

Accurate Eigenvalues for the Sturm-Liouville Problems, Involving Generalized and Periodic Ones

Chein-Shan Liu

Correspondence: Center of Excellence for Ocean Engineering, National Taiwan Ocean University, Keelung 20224, Taiwan

Received: May 29, 2022 Accepted: June 29, 2022 Online Published: July 4, 2022

doi:10.5539/jmr.v14n4p1

URL: <https://doi.org/10.5539/jmr.v14n4p1>

Abstract

In the paper, the eigenvalues of Sturm-Liouville problems (SLPs), generalized SLPs and periodic SLPs are solved. First, we propose a new method to transform the SLP with mixed boundary conditions to a generalized SLP for a transformed variable, for which the Dirichlet boundary conditions occur on two-side, but the coefficients are nonlinear functions of eigenvalue. To computing the eigenvalue and eigenfunction, we further recast the transformed system to an initial value problem for a new variable. In terms of the relative residual of two consecutive terminal values of the new variable a nonlinear equation is solved for seeking the eigenvalue by the fictitious time integration method (FTIM), which monotonically converges to the exact eigenvalue. We solve a numerically characteristic equation by the half-interval method (HIM) and a derivative-free iterative scheme LHL (Liu, Hong & Li, 2021) to achieve high precision eigenvalues. Next, the generalized SLP is transformed to a new one, so that the Dirichlet boundary condition happens on the right-end. By using the boundary shape function method and the uniqueness condition of the transformed variable, a definite initial value problem is derived for the new variable. To match the right-end Dirichlet boundary condition a numerically characteristic equation is obtained and solved by the HIM and LHL. Finally, new techniques for solving the periodic SLPs with three types periodic boundary conditions are proposed, which preserve the periodic boundary conditions with the aids of boundary shape functions. Three iterative algorithms are developed, which converge quickly. All the proposed iterative algorithms are identified by testing some examples.

Keywords: (Generalized, Periodic) Sturm-Liouville problem, Eigenvalues, Fictitious time integration method, Derivative-free iterative scheme, Half-interval method

1. Introduction

During 1836-37, Sturm and Liouville created a new mathematical analysis of second-order ordinary differential equation (ODE) equipped with the Sturm-Liouville boundary conditions (Lutzen, 1984). It is nowadays known as the Sturm-Liouville theory to deal with the Sturm-Liouville problem (SLP):

$$-(p(x)u'(x))' + q(x)u(x) = \lambda w(x)u(x), \quad x \in (a, b), \quad (1)$$

$$hu(a) - p(a)u'(a) = 0, \quad Hu(b) + p(b)u'(b) = 0, \quad (2)$$

where $p(x) > 0$ and $w(x) > 0$ are strictly positive functions of $x \in [a, b]$, and λ is a constant parameter to be determined. Eqs. (1) and (2) is a regular SLP, which allows non-trivial solutions only for certain values of λ , known as the eigenvalues.

The Sturm-Liouville problem is very important in partial differential equations, vibrations in continuum mechanics, heat conduction problems, Schrödinger equation, the transport of micro wave, and the Lax pair of KdV equation, etc. For example, as an application of the Sturm-Liouville problem,

$$\frac{1}{A(x)} \frac{\partial}{\partial x} \left(E(x)A(x) \frac{\partial u(x, t)}{\partial x} \right) = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \quad (3)$$

describes the longitudinal wave motion of a rod when the Young's modulus $E(x)$ is varying, with the cross section area $A(x)$, a constant density ρ of mass, and the axial displacement $u(x, t)$. Let $u(x, t) = e^{i\omega t}y(x)$ with ω a natural vibration frequency; Eq. (3) reduces to Eq. (1) with

$$p(x) = E(x)A(x), \quad q(x) = 0, \quad w(x) = \rho A(x), \quad \lambda = \omega^2. \quad (4)$$

In the design of engineering structure, knowing a first few frequencies ω is utmost important.

Besides some limited cases, there exists no general solution of the Sturm-Liouville problem. The Rayleigh minimal principle was often employed to solve the Sturm-Liouville problem, which is carried out in a large space and leaves a finite

number of free parameters to match the minimality condition, but it is effective only for a few lower-order eigenvalues (Gould, 1995; Hinton & Schaefer, 1997; Ghelardoni 1997). In general, for higher-order eigenvalues the accuracy is lost very fast.

Many numerical methods have been developed to solve the SLPs (El-Gamel & El-Hady, 2013; Alquran & Al-Khaled, 2010; Abbasbandy & Shirzadi, 2011; Ghelardoni & Gheri, 2001; Liu, 2008, 2010). The Sturm-Liouville theory is widely discussed and efficient and accurate codes have been developed, like as SLEIGN (Bailey, Gordon & Shampine, 1978; Bailey, Everitt & Zettl, 1991), SLEIGN2 (Bailey, Everitt & Zettl, 2001), SLEDGE (Pruess & Fulton, 1993), and SL02FM (Pruess & Marletta, 1992). An alternative code is MATSLISE (Ledoux, van Daele & vanden Berghe, 2005), which is based on the CP methods (Ixaru, de Meyer & vanden Berghe, 1999). In comparison with the corrected Numerov’s method, some competitive results have been achieved by the boundary value methods (Aceto, Ghelardoni & Magherini, 2009a, 2009b) and the high order difference schemes (Amodio & Settanni, 2015).

As to be proved in Theorem 1 below, the SLP with mixed boundary conditions can be transformed to a generalized Sturm-Liouville problem with the coefficients depending on λ nonlinearly. In the existent literature, the three problems of SLP, generalized SLP and periodic SLP were tackled separately. We will propose a unified treatment of these three SLPs by designing different boundary shape functions and transforming them to the corresponding initial value problems, and then some solvers of nonlinear equation are used to solve different target equations for computing the eigenvalues.

We arrange the paper as follows. In Section 2, we describe the SLP which with the Liouville transformation is recast to a canonical form equipped with a single potential function in the unit interval, and then we further transform it to a generalized SLP with the Dirichlet boundary conditions happened on two-side. With the aid of a simple transformation made in Section 3, we derive an initial value problem which together with the fictitious time integration method (FTIM) is used to determine the eigenvalue iteratively, and two numerical examples are given to test the FTIM. In Section 4, we propose a normalization condition for the transformed variable to ascertain the terminal value of the new variable explicitly, such that a definite initial value problem is derived for a new variable. To precisely match the right-end Dirichlet boundary condition, a numerically characteristic equation provides a medium to obtain the eigenvalues, which is solved by the half-interval method (HIM) and a derivative-free iterative scheme (LHL), and very accurate eigenvalues are revealed. In Section 5, we solve the generalized SLP which is transformed to a new generalized SLP with a simplified Dirichlet boundary condition on the right-end. By developing the boundary shape function method and a definite initial value problem method, we then apply the HIM and LHL to solve the numerically characteristic equation. In Section 6, we discuss the periodic SLP by developing the boundary shape function methods. The iterative algorithms are developed in Section 7 to determine the eigenvalues. The conclusions are drawn in Section 8.

2. A generalized Sturm-Liouville Problem with Two-Side Dirichlet Boundary Conditions

The Liouville transformation (Fu, Wang & Wei, 2015) is given by

$$v(t) = g(x)u(x), \quad g(x) = [w(x)p(x)]^{1/4}, \tag{5}$$

$$t = \frac{1}{c} \int_a^x \left(\frac{w(\tau)}{p(\tau)}\right)^{1/2} d\tau, \quad c = \int_a^b \left(\frac{w(\tau)}{p(\tau)}\right)^{1/2} d\tau, \tag{6}$$

which are used to transform Eqs. (1) and (2) to the following canonical form of the SLP, endowing with the mixed boundary conditions:

$$-\ddot{v}(t) + Q(t)v(t) = \mu v(t), \quad t \in (0, 1), \tag{7}$$

$$h_*v(0) - \dot{v}(0) = 0, \quad H_*v(1) + \dot{v}(1) = 0, \tag{8}$$

where

$$Q(t) = c^2 \frac{q(x)g(x) - [p(x)g'(x)]'}{w(x)g(x)}, \quad \mu = c^2\lambda, \tag{9}$$

$$h_* = cg(a)[hg(a) - p(a)g'(a)], \quad H_* = cg(b)[Hg(b) + p(b)g'(b)]. \tag{10}$$

The original SLP becomes a new SLP in the unit interval with the same eigenvalues multiplied by a constant factor c^2 , and only the potential function $Q(t)$ is required.

Therefore, we consider a specific SLP by

$$u''(x) + q(x, \lambda)u(x) = 0, \quad x \in (0, 1), \tag{11}$$

$$a_1u(0) + b_1u'(0) = 0, \quad a_2u(1) + b_2u'(1) = 0, \tag{12}$$

where $q(x, \lambda)$ is a linear function of λ . It is different from the Liouville transformation in Eq. (5) that a new one in terms of $u(x)$ and $u'(x)$ is derived below, which can transform the above SLP with mixed boundary conditions to a new generalized SLP with the Dirichlet boundary conditions happened on two-side.

Theorem 1 The Sturm-Liouville problem (11) and (12) can be transformed to a generalized Sturm-Liouville problem with two-side Dirichlet boundary conditions:

$$v''(x) = F_1(x, \lambda)v'(x) + F_2(x, \lambda)v(x), \tag{13}$$

$$v(0) = 0, \quad v(1) = 0, \tag{14}$$

in which the transformed variable $v(x)$ is related to $u(x)$ and $u'(x)$ by

$$v(x) = [a_2a_3(\lambda) - (1 - x)q(x, \lambda)b_2b_3]u(x) + [a_3(\lambda)b_2 + (1 - x)a_2b_3]u'(x), \tag{15}$$

and the coefficients $F_1(x, \lambda)$ and $F_2(x, \lambda)$ are given by

$$F_1(x, \lambda) = \frac{a_{11}(x, \lambda)a_{12}(x, \lambda)}{D(x, \lambda)} - \frac{a_{12}(x, \lambda)a_{21}(x, \lambda)}{D(x, \lambda)}, \tag{16}$$

$$F_2(x, \lambda) = \frac{1}{D(x, \lambda)}[a_{11}(x, \lambda)a_{21}(x, \lambda) + a'_{12}(x, \lambda)a_{21}(x, \lambda) + a_{12}(x, \lambda)a_{22}(x, \lambda)q(x, \lambda) - a_{22}(x, \lambda)a'_{11}(x, \lambda)], \tag{17}$$

where

$$a_3(\lambda) = a_1a_2 + b_1b_2q(0, \lambda), \tag{18}$$

$$b_3 = a_2b_1 - a_1b_2, \tag{19}$$

$$a_{11}(x, \lambda) = a_2a_3(\lambda) - (1 - x)q(x, \lambda)b_2b_3, \tag{20}$$

$$a_{12}(x, \lambda) = a_3(\lambda)b_2 + (1 - x)a_2b_3, \tag{21}$$

$$a_{21}(x, \lambda) = a'_{11}(x, \lambda) - 2a'_{12}(x, \lambda)q(x, \lambda) - a_{12}(x, \lambda)q'(x, \lambda) - a_{11}(x, \lambda)q(x, \lambda), \tag{22}$$

$$a_{22}(x, \lambda) = 2a'_{11}(x, \lambda) - a_{12}(x, \lambda)q(x, \lambda), \tag{23}$$

$$D(x, \lambda) = a_{11}(x, \lambda)[a_{11}(x, \lambda) + a'_{12}(x, \lambda)] + a_{12}(x, \lambda)[a_{12}(x, \lambda)q(x, \lambda) - a'_{11}(x, \lambda)]. \tag{24}$$

Proof. Inserting $x = 0$ and $x = 1$ into Eq. (15) and taking Eqs. (18), (19) and (12) into account, we have

$$\begin{aligned} v(0) &= [a_2a_3(\lambda) - q(0, \lambda)b_2b_3]u(0) + [a_3(\lambda)b_2 + a_2b_3]u'(0) \\ &= a_2[a_1a_2 + b_1b_2q(0, \lambda)]u(0) - q(0, \lambda)b_2[a_2b_1 - a_1b_2]u(0) \\ &\quad + b_2[a_1a_2 + b_1b_2q(0, \lambda)]u'(0) + a_2[a_2b_1 - a_1b_2]u'(0) \\ &= [a_2^2 + b_2^2q(0, \lambda)][a_1u(0) + b_1u'(0)] = 0, \end{aligned} \tag{25}$$

$$v(1) = a_2a_3(\lambda)u(1) + a_3(\lambda)b_2u'(1) = a_3(\lambda)[a_2u(1) + b_2u'(1)] = 0. \tag{26}$$

Hence, the homogeneous boundary conditions in Eq. (14) are verified. Eq. (15) is re-written as

$$v(x) = a_{11}(x, \lambda)u(x) + a_{12}(x, \lambda)u'(x), \tag{27}$$

where $a_{11}(x, \lambda)$ and $a_{12}(x, \lambda)$ are given by Eqs. (20) and (21). Differentiating Eq. (27) to x and inserting Eq. (11) for $u''(x)$, leads to

$$v'(x) = [a'_{11}(x, \lambda) - q(x, \lambda)a_{12}(x, \lambda)]u(x) + [a_{11}(x, \lambda) + a'_{12}(x, \lambda)]u'(x). \tag{28}$$

Expressing $u(x)$ and $u'(x)$ in terms of $v(x)$ and $v'(x)$ by using Eqs. (27) and (28), we can obtain the inverse transformations by

$$u(x) = \frac{a_{11}(x, \lambda) + a'_{12}(x, \lambda)}{D(x, \lambda)}v(x) - \frac{a_{12}(x, \lambda)}{D(x, \lambda)}v'(x), \tag{29}$$

$$u'(x) = \frac{a_{12}(x, \lambda)q(x, \lambda) - a'_{11}(x, \lambda)}{D(x, \lambda)}v(x) + \frac{a_{11}(x, \lambda)}{D(x, \lambda)}v'(x), \tag{30}$$

where $D(x, \lambda)$ is defined by Eq. (24). Further differentiating Eq. (28) to x , gives

$$v''(x) = a_{21}(x, \lambda)u(x) + a_{22}(x, \lambda)u'(x), \tag{31}$$

where $a_{21}(x, \lambda)$ and $a_{22}(x, \lambda)$ are defined by Eqs. (22) and (23). Inserting Eq. (29) for $u(x)$ and Eq. (30) for $u'(x)$ into Eq. (31), we can derive Eq. (13). The proof is completed. \square

Although $q(x, \lambda)$ as usual is a linear function of λ for the SLP, $F_1(x, \lambda)$ and $F_2(x, \lambda)$ as shown by Eqs. (16) and (17) are nonlinear functions of λ ; hence, the resultant system in Eqs. (13) and (14) is a generalized SLP.

3. Transforming to an Initial Value Problem

With the aid of Theorem 1, we derive new methods in Sections 3 and 4 to solve Eqs. (11) and (12), and give examples to test the proposed methods.

Theorem 2 For a new variable $y(x) \in C^2[0, 1]$, the transformed variable $v(x)$ is given by

$$v(x) = y(x) - (1 - x)y(0) - xy(1), \tag{32}$$

which satisfies the boundary conditions in Eq. (14).

Proof. It is straightforward to prove that $v(0) = y(0) - y(0) = 0$ and $v(1) = y(1) - y(1) = 0$. \square

3.1 Fictitious Time Integration Method

For solving a nonlinear equation:

$$F(\lambda) = 0, \tag{33}$$

we can transform it to a first-order differential equation (Liu & Atluri, 2008):

$$\lambda'(\xi) = \frac{\nu}{1 + \xi} F(\lambda(\xi)), \tag{34}$$

where ξ is a fictitious time variable and ν is a constant parameter. Integrating Eq. (34) by the forward Euler scheme, yields

$$\lambda(k + 1) = \lambda(k) + \frac{\nu \Delta \xi}{1 + \xi_k} F(\lambda(k)), \quad k = 0, 1, \dots, \tag{35}$$

where $\xi_k = k\Delta\xi$, $\Delta\xi > 0$ is an increment of fictitious time, and $\lambda(k)$ denotes the eigenvalue at the k th iteration. Eq. (35) is the fictitious time integration method (FTIM) developed by Liu and Atluri (2008) to solve nonlinear equation (33).

Now, we explore the FTIM (Liu & Atluri, 2008) to determine λ in Eqs. (11) and (12) with the aids of Theorems 1 and 2. It follows from Eqs. (13) and (32) that $y(x)$ replacing $v(x)$,

$$y''(x) = F_1(x, \lambda)[y'(x) + y(0) - y(1)] + F_2(x, \lambda)[y(x) - (1 - x)y(0) - xy(1)], \tag{36}$$

where $y(1)$ is an unknown constant to be determined by the iteration method. With the given initial values $y(0)$ and $y'(0)$, if y can be integrated from Eq. (36), Theorem 2 guarantees that $v(x)$ satisfies the Dirichlet boundary conditions in Eq. (14). Then, by Eqs. (25) and (26), $u(x)$ satisfies the mixed boundary conditions in Eq. (12).

Let $\lambda(k)$ be the eigenvalue λ and $y_k(1)$ be the terminal value of $y(1)$ in Eq. (36) at the k th iteration. We set

$$F(\lambda(k)) = |y_{k+1}(1) - y_k(1)|, \tag{37}$$

which is using the relative residual of the iterative sequence $y_k(1)$ as a driving force in Eq. (35), so that we can enforce the iterative sequence of $\lambda(k)$ tending to its limit point as an approximation to the exact eigenvalue λ with $F(\lambda) = 0$.

Based on FTIM and Theorems 1 and 2, the iterative algorithm for solving Eqs. (11) and (12) is depicted as follows. (i) Give $y(0)$, $y'(0)$, the initial guesses of $\lambda(0)$ and $y_0(1)$, $\Delta\xi$, ν , ϵ_1 , and N . (ii) For $k = 0, 1, 2, \dots$, integrate Eq. (36) with N steps from $x = 0$ to $x = 1$ to obtain $y_{k+1}(1)$, and iterate $\lambda(k)$ by Eqs. (35) and (37) until $F(\lambda(k)) < \epsilon_1$.

If λ is an eigenvalue to be sought, we can take $\lambda(0) < \lambda$ and $\nu > 0$, and Eqs. (35) and (37) generate a monotonically increasing sequence with $\lambda(k + 1) > \lambda(k)$, $k = 0, 1, \dots$ due to $\nu \Delta \xi F(\lambda(k)) / (1 + \xi_k) > 0$, which converges to the exact eigenvalue λ very fast.

Taking $x = 0$ and inserting $v(0) = 0$ and $v'(0) = y'(0) - y(1) + y(0)$ into Eqs. (29) and (30), leads to

$$u(0) = -\frac{a_{12}(0, \lambda)}{D(0, \lambda)} [y'(0) - y(1) + y(0)], \tag{38}$$

$$u'(0) = \frac{a_{11}(0, \lambda)}{D(0, \lambda)} [y'(0) - y(1) + y(0)]. \tag{39}$$

Using the initial values $u(0)$ and $u'(0)$ and integrating Eq. (11), the eigenfunction can be obtained.

For the SLP endowed with the mixed boundary conditions, we can transform it to the Dirichlet type generalized SLP as shown in Theorem 1. Two examples are given below to demonstrate the usefulness of Theorem 1 and the FTIM.

3.2 Example 1

To explore the efficiency and accuracy of the proposed iterative algorithm, we consider

$$u''(x) + \lambda u(x) = 0, \quad x \in (0, 1), \tag{40}$$

$$u(0) + \mu_1 u'(0) = 0, \quad u(1) + \mu_2 u'(1) = 0, \tag{41}$$

where $\lambda_k, k = 0, 1, 2, \dots$ are solved from the characteristic equation:

$$f(\lambda) = \mu_1 \mu_2 \lambda \sin \sqrt{\lambda} + (\mu_2 - \mu_1) \sqrt{\lambda} \cos \sqrt{\lambda} + \sin \sqrt{\lambda} = 0, \tag{42}$$

while the k th order eigenfunction is given by $u_k(x) = \mu_1 \sqrt{\lambda_k} \cos(\sqrt{\lambda_k}x) - \sin(\sqrt{\lambda_k}x)$.

With $\mu_1 = 1, \mu_2 = -2$, in the iterative algorithm we take $y(0) = 1, y'(0) = 0, y_0(1) = 10, \epsilon_1 = 10^{-10}, \Delta\xi = 0.001$ and $N = 500$ and iteratively determine λ by Eqs. (35) and (37), whose results are listed in Table 1 to show $\lambda(0)$, the computed λ , the exact λ , the absolute error of eigenvalue (EE), ν , the number of iterations (NI), the absolute error of right boundary condition (ERBC) and the maximum error (ME) of eigenfunction. Notice that the absolute error of left boundary condition is zero, so that it is not listed in the table.

Table 1. For Eqs. (40) and (41) solved by the FTIM with Eqs. (35) and (37), listing the computed eigenvalues and other parameters

| $\lambda(0)$ | Computed λ | Exact λ | EE | ν | NI | ERBC | ME |
|--------------|--------------------|-----------------|------------------------|-------|----|------------------------|------------------------|
| 6.5 | 6.65898104446 | 6.65898104424 | 2.27×10^{-10} | 2500 | 8 | 2.89×10^{-10} | 2.93×10^{-11} |
| 36 | 36.4377786495 | 36.4377786333 | 1.62×10^{-8} | 8900 | 6 | 1.64×10^{-8} | 1.07×10^{-9} |
| 80 | 85.8090551476 | 85.8090549627 | 1.23×10^{-7} | 19100 | 9 | 1.85×10^{-7} | 5.85×10^{-9} |

3.3 Example 2

We test the following SLP:

$$-u''(x) + \frac{2}{(x + g_0)^2} u(x) = \lambda u(x), \quad x \in (0, 1), \tag{43}$$

$$u(0) + \mu_1 u'(0) = 0, \quad u(1) + \mu_2 u'(1) = 0, \tag{44}$$

where $\mu_1 = g_0 = 1$ and $\mu_2 = 1 + g_0 = 2$. This problem has the eigenvalues $\lambda_k = (k + 1)^2 \pi^2, k = 0, 1, 2, \dots$. However, the exact form of the eigenfunction $u(x)$ does not exist.

We fix $y(0) = 1, y'(0) = 0, y_0(1) = 10, \epsilon_1 = 10^{-10}, \Delta\xi = 0.001$ and $N = 500$ and iteratively determine λ by Eqs. (35) and (37), whose results are shown in Table 2 for $\lambda(0)$, the computed λ , the exact λ , EE, ν , NI, and ERBC.

Table 2. For Eqs. (43) and (44) solved by the FTIM with Eqs. (35) and (37), listing the computed eigenvalues and other parameters

| $\lambda(0)$ | Computed λ | Exact λ | EE | ν | NI | ERBC |
|--------------|--------------------|-----------------|------------------------|-------|----|------------------------|
| 9 | 9.86960440127 | 9.86960440109 | 1.84×10^{-10} | 1820 | 8 | 2.78×10^{-13} |
| 30 | 39.4784176208 | 39.4784176044 | 1.64×10^{-8} | 7500 | 7 | 1.86×10^{-12} |
| 80 | 88.8264397964 | 88.8264396098 | 1.87×10^{-7} | 18800 | 9 | 1.38×10^{-12} |

The above two examples as shown in Tables 1 and 2 show that the FTIM is convergent very fast with small numbers of iterations. The parameter $\Delta\xi$ must be small enough to avoid over-shooting, while the value of the parameter ν can be chosen as large as possible to accelerate the convergence speed. However, the accuracy of eigenvalues can be further improved by other methods given below.

4. Numerically Characteristic Equation

4.1 Terminal Value $y(1)$ of the New Variable $y(x)$

In view of Eqs. (13) and (14), if $v(x)$ is a transformed eigenfunction, then $\gamma v(x)$, $\gamma \neq 0$ is also a transformed eigenfunction. For the uniqueness of $v(x)$ and then $u(x)$ by Eq. (29), we can consider a normalization condition as follows:

$$v'(0) = A_0, \tag{45}$$

where A_0 is a given nonzero constant. We fix $A_0 = 1$.

Theorem 3 If Eq. (45) is imposed for the uniqueness of $v(x)$, then the terminal value $y(1)$ in Eq. (32) is given by

$$y(1) = y(0) + y'(0) - A_0; \tag{46}$$

hence,

$$v(x) = y(x) - y(0) + x[A_0 - y'(0)]. \tag{47}$$

Proof. Differentiating Eq. (32) with respect to x and inserting $x = 0$, yields

$$v'(0) = y'(0) + y(0) - y(1), \tag{48}$$

which immediately leads to Eq. (46) by using $v'(0) = A_0$ in Eq. (45). Inserting Eq. (46) for $y(1)$ into Eq. (32), we can derive Eq. (47). \square

Therefore, we come to a definite second-order initial value problem from Eqs. (36) and (46) for the new variable $y(x)$:

$$y''(x) = F_1(x, \lambda)[y'(x) + A_0 - y'(0)] + F_2(x, \lambda)[y(x) - y(0) + A_0x - xy'(0)], \tag{49}$$

which is a nonhomogeneous ODE, such that the given initial values can be taken as $y(0) = 0$ and $y'(0) = 0$ without leading to $y(x) = 0$. Hence, Eq. (49) reduces to

$$y''(x) = F_1(x, \lambda)[y'(x) + 1] + F_2(x, \lambda)[y(x) + x], \tag{50}$$

where we have taken $A_0 = 1$. Notice the difference between Eqs. (36) and (50). Eq. (36) exists an unknown constant $y(1)$, which is disappeared in Eq. (50).

Up to here we have presented a new formulation of the SLP. When the original SLP in Eqs. (11) and (12) exist two unknowns: an unknown initial value $u(0)$ or $u'(0)$ and the eigenvalue λ , the present formulation in Eq. (50) only has one unknown λ to be determined below.

4.2 Half-Interval Method

With the given initial values $y(0) = y'(0) = 0$ and for any λ , we can integrate Eq. (50) from $x = 0$ to $x = 1$ to obtain $y(1)$; hence, inserting $x = 1$ into Eq. (47) and taking $y(0) = y'(0) = 0$ and $A_0 = 1$, yields

$$v(1; \lambda) = y(1) + 1. \tag{51}$$

Here, $y(1)$ is obtained by integrating Eq. (50) with the initial conditions $y(0) = y'(0) = 0$, which is no longer the one in Eq. (46).

The curve of $v(1; \lambda)$ vs. λ is an eigenvalue curve. We can adjust the value of λ until $v(1; \lambda)$ satisfies $|v(1; \lambda) - 0| < \epsilon_2$, where ϵ_2 is a given tolerance of the error to mismatch the right-end Dirichlet boundary condition in Eq. (14):

$$v(1; \lambda) = y(1) + 1 = 0, \tag{52}$$

which is a numerically characteristic equation to determine λ . We can apply the half-interval method to solve Eq. (52); emanating from $y(0) = y'(0) = 0$, we repeatedly integrate Eq. (50) with N steps from $x = 0$ to $x = 1$ to obtain $y(1)$ and then $v(1; \lambda) = y(1) + 1$ is available for each λ . In the original SLP of Eqs. (11) and (12) the target equation $a_2u(1) + b_2u'(1) = 0$ upon comparing to Eq. (52) is more complicated to determine the eigenvalue. In the present formulation of Eq. (50), the target equation $y(1) + 1 = 0$ is simple to determine the eigenvalue.

Like Eq. (42) for Example 1, Eq. (52) can be used to determine the eigenvalue λ . However, when Eq. (42) with its explicit form can be effectively solved by the Newton iteration method, Eq. (52) has to be solved by other methods, since it is an implicit form nonlinear equation of λ . The half-interval method (HIM) is described as follows. If λ_d is a

desired eigenvalue, we can detect the eigenvalue curve and decide two initial values λ_0 and λ with $\lambda_0 < \lambda_d < \lambda$, such that $v(1; \lambda_0)v(1; \lambda) < 0$. Then, we repeat the following processes until convergence:

```

 $v_0 = v(1; \lambda_0), v_\lambda = v(1; \lambda)$ 
 $L = 0$ 
1  $\lambda_1 = (\lambda_0 + \lambda)/2$ 
 $L = L + 1$ 
If  $L > N_u$  Then
Return
Endif
 $v_1 = v(1; \lambda_1)$ 
If  $v_0 v_1 \leq 0$  Then
 $\lambda = \lambda_1$ 
 $v_\lambda = v(1; \lambda)$ 
Else
 $\lambda_0 = \lambda_1$ 
 $v_0 = v(1; \lambda_0)$ 
Endif
If  $|v_0| < \epsilon_2$  And  $|v_\lambda| < \epsilon_2$  Then
 $\lambda = (\lambda_0 + \lambda)/2$ 
Return
Else
Go To 1
Endif
    
```

N_u is the upper bound of the number of iterations, which is set to be $N_u = 100$. Besides the first two integrations, twice integrations of Eq. (50) in each iteration are required in the HIM.

4.3 A Derivative-Free Iterative Scheme

As to be shown below by numerical examples, the HIM is convergent slowly. Liu, Hong & Lee (2021) have proposed an iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{\alpha + \beta f(x_n)}, \tag{53}$$

where

$$\alpha = f'(x^*), \beta = \frac{f''(x^*)}{2f'(x^*)}, \tag{54}$$

in which x^* is a simple root of the nonlinear equation $f(x) = 0$. The iterative scheme (53) has third-order convergence, which is shortened as LHL.

Like the half-interval method the first step used in the LHL is choosing two initial guesses x_0 and x_2 such that $f(x_0)f(x_2) < 0$ to render $x^* \in (x_0, x_2)$. Then, we take $x_1 = (x_0 + x_2)/2$. As the approximations of α and β in Eq. (54), we can use the technique of finite differences:

$$\alpha = \frac{f(x_2) - f(x_0)}{x_2 - x_0}, \beta = \frac{1}{2\alpha} \frac{f(x_2) - 2f(x_1) + f(x_0)}{(x_1 - x_0)^2}. \tag{55}$$

In summary, the procedures of the LHL are given as follows: (i) Give the initial guesses of x_0 and x_2 to render $f(x_0)f(x_2) < 0$. (ii) Compute α and β by Eq. (55). (iii) For $n=0,1,\dots$, doing

$$x_{n+1} = x_n - \frac{f(x_n)}{\alpha + \beta f(x_n)}, \text{ until } |f(x_n)| < \epsilon_3.$$

We will apply the LHL to solve Eq. (52), which is a derivative-free iterative scheme. We fix the convergence criterion ϵ_1 for the FTIM, ϵ_2 for the HIM and ϵ_3 for the LHL.

4.4 Examples 1 and 2

For example 1, we fix $\epsilon_2 = \epsilon_3 = 10^{-15}$ and $N = 5000$ and iteratively determine λ by applying the half-interval method (HIM) and LHL to Eq. (52), and the results are listed in Table 3 to show the computed λ , the exact λ , EE, NI_a , NI_b , ERBC and ME. Since the convergence criterion and N used in the HIM and LHL are the same, they obtained the same results, but the numbers of iterations for HIM denoted as NI_a are much larger than that for LHL denoted as NI_b . When the CPU time spent in the LHL is about 4.28 s, the HIM needs 43.38 s. The LHL is much saving CPU time than the HIM.

Table 3. For Eqs. (40) and (41) solved by the the HIM and LHL to Eq. (52), listing the computed eigenvalues and other parameters

| Computed λ | Exact λ | EE | NI_a | NI_b | ERBC | ME |
|--------------------|-----------------|------------------------|--------|--------|------------------------|------------------------|
| 6.65898104424 | 6.65898104424 | 5.68×10^{-14} | 43 | 6 | 1.08×10^{-13} | 1.91×10^{-14} |
| 36.4377786333 | 36.4377786333 | 1.71×10^{-12} | 42 | 5 | 2.04×10^{-12} | 2.03×10^{-13} |
| 85.8090549628 | 85.8090549627 | 1.82×10^{-11} | 37 | 4 | 9.67×10^{-12} | 9.15×10^{-13} |

For example 2, we fix $\epsilon_2 = \epsilon_3 = 10^{-15}$ and $N = 5000$, and the results are listed in Table 4 to show the computed λ , the exact λ , EE, NI_a , NI_b , and ERBC.

Table 4. For Eqs. (43) and (44) solved by the HIM and LHL to Eq. (52), listing the computed eigenvalues and other parameters

| Computed λ | Exact λ | EE | NI_a | NI_b | ERBC |
|--------------------|-----------------|------------------------|--------|--------|------------------------|
| 9.86960440109 | 9.86960440109 | 5.68×10^{-14} | 44 | 8 | 1.11×10^{-16} |
| 39.4784176044 | 39.4784176044 | 1.71×10^{-12} | 43 | 6 | 3.19×10^{-17} |
| 88.8264396098 | 88.8264396098 | 1.91×10^{-11} | 40 | 6 | 3×10^{-17} |

Upon comparing Tables 3 and 4 to Tables 1 and 2, the improvement of the accuracy in every aspect is apparent with about three orders. For the purpose to achieve the high precision in Tables 3 and 4, N is increased to $N = 5000$. The NIs in the half-interval method (HIM) is increased and the CPU time for integrating Eq. (50) with $2(NI+1)$ times is increased, but in the FTIM it needs NI times integrations of the ODE with a less CPU time. Although the CPU time is increased for the HIM, but this is deserved because of the increase of accuracy with three orders than that for the FTIM.

4.5 Example 3

We consider a singular SLP (El-Gamel & El-Hady, 2013; Bailey, Everitt & Zettl, 1991):

$$u''(x) + \lambda u(x) + \frac{1}{x}u(x) = 0, \quad x \in (0, 1), \tag{56}$$

$$u(0) = 0, \quad u(1) = 0, \tag{57}$$

which is already in the form of Theorem 1, and thus $v(x)$ is replaced by $u(x)$ in Sections 4.1 and 4.2.

We fix $\epsilon_2 = \epsilon_3 = 10^{-15}$, $\Delta x = 1/5000$ and $N = 5000$, and the results are listed in Table 5 to show the computed λ , the exact λ , EE, NI_a , and NI_b . Notice that ERBC is zero. The accuracy is competitive to that in (El-Gamel & El-Hady, 2013), but is more accurate than that in (Bailey, Everitt & Zettl, 1991) about four orders, where the code SLEIGN2 was used.

Table 5. For Eqs. (56) and (57) solved by the HIM and LHL to Eq. (52), listing the computed eigenvalues and other parameters

| Computed λ | Exact λ | EE | NI_a | NI_b |
|--------------------|-----------------|------------------------|--------|--------|
| 7.373985015175 | 7.37398501517 | 5.14×10^{-12} | 39 | 7 |
| 36.33601959523 | 36.33601959523 | 3.48×10^{-12} | 38 | 5 |
| 85.29258209416 | 85.29258209413 | 2.57×10^{-11} | 35 | 5 |
| 154.0986237399 | 154.0986237398 | 1.1×10^{-10} | 36 | 5 |

4.6 Example 4

To further test the accuracy, we consider (El-Gamel & El-Hady, 2013)

$$u''(x) = \tanh xu'(x) - \lambda \cosh^2 xu(x), \quad x \in (0, 1), \tag{58}$$

$$u(0) = 0, \quad u(1) = 0. \tag{59}$$

We fix $\epsilon_2 = \epsilon_3 = 10^{-15}$ and $\Delta x = 1/N = 1/8000$, and the results are tabulated in Table 6 to show the computed λ , the exact λ , EE, NI_a , and NI_b . The accuracy is slightly better than (El-Gamel & El-Hady, 2013) in the first three eigenvalues.

Table 6. For Eqs. (58) and (59) solved by the HIM and LHL to Eq. (52), listing the computed eigenvalues and other parameters

| Computed λ | Exact λ | EE | NI_a | NI_b |
|--------------------|-----------------|------------------------|--------|--------|
| 7.146202155733 | 7.146202155733 | 9.77×10^{-15} | 41 | 7 |
| 28.58480862293 | 28.58480862293 | 1.88×10^{-13} | 40 | 8 |
| 64.31581940160 | 64.31581940160 | 2.36×10^{-12} | 38 | 6 |
| 114.3392344917 | 114.3392344917 | 1.40×10^{-11} | 35 | 5 |
| 178.6550538934 | 178.6550538933 | 5.37×10^{-11} | 37 | 6 |

5. Generalized Sturm-Liouville Problems

Let us consider Eq. (11) with the λ -dependent mixed boundary conditions:

$$a_1(\lambda)u(0) + b_1(\lambda)u'(0) = 0, \quad a_2(\lambda)u(1) + b_2(\lambda)u'(1) = 0, \tag{60}$$

which is a generalized Sturm-Liouville problem.

Solving the generalized SLP is more difficult than that to solve the SLP. Some numerical methods were developed to tackle the eigen-parameter dependent boundary conditions problems (Liu, 2010, 2012; Aliyev & Kerimov, 2008; Binding & Browne, 1997; Chanane, 2005, 2007, 2008; Reutskiy, 2008, 2010; Annaby & Tharwat, 2006; El-Gamel, 2014).

5.1 A Main Result

Theorem 4 The generalized Sturm-Liouville problem (11) and (60) with $a_2(\lambda)b_2(\lambda) \neq 0$ can be transformed to a new one with the Dirichlet boundary condition on the right-end:

$$w''(x) = F_1(x, \lambda)w'(x) + F_2(x, \lambda)w(x), \tag{61}$$

$$a_3(\lambda)w(0) + b_3(\lambda)w'(0) = 0, \quad w(1) = 0, \tag{62}$$

where

$$\begin{aligned} w(x) &= a_2(\lambda)u(x) + b_2(\lambda)u'(x), \\ a_3(\lambda) &= a_1(\lambda)a_2(\lambda) + b_1(\lambda)b_2(\lambda)q(0, \lambda), \\ b_3(\lambda) &= a_2(\lambda)b_1(\lambda) - a_1(\lambda)b_2(\lambda), \end{aligned} \tag{63}$$

$$\begin{aligned} D(x, \lambda) &= a_2^2(\lambda) + q(x, \lambda)b_2^2(\lambda), \\ F_1(x, \lambda) &= \frac{q'(x, \lambda)b_2^2(\lambda)}{D(x, \lambda)}, \end{aligned}$$

$$F_2(x, \lambda) = -q(x, \lambda) - \frac{q'(x, \lambda)a_2(\lambda)b_2(\lambda)}{D(x, \lambda)}, \tag{64}$$

in which the primes in $q'(x, \lambda)$ denotes the differential with respect to x .

Proof. It is easy to check that

$$w(x) = a_2(\lambda)u(x) + b_2(\lambda)u'(x) \tag{65}$$

satisfies the second condition in Eq. (62), when $u(x)$ satisfies Eq. (60).

Differentiating Eq. (65) with respect to x and inserting Eq. (11) for $u''(x)$, we have

$$w'(x) = a_2(\lambda)u'(x) + b_2(\lambda)u''(x) = a_2(\lambda)u'(x) - q(x, \lambda)b_2(\lambda)u(x). \tag{66}$$

From Eqs. (65) and (66), we can express $u(x)$ and $u'(x)$ in terms of $w(x)$ and $w'(x)$ as follows:

$$u(x) = \frac{1}{D(x, \lambda)} [a_2(\lambda)w(x) - b_2(\lambda)w'(x)], \tag{67}$$

$$u'(x) = \frac{1}{D(x, \lambda)} [q(x, \lambda)b_2(\lambda)w(x) + a_2(\lambda)w'(x)], \tag{68}$$

where $D(x, \lambda)$ is defined in Eq. (64).

Inserting Eqs. (67) and (68) with $x = 0$ into Eq. (60), yields

$$\begin{aligned} \frac{a_1(\lambda)}{D(0, \lambda)} [a_2(\lambda)w(0) - b_2(\lambda)w'(0)] + \frac{b_1(\lambda)}{D(0, \lambda)} [q(0, \lambda)b_2(\lambda)w(0) + a_2(\lambda)w'(0)] &= 0, \\ [a_1(\lambda)a_2(\lambda) + q(0, \lambda)b_1(\lambda)b_2(\lambda)]w(0) + [a_2(\lambda)b_1(\lambda) - a_1(\lambda)b_2(\lambda)]w'(0) &= 0. \end{aligned}$$

Thus, we prove the first one in Eq. (62) with $a_3(\lambda)$ and $b_3(\lambda)$ defined in Eq. (63).

Differentiating Eq. (66) with respect to x , and using Eq. (11) for $u''(x)$, Eq. (67) for $u(x)$ and Eq. (68) for $u'(x)$, we have

$$\begin{aligned} w''(x) &= [a_2(\lambda)q(x, \lambda) + q'(x, \lambda)b_2(\lambda)] \frac{b_2(\lambda)w'(x) - a_2(\lambda)w(x)}{D(x, \lambda)} \\ &\quad - \frac{q(x, \lambda)b_2(\lambda)}{D(x, \lambda)} [q(x, \lambda)b_2(\lambda)w(x) + a_2(\lambda)w'(x)] \\ &= \frac{q'(x, \lambda)b_2^2(\lambda)}{D(x, \lambda)} w'(x) - \frac{1}{D(x, \lambda)} [q(x, \lambda)a_2^2(\lambda) + q'(x, \lambda)a_2(\lambda)b_2(\lambda) + q^2(x, \lambda)b_2^2(\lambda)]w(x) \\ &= \frac{q'(x, \lambda)b_2^2(\lambda)}{D(x, \lambda)} w'(x) - q(x, \lambda)w(x) - \frac{q'(x, \lambda)a_2(\lambda)b_2(\lambda)}{D(x, \lambda)} w(x). \end{aligned} \tag{69}$$

We prove Eq. (61) with $F_1(x, \lambda)$ and $F_2(x, \lambda)$ given in Eq. (64). □

5.2 Boundary Shape Function Method

Eqs. (61) and (62) constitute a highly nonlinear generalized SLP for λ . To solve it, we can derive the following two linear shape functions in terms of λ :

$$s_1(x, \lambda) = \frac{1 - x}{a_3(\lambda) - b_3(\lambda)}, \quad s_2(x, \lambda) = \frac{a_3(\lambda)x - b_3(\lambda)}{a_3(\lambda) - b_3(\lambda)}. \tag{70}$$

Let

$$G(x, \lambda) := s_1(x, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] + s_2(x, \lambda)y(1). \tag{71}$$

Theorem 5 For $y(x) \in C^2[0, 1]$, if the boundary shape function $w(x)$ is given by

$$w(x) = y(x) - G(x, \lambda) = y(x) - s_1(x, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] - s_2(x, \lambda)y(1), \tag{72}$$

the boundary conditions in Eq. (62) hold for $w(x)$.

Proof. From Eq. (72), it is obvious that

$$w(1) = y(1) - s_1(1, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] - s_2(1, \lambda)y(1) = y(1) - y(1) = 0,$$

by using Eq. (70).

Inserting $x = 0$ into Eq. (72), differentiating Eq. (72) and inserting $x = 0$, we have

$$\begin{aligned} w(0) &= y(0) - s_1(0, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] - s_2(0, \lambda)y(1), \\ w'(0) &= y'(0) - s'_1(0, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] - s'_2(0, \lambda)y(1). \end{aligned}$$

Obviously, by using Eq. (70) the following combination holds:

$$\begin{aligned} a_3(\lambda)w(0) + b_3(\lambda)w'(0) &= a_3(\lambda)y(0) + b_3(\lambda)y'(0) \\ &\quad - [a_3(\lambda)s_1(0, \lambda) + b_3(\lambda)s'_1(0, \lambda)][a_3(\lambda)y(0) + b_3(\lambda)y'(0)] \\ &\quad - [a_3(\lambda)s_2(0, \lambda) + b_3(\lambda)s'_2(0, \lambda)]y(1) \\ &= a_3(\lambda)y(0) + b_3(\lambda)y'(0) - [a_3(\lambda)y(0) + b_3(\lambda)y'(0)] = 0. \end{aligned}$$

We complete the proof. \square

It follows from Eqs. (61), (64) and (72) that

$$\begin{aligned}
 y''(x) &= G''(x, \lambda) + F_1(x, \lambda)[y'(x) - G'(x, \lambda)] + F_2(x, \lambda)[y(x) - G(x, \lambda)] \\
 &= G''(x, \lambda) + \frac{q'(x, \lambda)b_2^2(\lambda)}{D(x, \lambda)}[y'(x) - G'(x, \lambda)] \\
 &\quad - \left[q(x, \lambda) + \frac{q'(x, \lambda)a_2(\lambda)b_2(\lambda)}{D(x, \lambda)} \right] [y(x) - G(x, \lambda)].
 \end{aligned} \tag{73}$$

If we can solve y with the initial values $y(0)$ and $y'(0)$ being given, Theorem 5 guarantees that $w(x)$ satisfies the boundary conditions in Eq. (62). However, $y(1)$ in $G(x, \lambda)$ of Eq. (71) is unknown, whose exact value can be determined below.

5.3 A Definite Initial Value Problem

In view of Eqs. (61) and (62), if $w(x)$ is a transformed eigenfunction, then $\gamma w(x)$, $\gamma \neq 0$ is also a transformed eigenfunction. For the uniqueness of $w(x)$ and then $u(x)$ by Eq. (67), we can consider a normalization condition as follows:

$$w'(0) = A_0, \tag{74}$$

where A_0 is a given nonzero constant.

Theorem 6 If Eq. (74) is imposed for the uniqueness of $w(x)$, then the right-end value $y(1)$ in Eq. (72) is given by

$$y(1) = y(0) + y'(0) - A_0 + \frac{b_3(\lambda)A_0}{a_3(\lambda)}. \tag{75}$$

Proof. Differentiating Eq. (72) with respect to x and inserting $x = 0$, yields

$$w'(0) = y'(0) + \frac{1}{a_3(\lambda) - b_3(\lambda)} [a_3(\lambda)y(0) + b_3(\lambda)y'(0)] - \frac{a_3(\lambda)}{a_3(\lambda) - b_3(\lambda)} y(1), \tag{76}$$

which immediately leads to Eq. (75) by using $w'(0) = A_0$ in Eq. (74). \square

Therefore, we come to a definite second-order ODE from Eqs. (73), (71) and (75) for the new variable $y(x)$:

$$y''(x) = F_1(x, \lambda)[y'(x) - G'(x, \lambda)] + F_2(x, \lambda)[y(x) - G(x, \lambda)], \tag{77}$$

where

$$G(x, \lambda) := s_1(x, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] + s_2(x, \lambda) \left[y(0) + y'(0) - A_0 + \frac{b_3(\lambda)A_0}{a_3(\lambda)} \right], \tag{78}$$

and $G''(x, \lambda) = 0$ was taken into account. Upon giving the initial values $y(0)$ and $y'(0)$ and fixing $A_0 = 1$, Eq. (77) is a definite initial value problem with only unknown eigenvalue λ involved.

We can integrate Eq. (77) to obtain $y(1)$; hence, inserting $x = 1$ into Eq. (72), yields

$$w(1; \lambda) = -s_1(1, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] + [1 - s_2(1, \lambda)]y(1), \tag{79}$$

where $y(1)$ is obtained from the integration of Eq. (77) with the given initial conditions $y(0)$ and $y'(0)$, which is no longer the one in Eq. (75).

To match the right-end Dirichlet boundary condition in Eq. (62), we can derive

$$w(1; \lambda) = [1 - s_2(1, \lambda)]y(1) - s_1(1, \lambda)[a_3(\lambda)y(0) + b_3(\lambda)y'(0)] = 0, \tag{80}$$

which is a numerically characteristic equation to determine λ . We can adjust the value of λ until $w(1; \lambda)$ satisfies $|w(1; \lambda)| < \epsilon_2$, by applying the HIM or LHL to solve Eq. (80) to determine the eigenvalue λ .

In the original generalized SLP in Eqs. (11) and (60) the initial value $u(0)$ or $u'(0)$ is missing and the target equation $a_2(\lambda)u(1) + b_2(\lambda)u'(1) = 0$ is complicated to determine the eigenvalue. The present formulation in Eq. (77) overcomes these difficulties with the initial values $y(0) = 0$ and $y'(0) = 0$ given and the target equation $w(1; \lambda) = 0$ in Eq. (80) is simplified to determine the eigenvalue more effectively. The two unknown values of a missing $u(0)$ or $u'(0)$ and λ in Eqs. (11) and (60) are reduced to one with λ being the only unknown in the current formulation. The solution of Eq. (80) can be obtained via the HIM or the LHL.

5.4 Example 5

From (Binding & Browne, 1997; Chanane, 2005), we consider

$$u''(x) + \lambda u(x) = 0, \quad 0 < x < 1, \tag{81}$$

$$u(0) + (\lambda - 4\pi^2)u'(0) = 0, \quad u(1) - \lambda u'(1) = 0. \tag{82}$$

In (Chanane, 2005; Liu, 2012) the characteristic equation is derived as

$$f(\lambda) = (1 + 4\pi^2\lambda^2 - \lambda^3) \sin \sqrt{\lambda} - \sqrt{\lambda}(2\lambda - 4\pi^2) \cos \sqrt{\lambda} = 0, \tag{83}$$

which can be used to determine λ of Eqs. (81) and (82).

We fix $\epsilon_2 = \epsilon_3 = 10^{-15}$ and $N = 5000$, and iteratively determine λ by applying the HIM to and LHL Eq. (80), and the results are listed in Table 7 to show the computed λ , the exact λ , EE, N , NI_a , NI_b , and ERBC. For the first eigenvalue, $u(0) = 0.010557$ and $u'(0) = 3.54887 \times 10^{-4}$ are obtained. For the second eigenvalue, $u(0) = 1.2692117 \times 10^{-4}$ and $u'(0) = -2.5752548 \times 10^{-6}$ are obtained.

Table 7. For example 5 solved by the HIM and LHL to Eq. (80), listing the computed eigenvalues and other parameters

| Computed λ | Exact λ | EE | NI_a | NI_b | ERBC |
|--------------------|-----------------|------------------------|--------|--------|------------------------|
| 9.73088657821 | 9.73088657821 | 5.68×10^{-14} | 44 | 7 | 2.95×10^{-15} |
| 88.7633162526 | 88.7633162526 | 1.86×10^{-11} | 41 | 5 | 1.5×10^{-16} |

5.5 Example 6

We consider (Chanane, 2005; Reutskiy, 2008, 2010):

$$u''(x) + (\lambda - e^x)u(x) = 0, \quad 0 < x < 1, \tag{84}$$

$$u(0) = 0, \quad -\sqrt{\lambda} \sin \sqrt{\lambda} u(1) + \cos \sqrt{\lambda} u'(1) = 0. \tag{85}$$

We fix $\epsilon_2 = \epsilon_3 = 10^{-15}$ and $N = 5000$, and iteratively determine λ by applying the HIM and LHL to Eq. (80), and the results are listed in Table 8 to show the computed λ , λ obtained from (Chanane, 2005), N , NI_a , NI_b , and ERBC.

Table 8. For Eqs. (84) and (85) solved by the HIM and LHL to Eq. (80), listing the computed eigenvalues and other parameters

| Computed λ | λ in (Chanane, 2005) | NI_a | NI_b | ERBC |
|--------------------|------------------------------|--------|--------|------------------------|
| 0.929062028579 | 0.929062028570 | 49 | 32 | 2.44×10^{-15} |
| 6.74788117822 | 6.747881178 | 45 | 10 | 1.22×10^{-15} |
| 16.1245477259 | 16.12454773 | 46 | 7 | 7.22×10^{-16} |
| 31.2202765051 | 31.22027651 | 38 | 7 | 2.42×10^{-15} |
| 50.7339278392 | 50.73392784 | 43 | 7 | 2.43×10^{-15} |

5.6 Example 7

Consider (Chanane, 2005)

$$u''(x) + \lambda u(x) = 0, \quad 0 < x < 1, \tag{86}$$

$$u(0) - 2u'(0) = 0, \quad (1 + \sqrt{\lambda})u(1) + (1 - \lambda)u'(1) = 0, \tag{87}$$

where the exact eigenvalues can be obtained from (Chanane, 2005) by

$$f(\lambda) = 2 \cos \sqrt{\lambda} + \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} + (1 - \sqrt{\lambda})(\cos \sqrt{\lambda} - 2\sqrt{\lambda} \sin \sqrt{\lambda}) = 0. \tag{88}$$

In order to raise the accuracy, we fix $\epsilon_2 = \epsilon_3 = 10^{-15}$ and $\Delta x = 1/5000$, and iteratively determine λ by applying the HIM and LHL to Eq. (80), and the results are listed in Table 9 to show the computed λ , the exact λ , EE, NI_a , NI_b , and ERBC.

Table 9. For Eqs. (86) and (87) solved by HIM and LHL, listing the computed eigenvalues and other parameters

| Computed λ | Exact λ | EE | NI _a | NI _b | ERBC |
|--------------------|-----------------|------------------------|-----------------|-----------------|------------------------|
| 9.938743414 | 9.938743414 | 2.84×10^{-14} | 43 | 11 | 2.43×10^{-15} |
| 40.09896973 | 40.09896973 | 1.59×10^{-12} | 40 | 9 | 1.5×10^{-19} |
| 89.58763453 | 89.58763453 | 1.95×10^{-11} | 42 | 8 | 2.13×10^{-15} |

6. Periodic Sturm-Liouville Problem

Here, we start from the periodic Sturm-Liouville problem (Andrew, 1989):

$$y''(x) = q(x)y(x) - \lambda y(x), \quad 0 < x < \ell, \tag{89}$$

and consider three types periodic boundary conditions:

$$\text{case (a) : } y(0) = y(\ell), \quad y'(0) = y'(\ell), \tag{90}$$

$$\text{case (b) : } y(0) = y(\ell), \quad y'(0) = y'(\ell) = B_0, \tag{91}$$

$$\text{case (c) : } y(0) = y(\ell) = A_0, \quad y'(0) = y'(\ell) = B_0. \tag{92}$$

For case (a), A_0 and B_0 are not given, such that three unknown values λ , $y(0)$ and $y'(0)$ are to be determined. For case (b), A_0 is not given and B_0 is given, two unknown values λ and $y(0)$ to be determined. For case (c), A_0 and B_0 are given, one unknown value λ to be determined. Obviously, the case (a) is more difficult than the cases (b) and (c).

Multiplying Eq. (89) by $y(x)$ and using $y''(x)y(x) = [y'(x)y(x)]' - y'(x)^2$, renders

$$\lambda y^2(x) = q(x)y^2(x) + y'(x)^2 - [y(x)y'(x)]'. \tag{93}$$

Integrating Eq. (93) from $x = 0$ to $x = \ell$ and using the periodic conditions in Eq. (90), (91) or (92), so that $y(\ell)y'(\ell) - y(0)y'(0) = 0$, and the eigenvalue is determined by

$$\lambda = \frac{\int_0^\ell [q(x)y^2(x) + y'(x)^2]dx}{\int_0^\ell y^2(x)dx}. \tag{94}$$

To guarantee that the eigenfunction $y(x)$ of Eq. (89) can automatically satisfy Eq. (90), we introduce the following shape functions:

$$s_1(x) = 1 - \frac{x}{\ell}, \quad s_1(0) - s_1(\ell) = 1, \quad s'_1(0) - s'_1(\ell) = 0, \tag{95}$$

$$s_2(x) = \frac{x}{2\ell} - \frac{x^2}{2\ell^2}, \quad s_2(0) - s_2(\ell) = 0, \quad s'_2(0) - s'_2(\ell) = 1. \tag{96}$$

Theorem 7 For any function $w(x) \in C^1[0, \ell]$, the boundary shape function

$$y(x) = w(x) - s_1(x)[w(0) - w(\ell)] - s_2(x)[w'(0) - w'(\ell)] \tag{97}$$

fulfills Eq. (90) for case (a) of the periodic Sturm-Liouville problem.

Proof. We first prove $y(0) = y(\ell)$. It follows from Eq. (97) that

$$y(0) = w(0) - s_1(0)[w(0) - w(\ell)] - s_2(0)[w'(0) - w'(\ell)],$$

$$y(\ell) = w(\ell) - s_1(\ell)[w(0) - w(\ell)] - s_2(\ell)[w'(0) - w'(\ell)].$$

Subtracting the first equation by the second equation, leads to

$$y(0) - y(\ell) = w(0) - w(\ell) - [s_1(0) - s_1(\ell)][w(0) - w(\ell)] - [s_2(0) - s_2(\ell)][w'(0) - w'(\ell)],$$

using Eqs. (95) and (96), which becomes

$$y(0) - y(\ell) = w(0) - w(\ell) - [w(0) - w(\ell)] = 0.$$

Thus, we prove the first one in Eq. (90). Similarly, taking the differential of Eq. (97) with respect to x and inserting $x = 0$ and $x = \ell$, we have

$$y'(0) = w'(0) - s'_1(0)[w(0) - w(\ell)] - s'_2(0)[w'(0) - w'(\ell)],$$

$$y'(\ell) = w'(\ell) - s'_1(\ell)[w(0) - w(\ell)] - s'_2(\ell)[w'(0) - w'(\ell)].$$

Subtracting the first equation by the second equation, generates

$$y'(0) - y'(\ell) = w'(0) - w'(\ell) - [s'_1(0) - s'_1(\ell)][w(0) - w(\ell)] - [s'_2(0) - s'_2(\ell)][w'(0) - w'(\ell)],$$

which by Eqs. (95) and (96) becomes

$$y'(0) - y'(\ell) = w'(0) - w'(\ell) - [w'(0) - w'(\ell)] = 0.$$

Thus, the second one in Eq. (90) is proved. □

Eqs. (89) and (91) in terms of $y_1(x) = y(x)$ and $y_2(x) = y'(x)$ are written as

$$y'_1(x) = y_2(x), \quad y'_2(x) = q(x)y_1(x) - \lambda y_1(x), \tag{98}$$

$$y_1(0) = y_1(\ell), \quad y_2(0) = y_2(\ell) = B_0. \tag{99}$$

Theorem 8 For any functions $z_1(x), z_2(x) \in C^1[0, \ell]$, we set the boundary shape functions to be

$$y_1(x) = z_1(x) - \left(s_0 - \frac{x}{\ell}\right)[z_1(0) - z_1(\ell)], \tag{100}$$

$$y_2(x) = z_2(x) - \left(s_0 - \frac{x}{\ell}\right)[z_2(0) - z_2(\ell)], \tag{101}$$

where $s_0 \neq 0$. If

$$z_2(\ell) = \frac{B_0 + (s_0 - 1)z_2(0)}{s_0}, \tag{102}$$

then $y_1(x)$ and $y_2(x)$ satisfy Eq. (99) for case (b) of the periodic Sturm-Liouville problem.

Proof. Inserting $x = 0$ and $x = \ell$ into Eqs. (100) and (101), leads to

$$\begin{aligned} y_1(0) &= z_1(0) - s_0[z_1(0) - z_1(\ell)], & y_2(0) &= z_2(0) - s_0[z_2(0) - z_2(\ell)], \\ y_1(\ell) &= z_1(\ell) - (s_0 - 1)[z_1(0) - z_1(\ell)], & y_2(\ell) &= z_2(\ell) - (s_0 - 1)[z_2(0) - z_2(\ell)]. \end{aligned}$$

Subtracting equations at the first row by those at the second row, yields

$$\begin{aligned} y_1(0) - y_1(\ell) &= z_1(0) - z_1(\ell) - (s_0 - s_0 + 1)[z_1(0) - z_1(\ell)] \\ &= z_1(0) - z_1(\ell) - [z_1(0) - z_1(\ell)] = 0, \\ y_2(0) - y_2(\ell) &= z_2(0) - z_2(\ell) - (s_0 - s_0 + 1)[z_2(0) - z_2(\ell)] \\ &= z_2(0) - z_2(\ell) - [z_2(0) - z_2(\ell)] = 0. \end{aligned}$$

Inserting $x = 0$ into Eq. (101) and using Eq. (102), yields

$$\begin{aligned} y_2(0) &= z_2(0) - s_0[z_2(0) - z_2(\ell)] = z_2(0)(1 - s_0) + s_0z_2(\ell) \\ &= z_2(0)(1 - s_0) + s_0 \frac{B_0 + (s_0 - 1)z_2(0)}{s_0} = B_0. \end{aligned}$$

Thus, we complete the proof that $y_1(x)$ and $y_2(x)$ in Eqs. (100) and (101) satisfy Eq. (99), if Eq. (102) holds. □

In terms of $y_1 = y$ and $y_2 = y'$, Eq. (92) is recast to

$$y_1(0) = y_1(\ell) = A_0, \quad y_2(0) = y_2(\ell) = B_0. \tag{103}$$

Theorem 9 In Eqs. (100) and (101), if $z_1(\ell)$ and $z_2(\ell)$ are given by

$$z_1(\ell) = \frac{A_0 + (s_0 - 1)z_1(0)}{s_0}, \tag{104}$$

$$z_2(\ell) = \frac{B_0 + (s_0 - 1)z_2(0)}{s_0}, \tag{105}$$

for $s_0 \neq 0$, then $y_1(x)$ and $y_2(x)$ satisfy Eq. (103) for case (c) of the periodic Sturm-Liouville problem.

Proof. The proofs of $y_1(0) = y_1(\ell)$ and $y_2(0) = y_2(\ell)$ are the same to that in Theorem 7. Inserting $x = 0$ into Eq. (100) and using Eq. (104), leads to

$$\begin{aligned} y_1(0) &= z_1(0) - s_0[z_1(0) - z_1(\ell)] = z_1(0)(1 - s_0) + s_0z_1(\ell) \\ &= z_1(0)(1 - s_0) + s_0 \frac{A_0 + (s_0 - 1)z_1(0)}{s_0} = A_0. \end{aligned}$$

Similarly, inserting $x = 0$ into Eq. (101) and using Eq. (105), leads to

$$\begin{aligned} y_2(0) &= z_2(0) - s_0[z_2(0) - z_2(\ell)] = z_2(0)(1 - s_0) + s_0z_2(\ell) \\ &= z_2(0)(1 - s_0) + s_0 \frac{B_0 + (s_0 - 1)z_2(0)}{s_0} = B_0. \end{aligned}$$

Thus, we complete the proof that $y_1(x)$ and $y_2(x)$ in Eqs. (100) and (101) satisfy Eq. (103), if Eqs. (104) and (105) hold. \square

7. Iterative Algorithms

Based on the boundary shape functions in Theorems 7-9, we can develop iterative algorithms to determine the eigenvalue and eigenfunction of Eqs. (89)-(92), namely the boundary shape function methods.

7.1 First Iterative Algorithm for Case (a)

By Theorem 7, $y(x)$ automatically satisfies the periodic boundary conditions in Eq. (90) for case (a), if we can solve $w(x)$, which suggests the following transformation of variables from $y(x)$ to $w(x)$:

$$w(x) = y(x) + G(x), \tag{106}$$

where

$$G(x) = s_1(x)[w(0) - w(\ell)] + s_2(x)[w'(0) - w'(\ell)]. \tag{107}$$

Let

$$\begin{aligned} w_3(x) &:= \int_0^x [q(\xi)y^2(\xi) + y'(\xi)^2]d\xi \\ &= \int_0^x \{q(\xi)[w(\xi) - G(\xi)]^2 + [w'(\xi) - G'(\xi)]^2\}d\xi, \end{aligned} \tag{108}$$

$$w_4(x) := \int_0^x y^2(\xi)d\xi = \int_0^x [w(\xi) - G(\xi)]^2d\xi, \tag{109}$$

where Eq. (106) was used. Then, the eigenvalue λ in Eq. (94) can be computed by

$$\lambda = \frac{w_3(\ell)}{w_4(\ell)}. \tag{110}$$

In terms of $w_1(x) = w(x)$, $w_2(x) = w'(x)$ and from Eqs. (89), (106), (108) and (109), we have

$$\begin{aligned} w'_1(x) &= w_2(x), \quad w'_2(x) = G''(x) + [q(x) - \lambda][w_1(x) - G(x)], \\ w'_3(x) &= q(x)[w_1(x) - G(x)]^2 + [w_2(x) - G'(x)]^2, \quad w_3(0) = 0, \\ w'_4(x) &= [w_1(x) - G(x)]^2, \quad w_4(0) = 0, \end{aligned} \tag{111}$$

which are initial value problems for w_k , $k = 1, \dots, 4$ by giving the initial values $w_1(0)$ and $w_2(0)$. The initial values $w_3(0) = 0$ and $w_4(0) = 0$ are the direct results of Eqs. (108) and (109) after inserting $x = 0$.

The above $w(\ell)$ denoted by c and $w'(\ell)$ by d in the function $G(x)$ of Eq. (107) are unknown constants. According to Theorem 7, the first iterative algorithm for Eqs. (89) and (90) for case (a) is summarized as follows. (i) Give ℓ , $w_1(0)$, $w_2(0)$, the initial guesses of $\lambda(0)$, c_0 , d_0 , the convergence criterion ϵ_1 , and N in $\Delta x = \ell/N$. (ii) For $k = 0, 1, 2, \dots$, integrate Eq. (111) with N steps from $x = 0$ to $x = \ell$, and take

$$c_{k+1} = w_1(\ell), \quad d_{k+1} = w_2(\ell), \quad \lambda(k) = \frac{w_3(\ell)}{w_4(\ell)},$$

until c_{k+1} , d_{k+1} and $\lambda(k + 1)$ abide

$$\sqrt{(c_{k+1} - c_k)^2 + (d_{k+1} - d_k)^2 + [\lambda(k + 1) - \lambda(k)]^2} < \epsilon_1.$$

To accelerate the convergence, we can relax the convergence criterion to $|\lambda(k + 1) - \lambda(k)| < \epsilon_1$ and using the initial guesses $c_0 = u_0$ and $d_0 = 2u_0 + 2\ell v_0 + (1 - 2\ell)w'(0) - 2w(0)$ which are derived from Eqs. (106), (107), (95) and (96), where u_0 and v_0 are initial guesses of $y(0)$ and $y'(0)$. When λ is available, we can determine $y(x)$ and $y'(x)$ by integrating Eq. (89) using the initial conditions $y(0) = w(0) - G(0)$ and $y'(0) = w'(0) - G'(0)$.

7.2 Second Iterative Algorithm for Case (b)

Based on Theorem 8, the second iterative algorithm for Eqs. (89) and (91) of case (b) is coined as follows. (i) Give $\ell, B_0, s_0, z_1(0), z_2(0)$, the initial guesses of $\lambda(0)$ and c_0, ϵ_2, N , and compute $z_2(\ell)$ by Eq. (102). (ii) For $k = 0, 1, 2, \dots$, integrate

$$\begin{aligned} z_1'(x) &= -\frac{1}{\ell}[z_1(0) - z_1(\ell)] + z_2(x) - \left(s_0 - \frac{x}{\ell}\right)[z_2(0) - z_2(\ell)], \\ z_2'(x) &= -\frac{1}{\ell}[z_2(0) - z_2(\ell)] + [q(x) - \lambda] \left[z_1(x) - \left(s_0 - \frac{x}{\ell}\right)[z_1(0) - z_1(\ell)]\right], \\ z_3'(x) &= q(x) \left[z_1(x) - \left(s_0 - \frac{x}{\ell}\right)[z_1(0) - z_1(\ell)]\right]^2 \\ &+ \left[z_2(x) - \left(s_0 - \frac{x}{\ell}\right)[z_2(0) - z_2(\ell)]\right]^2, \quad z_3(0) = 0, \\ z_4'(x) &= \left[z_1(x) - \left(s_0 - \frac{x}{\ell}\right)[z_1(0) - z_1(\ell)]\right]^2, \quad z_4(0) = 0, \end{aligned} \tag{112}$$

and take

$$c_{k+1} = z_1(\ell), \quad \lambda(k) = \frac{z_3(\ell)}{z_4(\ell)}.$$

If c_{k+1} and $\lambda(k + 1)$ converge with

$$\sqrt{(c_{k+1} - c_k)^2 + [\lambda(k + 1) - \lambda(k)]^2} < \epsilon_2,$$

then stop; otherwise, go to step (ii). We can determine $y(x)$ and $y'(x)$ by integrating Eq. (89) using $y(0) = (1 - s_0)z_1(0) + s_0z_1(\ell)$ and $y'(0) = (1 - s_0)z_2(0) + s_0z_2(\ell)$ as the initial conditions.

7.3 Third Iterative Algorithm for Case (c)

Based on Theorem 9, the third iterative algorithm for Eqs. (89) and (92) of case (c) is depicted as follows. (i) Give $\ell, A_0, B_0, s_0, w_1(0), w_2(0)$, the initial guess of $\lambda(0), \epsilon_3, N$ and compute $z_1(\ell)$ and $z_2(\ell)$ from Eqs. (104) and (105). (ii) For $k = 0, 1, 2, \dots$, integrate Eq. (112) and take $\lambda(k) = z_3(\ell)/z_4(\ell)$. If $\lambda(k + 1)$ converges to satisfy $|\lambda(k + 1) - \lambda(k)| < \epsilon_3$, then stop; otherwise, go to step (ii).

7.4 Numerical Results

We first consider Mathieu’s equation with $q(x) = 2 \cos 2x$ and $\ell = 2\pi$ (Liu, 2020). The first eigenvalues obtained by the three iterative algorithms are close to that tabulated in (Abramowitz, 1972). In the first iterative algorithm, with $w(0) = 1, w'(0) = 0, u_0 = 1, v_0 = 0, \epsilon_1 = 10^{-8}$, and $N = 500$, we iteratively determine λ with different $\lambda(0)$, whose results as listed in Table 10 are $n, \lambda_n, \text{NI}, y(0), y'(0)$, the absolute error of first boundary condition (EFBC) for $y(0) = y(\ell)$, and the absolute error of second boundary condition (ESBC) for $y'(0) = y'(\ell)$.

Table 10. For $q = 2 \cos 2x$ and $\ell = 2\pi$, the eigenvalues computed by the first iterative algorithm

| n | λ_n | NI | $y(0)$ | $y'(0)$ | EFBC | ESBC |
|-----|-------------|----|---------|------------------------|-----------------------|-----------------------|
| 1 | 1.859108093 | 15 | 0.99957 | 6.91×10^{-5} | 2.05×10^{-4} | 7.10×10^{-8} |
| 2 | 4.371301085 | 26 | 1.00368 | -5.85×10^{-4} | 2.60×10^{-4} | 1.82×10^{-7} |
| 3 | 9.078369555 | 6 | 1.06139 | -9.78×10^{-3} | 1.21×10^{-4} | 1.14×10^{-6} |

In the second iterative algorithm, we fix $B_0 = 0, w(0) = 1, w'(0) = 0, s_0 = 0.5, \epsilon_2 = 10^{-8}, N = 500$, whose results are listed in Table 11 to show $n, \lambda_n, y(0), \text{NI}, \text{EFBC}$ and ESBC .

Table 11. For $q = 2 \cos 2x$ and $\ell = 2\pi$, the eigenvalues computed by the second iterative algorithm

| n | λ_n | NI | $y(0)$ | EFBC | ESBC |
|-----|-------------|----|---------|-----------------------|-----------------------|
| 1 | 1.859108082 | 7 | 0.16471 | 2.39×10^{-7} | 8.17×10^{-8} |
| 2 | 4.371301025 | 4 | 0.51282 | 5.61×10^{-7} | 1.38×10^{-6} |
| 3 | 9.078369170 | 6 | 0.36703 | 1.40×10^{-7} | 9.67×10^{-6} |

In the third iterative algorithm, we fix $A_0 = 0.5, B_0 = 0, w(0) = w'(0) = 0, s_0 = 0.5, \epsilon_3 = 10^{-10}, N = 500$, whose results are listed in Table 12 to show $n, \lambda_n, NI, EFBC$ and $ESBC$.

Table 12. For $q = 2 \cos 2x$ and $\ell = 2\pi$, the eigenvalues computed by the third iterative algorithm

| n | λ_n | NI | EFBC | ESBC |
|-----|-------------|----|-----------------------|-----------------------|
| 1 | 1.859108081 | 8 | 1.5×10^{-8} | 1.0×10^{-8} |
| 2 | 4.371301019 | 5 | 1.51×10^{-9} | 3.23×10^{-9} |
| 3 | 9.078369170 | 3 | 5.56×10^{-9} | 4.98×10^{-9} |

Next, we consider a non-symmetric potential $q(x) = x^2(\pi - x)$ and $\ell = \pi$, which has been calculated in (Andrew, 1989) using the correction of finite difference method. We apply the iterative algorithms with the same parameters as those in the above to determine the eigenvalues, which are tabulated in Table 13 with the NIs for the third iterative algorithm.

Table 13. For $q = x^2(\pi - x)$ and $\ell = \pi$, the eigenvalues computed by the three iterative algorithms

| n | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|
| λ_n in (Andrew, 1989) | × | 6.5005 | 18.5848 | 38.5816 | 66.5821 | 102.5825 |
| λ_n (1st) | 3.447673269 | 6.500490560 | 18.58499500 | 38.58176373 | 66.58211038 | 102.5825800 |
| λ_n (2nd) | 3.448524148 | 6.500490559 | 18.58499502 | 38.58176416 | 66.58211373 | 102.5825801 |
| λ_n (3rd) | 3.448524148 | 6.550874477 | 18.58499502 | 38.58176416 | 66.58211373 | 102.5825801 |
| NI | 5 | 6 | 4 | 4 | 4 | 4 |

As shown in Tables 10-13, the high efficiency and accuracy of the proposed iterative algorithms can be appreciated. In Table 10, the NIs are larger than that in Tables 11-13, because case (a) is a more difficult periodic Sturm-Liouville problem than that of cases (b) and (c). Although we have relaxed the convergence criterion in the first iterative algorithm, the accuracy for the preservation of periodic boundary conditions is acceptable in the order 10^{-4} . The success of the proposed iterative algorithms is attributed to the good properties of the boundary shape functions in Theorems 7-9.

8. Conclusions

A new result is transformed the Sturm-Liouville problem with mixed boundary conditions into a generalized Sturm-Liouville eigenvalue problem with the Dirichlet boundary conditions happened on two sides, which greatly enhanced the convergence speed of the iterative algorithm to determine the eigenvalues. Combining to the fictitious time integration method (FTIM), an iterative algorithm was presented by transforming the problem to an initial value problem for the new variable. The eigenvalue and a terminal value of the new variable are determined iteratively by using the FTIM, from which the iterative sequence of eigenvalues are monotonically converged to its limit point to approximate the true eigenvalue when the relative residual reduced to zero. We proposed a normalization condition for the transformed variable and derived the terminal value of the new variable explicitly, such that we can obtain a definite initial value problem for the new variable. In order to precisely meet the right-end Dirichlet boundary condition, a numerically characteristic equation was provided which was solved by the half-interval method (HIM) and a derivative-free iterative method LHL to obtain the eigenvalues with high precision. For the generalized Sturm-Liouville problem it needs to solve two unknowns: a missing initial value and the eigenvalue. The new formulation presented in the paper overcame this difficulty by transforming the generalized Sturm-Liouville problem to a definite initial value problem with the initial values given, and the Dirichlet boundary condition on the right-end acting as a target equation. Then, the uniqueness condition of the eigenfunction is an extra requirement to ascertain $y(1)$ in Eq. (75), such that only the eigenvalues are unknown to be determined. The sequence of iterative eigenvalues generated from the numerically characteristic equation converged very fast to the true eigenvalues by using the HIM and LHL. For the periodic Sturm-Liouville problems, we considered three types periodic boundary conditions and utilized the boundary shape functions methods to transform them into initial value problems involved unknown terminal values of the new variables. Three iterative algorithms were developed to compute the eigenvalues, whose high efficiency and accuracy can be appreciated.

In the whole paper the boundary shape function methods and the initial value problem methods are developed to simplify the three types Sturm-Liouville problems, involving only unknown eigenvalues to be determined. The Dirichlet type target equations are solved by the HIM and LHL. Both methods are effectively to determine the eigenvalues; however, we found that the LHL is convergent faster than the HIM. Many numerical examples assured the high-performance of the proposed iterative algorithms through a few iterations and with high accuracy of the obtained eigenvalues and eigenfunctions to preserve the mixed boundary conditions or periodic boundary conditions.

Overall, the relative errors of the eigenvalues obtained by the HIM and LHL are located between 10^{-13} and 10^{-15} as shown in Tables 3-9, which are more accurate than that presented in the literature. Owing to the high-accuracy of the proposed methods, they might be applied to solve more difficult nonlinear Sturm-Liouville problems and inverse Sturm-Liouville problems in the future.

References

- Abbasbandy, S., & Shirzadi, A. (2011). A new application of the homotopy analysis method: solving the Sturm-Liouville problems. *Commun. Nonlinear Sci. Numer. Simulat.*, *16*, 112-126. <https://doi.org/10.1016/j.cnsns.2010.04.004>
- Abramowitz, M., & Stegun, I. (1972). *Handbook of Mathematical Functions*. Dover, New York. <https://doi.org/10.1016/B978-012382256-7/50005-1>
- Aceto, L., Ghelardoni, P., & Magherini, C. (2009a). Boundary value methods as an extension of Numerov's method for Sturm-Liouville eigenvalue estimates. *Appl. Numer. Math.*, *59*, 1644-1656. <https://doi.org/10.1016/j.apnum.2008.11.005>
- Aceto, L., Ghelardoni, P., & Magherini, C. (2009b). BVMs for Sturm-Liouville eigenvalue estimates with general boundary conditions. *JNAIAM J. Numer. Anal. Ind. Appl. Math.*, *4*, 113-127.
- Aliyev, Y. N., & Kerimov, N. (2008). The basis property of Sturm-Liouville problems with boundary conditions depending quadratically on the eigenparameter. *Arabian J. Sci. Eng.*, *33*, 123-136. <http://hdl.handle.net/20.500.12323/4645>
- Alquran, M., & Al-Khaled, K. (2010). Approximations of Sturm-Liouville eigenvalues using sinc-Galerkin and differential transform methods. *Appl. Appl. Math.*, *5*, 128-147. <https://digitalcommons.pvamu.edu/aam/vol5/iss1/11>
- Amodio, P., & Settanni, G. (2015). Variable-step finite difference schemes for the solution of Sturm-Liouville problems. *Commun. Nonlinear Sci. Numer. Simulat.*, *20*, 641-649. <https://doi.org/10.1016/j.cnsns.2014.05.032>
- Andrew, A. L. (1989). Correction of finite difference eigenvalues of periodic Sturm-Liouville problems. *J. Austral. Math. Soc. B*, *30*, 460-469. <https://doi.org/10.1017/S0334270000006391>
- Annaby, M. H., & Tharwat, M. M. (2006). On sampling theory and eigenvalue problems with an eigenparameter in the boundary conditions. *SUT. J. Math*, *42*, 157-175. <https://doi.org/10.20604/00001050>
- Bailey, P. B., Gordon, M. K., & Shampine, L. F. (1978). Automatic solution of the Sturm-Liouville problem. *ACM Trans. Math. Software*, *4*, 193-208. <https://dl.acm.org/doi/pdf/10.1145/355791.355792>
- Bailey, P. B., Everitt, W. N., & Zettl, A. (1991). Computing eigenvalues of singular Sturm-Liouville problems. *Results Math*, *20*, 391-423. <https://doi.org/10.1007/BF03323182>
- Bailey, P. B., Everitt, W. N., & Zettl, A. (2001). Algorithm 810: the SLEIGN2 Sturm-Liouville code. *ACM Trans. Math. Software*, *27*, 143-193. <https://doi.org/10.1145/383738.383739>
- Binding, P. A., & Browne, P. J. (1997). Oscillation theory for indefinite Sturm-Liouville problems with eigenparameter-dependent boundary conditions. *Proc. Royal Soc. Edinburgh*, *127A*, 1123-1136. <https://doi.org/10.1017/S0308210500026974>
- Chanane, B. (2005). Computation of the eigenvalues of Sturm-Liouville problems with parameter dependent boundary conditions using the regularized sampling method. *Math. Comp.*, *74*, 1793-1801. <https://doi.org/10.1090/S0025-5718-05-01717-5>
- Chanane, B. (2007). Computing the spectrum of non-self-adjoint Sturm-Liouville problems with parameter-dependent boundary conditions. *J. Comp. Appl. Math*, *206*, 229-237. <https://doi.org/10.1016/j.cam.2006.06.014>
- Chanane, B. (2008). Sturm-Liouville problems with parameter dependent potential and boundary conditions. *J. Comp. Appl. Math*, *212*, 282-290. <https://doi.org/10.1016/j.cam.2006.12.006>
- El-Gamel, M. (2014). Numerical comparison of sinc-collocation and Chebychev-collocation methods for determining the eigenvalues of Sturm-Liouville problems with parameter-dependent boundary conditions. *SeMA J.*, *66*, 29-42. <https://doi.org/10.1007/s40324-014-0022-9>
- El-Gamel, M., & El-Hady, M. A. (2013). Two very accurate and efficient methods for computing eigenvalues of Sturm-Liouville problems. *Appl. Math. Model.*, *37*, 5039-5046. <https://doi.org/10.1016/j.apm.2012.10.019>
- Fu, S. Z., Wang, Z., & Wei, G. S. (2015). *Sturm-Liouville Problem and its Inverse Problem*. Science Press, Beijing, China (in Chinese).
- Ghelardoni, P. (1997). Approximations of Sturm-Liouville eigenvalues using boundary value methods. *Appl. Numer.*

- Math.*, 23, 311-325. [https://doi.org/10.1016/S0168-9274\(96\)00073-6](https://doi.org/10.1016/S0168-9274(96)00073-6)
- Ghelardoni, P., & Gheri, G. (2001). Improved shooting technique for numerical computations of eigenvalues in Sturm-Liouville problems. *Nonlinear Anal.*, 47, 885-896. [https://doi.org/10.1016/S0362-546X\(01\)00231-0](https://doi.org/10.1016/S0362-546X(01)00231-0)
- Gould, S. H. (1995). *Variational Methods for Eigenvalues Problems; An Introduction to the Methods of Rayleigh, Ritz, Weinstein, and Aronszain*. Dover Publication, New York.
- Hinton, D., & Schaefer, P. W., Eds. (1997), *Spectral Theory & Computational Methods of Sturm-Liouville Problems*. Marcel Dekker, New York.
- Ixaru, L. G., de Meyer, H., & vanden Berghe, G. (1999). SLCPM12—A program for solving regular Sturm-Liouville problems. *Comput. Phys. Commun.*, 118, 259-277. [https://doi.org/10.1016/S0010-4655\(98\)00181-7](https://doi.org/10.1016/S0010-4655(98)00181-7)
- Ledoux, V., van Daele, M., & vanden Berghe, G. (2005). MATSLISE: a MATLAB package for the numerical solution of Sturm-Liouville and Schrödinger equation. *ACM Trans. Math. Software*, 31, 532-554. <https://doi.org/10.1145/1114268.1114273>
- Liu, C.-S. (2008) A Lie-group shooting method for computing eigenvalues and eigenfunctions of Sturm-Liouville problems. *Comput. Model. Eng. Sci.*, 26, 157-168. <https://doi.org/10.3970/CMES.2008.026.157>
- Liu, C.-S. (2010) The Lie-group shooting method for computing the generalized Sturm-Liouville problems. *Comput. Model. Eng. Sci.*, 56, 85-112. <https://doi.org/10.3970/CMES.2010.056.85>
- Liu, C.-S. (2012). Computing the eigenvalues of the generalized Sturm-Liouville problems based on the Lie-group $SL(2, \mathbb{R})$. *J. Comput. Appl. Math.*, 236, 4547-4560. <https://doi.org/10.1016/j.cam.2012.05.006>
- Liu, C.-S. (2020). Analytic solutions of the eigenvalues of Mathieu's equation. *J. Math. Research*, 12(1), 1-11. <https://doi.org/10.5539/jmr.v12n1p1>
- Liu, C.-S., & Atluri, S. N. (2008). A novel time integration method for solving a large system of non-linear algebraic equations. *Comput. Model. Eng. Sci.* 31, 71-83. <https://doi.org/10.3970/CMES.2008.031.71>
- Liu, C.-S., Hong, H. K., & Lee, T. L. (2021). A splitting method to solve a single nonlinear equation with derivative-free iterative schemes. *Math. Comput. Simul.*, 190, 837-847. <https://doi.org/10.1016/j.matcom.2021.06.019>
- Lutzen, J. (1984). Sturm and Liouville's work on ordinary linear differential equations. The emergence of Sturm-Liouville theory. *Arch. History Exact Sci.*, 29, 309-376. <https://doi.org/10.1007/BF00348405>
- Pruess, S., & Fulton, C. T. (1993). Mathematical software for Sturm-Liouville problems. *ACM Trans. Math. Software*, 19, 360-376. <https://doi.org/10.1145/155743.155791>
- Pruess, S., & Marletta, M. (1992). Automatic solution of Sturm-Liouville problems using the Pruess method. *J. Comput. Appl. Math.*, 39, 57-78. [https://doi.org/10.1016/0377-0427\(92\)90222-J](https://doi.org/10.1016/0377-0427(92)90222-J)
- Reutskiy, S. Y. (2008). A meshless method for nonlinear, singular and generalized Sturm-Liouville problems. *Comp. Model. Eng. Sci.*, 34, 227-252. <https://doi.org/10.3970/CMES.2008.034.227>
- Reutskiy, S. Y. (2010). The method of external excitation for solving generalized Sturm-Liouville problems. *J. Comp. Appl. Math.*, 233, 2374-2386. <https://doi.org/10.1016/j.cam.2009.10.022>

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (<http://creativecommons.org/licenses/by/4.0/>).