the Fractional Navier-Stokes System in Dimension 2

We consider the following fractional Navier-Stokes system

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} - \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = g \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} - \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases} \tag{55}$$

with the following initial condition

$$(u(x,y,0), v(x,y,0)) = (-\cos(x+y), \cos(x+y)) \tag{56}$$

We will solve the problem (55)-(56) by the SBA method.

Using the operators, the first two equations of the system (55) yield the following system.

$$\begin{cases} D_t^\alpha u &= g + \nu Ru - Nu \\ D_t^\alpha v &= -g + \nu Rv - Nv \end{cases} \tag{57}$$

Where $D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$. Applying $I_{0,t}^\alpha$ to the system (57), we obtain

$$\begin{cases} u(x,y,t) - \sum_{k=0}^{m-1} \dfrac{u_t^{(k)}(x,y,0)}{\Gamma(k+1)} t^k = I_{0,t}^\alpha g + \nu I_{0,t}^\alpha Ru - I_{0,t}^\alpha Nu \\ v(x,y,t) - \sum_{k=0}^{m-1} \dfrac{v_t^{(k)}(x,y,0)}{\Gamma(k+1)} t^k = -I_{0,t}^\alpha g + \nu I_{0,t}^\alpha Rv - I_{0,t}^\alpha Nv, \end{cases} \tag{58}$$

or

$$\begin{cases} u(x,y,t) &= u(x,y,0) + I_{a,t}^\alpha g + \nu I_{a,t}^\alpha Ru - I_{a,t}^\alpha Nu \\ v(x,y,t) &= u(x,y,0) - I_{0,t}^\alpha g + \nu I_{0,t}^\alpha Rv - I_{0,t}^\alpha Nv \end{cases} \tag{59}$$

Applying the method of successive approximation to (59), we obtain

$$\begin{cases} u^k(x,y,t) = -\cos(x+y) + I_{0,t}^\alpha g + \nu I_{0,t}^\alpha Ru^k - I_{0,t}^\alpha Nu^{k-1}, \ k \geq 1 \\ v^k(x,y,t) = \cos(x+y) - I_{0,t}^\alpha g + \nu I_{0,t}^\alpha Rv^k - I_{0,t}^\alpha Nv^{k-1}, \ k \geq 1 \end{cases} \tag{60}$$

Let us apply to (60) the Adomian algorithm

$$\begin{cases} u_0^k(x,y,t) = -\cos(x+y) + I_{0,t}^\alpha g - I_{0,t}^\alpha Nu^{k-1}, \ k \geq 1 \\ v_0^k(x,y,t) = \cos(x+y) - I_{0,t}^\alpha g - I_{0,t}^\alpha Nv^{k-1}, \ k \geq 1 \\ u_n^k(x,y,t) = \nu I_{0,t}^\alpha Ru_{n-1}^k, \ n \geq 1 \\ v_n^k(x,y,t) = \nu I_{0,t}^\alpha Rv_{n-1}^k, \ n \geq 1 \end{cases} \tag{61}$$

Then we apply to (61) Picard's principle: we take $u^0(x,y,t) = 0$ a root of the equation $Nu = 0$.

First step

For $k = 1$; we compute $u^1(x,y,t)$ using the following algorithm:

$$\begin{cases} u_0^1(x,y,t) = -\cos(x+y) + I_{0,t}^\alpha g \\ v_0^1(x,y,t) = \cos(x+y) - I_{0,t}^\alpha g \\ u_n^1(x,y,t) = \nu I_{0,t}^\alpha Ru_{n-1}^1, \ n \geq 1 \\ v_n^1(x,y,t) = \nu I_{0,t}^\alpha Rv_{n-1}^1, \ n \geq 1 \end{cases} \tag{62}$$

We have

$$\begin{cases} u_0^1(x,y,t) = -\cos(x+y) + \dfrac{g}{\Gamma(\alpha+1)} t^\alpha \\ v_0^1(x,y,t) = \cos(x+y) - \dfrac{g}{\Gamma(\alpha+1)} t^\alpha \end{cases} \tag{63}$$

- For $n = 1$, we have:

$$\begin{cases} u_1^1(x,y,t) = \nu I_{0,t}^\alpha Ru_0^1 = \nu I_{0,t}^\alpha (2\cos(x+y)) = 2\nu\cos(x+y) \dfrac{1}{\Gamma(\alpha+1)} t^\alpha \\ v_1^1(x,y,t) = \nu I_{0,t}^\alpha Rv_0^1 = \nu I_{0,t}^\alpha (-2\cos(x+y)) = -2\nu\cos(x+y) \dfrac{1}{\Gamma(\alpha+1)} t^\alpha \end{cases} \tag{64}$$

- For $n = 2$, we have:

$$\begin{aligned} u_2^1(x,y,t) &= \nu I_{0,t}^{\alpha} R u_1^1 = \nu I_{0,t}^{\alpha}\left(-4\nu\cos(x+y)\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right) \\ &= -4\nu^2\cos(x+y)\frac{1}{\Gamma(\alpha+1)} I_{0,t}^{\alpha}\left(t^{\alpha}\right) \\ &= -(2\nu)^2\cos(x+y)\frac{1}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tau^{\alpha} d\tau \end{aligned} \tag{65}$$

and

$$\begin{aligned} v_2^1(x,y,t) &= \nu I_{0,t}^{\alpha} R v_1^1 = \nu I_{0,t}^{\alpha}\left(4\nu\cos(x+y)\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right) \\ &= (2\nu)^2\cos(x+y)\frac{1}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tau^{\alpha} d\tau \end{aligned} \tag{66}$$

Performing the change of variable $\tau = wt$, we obtain:

$$\begin{aligned} u_2^1(x,y,t) &= -(2\nu)^2\cos(x+y)\frac{1}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{2\alpha}\,(1-w)^{\alpha-1}\,w^{\alpha} dw \\ &= -(2\nu)^2\cos(x+y)\frac{1}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha)}t^{2\alpha}B(\alpha,\alpha+1) \\ &= -(2\nu)^2\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} \end{aligned} \tag{67}$$

$$v_2^1(x,y,t) = (2\nu)^2\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} \tag{68}$$

- For $n = 3$, we have:

$$\begin{aligned} u_3^1(x,y,t) &= \nu I_{0,t}^{\alpha} R u_2^1 = \nu I_{0,t}^{\alpha}\left(2(2\nu)^2\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}\right) \\ &= (2\nu)^3\cos(x+y)\frac{1}{\Gamma(2\alpha+1)} I_{0,t}^{\alpha}\left(t^{2\alpha}\right) \\ &= (2\nu)^3\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tau^{2\alpha} d\tau \end{aligned} \tag{69}$$

and

$$\begin{aligned} v_3^1(x,y,t) &= \nu I_{0,t}^{\alpha} R u_2^1 = \nu I_{0,t}^{\alpha}\left(-2(2\nu)^2\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}\right) \\ &= -(2\nu)^3\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tau^{2\alpha} d\tau \end{aligned} \tag{70}$$

Performing the change of variable $\tau = wt$, we obtaint:

$$\begin{aligned} u_3^1(x,y,t) &= (2\nu)^3\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{3\alpha}\,(1-w)^{\alpha-1}\,w^{2\alpha} dw \\ &= (2\nu)^3\cos(x+y)\frac{1}{\Gamma(2\alpha+1)}\frac{1}{\Gamma(\alpha)}t^{3\alpha}B(\alpha,2\alpha+1) \\ &= (2\nu)^3\cos(x+y)\frac{1}{\Gamma(3\alpha+1)}t^{3\alpha} \end{aligned} \tag{71}$$

and

$$v_3^1(x,y,t) = -(2\nu)^3\cos(x+y)\frac{1}{\Gamma(3\alpha+1)}t^{3\alpha} \tag{72}$$

We note that for any integer $n \geq 1$:

$$\begin{cases} u_n^1(x,y,t) & = (-1)^{n-1}(2\nu)^n \cos(x+y)\dfrac{1}{\Gamma(n\alpha+1)}t^{n\alpha} = -\dfrac{(-2\nu t^{\alpha})^n}{\Gamma(n\alpha+1)}\cos(x+y) \\ v_n^1(x,y,t) & = (-1)^n(2\nu)^n \cos(x+y)\dfrac{1}{\Gamma(n\alpha+1)}t^{n\alpha} = \dfrac{(-2\nu t^{\alpha})^n}{\Gamma(n\alpha+1)}\cos(x+y) \end{cases} \tag{73}$$

We have

$$u^1(x,y,t) = \sum_{n=0}^{+\infty} u_n^1(x,y,t) = u_0^1(x,y,t) + \sum_{n=1}^{+\infty} u_n^1(x,y,t) \tag{74}$$

$$= -\cos(x+y) + \frac{g}{\Gamma(\alpha+1)}t^{\alpha} - \cos(x+y)\sum_{n=1}^{+\infty}\frac{(-2\nu t^{\alpha})^n}{\Gamma(n\alpha+1)}$$

$$= \frac{g}{\Gamma(\alpha+1)}t^{\alpha} - \cos(x+y)\sum_{n=0}^{+\infty}\frac{(-2\nu t^{\alpha})^n}{\Gamma(n\alpha+1)} \tag{75}$$

$$= \frac{g}{\Gamma(\alpha+1)}t^{\alpha} - \cos(x+y)$$

$$= \frac{g}{\Gamma(\alpha+1)}t^{\alpha} - \cos(x+y)E_{\alpha}(-2\nu t^{\alpha})$$

$$v^1(x,y,t) = \sum_{n=0}^{+\infty} v_n^1(x,y,t) = v_0^1(x,y,t) + \sum_{n=1}^{+\infty} v_n^1(x,y,t) \tag{76}$$

$$= \cos(x+y) - \frac{g}{\Gamma(\alpha+1)}t^{\alpha} + \cos(x+y)\sum_{n=1}^{+\infty}\frac{(-2\nu t^{\alpha})^n}{\Gamma(n\alpha+1)} \tag{77}$$

$$= -\frac{g}{\Gamma(\alpha+1)}t^{\alpha} + \cos(x+y)E_{\alpha}(-2\nu t^{\alpha})$$

The approximate solution of the problem (57)-(56) in step1 is thus

$$\left(u^1, v^1\right) = \left(\frac{g}{\Gamma(\alpha+1)}t^{\alpha} - \cos(x+y)E_{\alpha}(-2\nu t^{\alpha}); \frac{g}{\Gamma(\alpha+1)}t^{\alpha} + \cos(x+y)E_{\alpha}(-2\nu t^{\alpha})\right) \tag{78}$$

Step 2: we have

$$N(u^1) = u^1\frac{\partial u^1}{\partial x} + v^1\frac{\partial u^1}{\partial y}$$

$$= \left(\frac{g}{\Gamma(\alpha+1)}t^{\alpha} - \cos(x+y)E_{\alpha}(-2\nu t^{\alpha})\right)(\sin(x+y)E_{\alpha}(-2\nu t^{\alpha})) \tag{79}$$

$$+ \left(-\frac{g}{\Gamma(\alpha+1)}t^{\alpha} + \cos(x+y)E_{\alpha}(-2\nu t^{\alpha})\right)(\sin(x+y)E_{\alpha}(-2\nu t^{\alpha})) = 0$$

and

$$N(v^1) = u^1\frac{\partial v^1}{\partial x} + v^1\frac{\partial v^1}{\partial y}$$

$$= \left(\frac{g}{\Gamma(\alpha+1)}t^{\alpha} - \cos(x+y)E_{\alpha}(-2\nu t^{\alpha})\right)(-\sin(x+y)E_{\alpha}(-2\nu t^{\alpha})) \tag{80}$$

$$+ \left(-\frac{g}{\Gamma(\alpha+1)}t^{\alpha} + \cos(x+y)E_{\alpha}(-2\nu t^{\alpha})\right)(-\sin(x+y)E_{\alpha}(-2\nu t^{\alpha})) = 0$$

Therefore, according to Proposition1, the problem (57)-(56) admits a unique solution

$$\begin{aligned}(u(x,y,t),v(x,y,t)) &= (u^1,v^1) \\ &= \left(\frac{g}{\Gamma(\alpha+1)}t^{\alpha}-\cos(x+y)E_{\alpha}(-2\nu t^{\alpha});-\frac{g}{\Gamma(\alpha+1)}t^{\alpha}+\cos(x+y)E_{\alpha}(-2\nu t^{\alpha})\right)\end{aligned} \quad (81)$$

The pair (81) checks the incompressibility condition. Therefore, the exact solution of the problem (55)-(56) is

$$(u(x,y,t),v(x,y,t))=\left(\frac{g}{\Gamma(\alpha+1)}t^{\alpha}-\cos(x+y)E_{\alpha}(-2\nu t^{\alpha});-\frac{g}{\Gamma(\alpha+1)}t^{\alpha}+\cos(x+y)E_{\alpha}(-2\nu t^{\alpha})\right) \quad (82)$$

If we replace α with 1 in (82), then we get

$$\begin{aligned}(u(x,y,t),v(x,y,t)) &= (\frac{g}{\Gamma(2)}t-\cos(x+y)E_1(-2\nu t);-\frac{g}{\Gamma(2)}t+\cos(x+y)E_1(-2\nu t)) \\ &= \left(gt-\cos(x+y)e^{-2\nu t},-gt+\cos(x+y)e^{-2\nu t}\right)\end{aligned} \quad (83)$$

5. Conclusion

Under certain conditions imposed on the equations of the classical Navier-Stokes system, the SBA method which is an asymptotic method, allowed us to obtain the exact solution of the system with a Cauchy type initial condition. Under these same conditions, it also allowed us to obtain the exact solution of the fractional Navier-Stokes system. The exact solution of the fractional Navier-Stokes system obtained coincides with the exact solution of the classical Navier-Stokes system obtained when the fractional derivation order $\alpha = 1$. The SBA method is efficient for solving classical nonlinear partial differential equations with initial conditions and is easily extendable to nonlinear fractional partial differential equations.

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