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On Kostant-Kirillov Symplectic Structure and Quasi-Poisson Structures of the Euler-Arnold Systems

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Abstract

The concept of symplectic structure emerged between 1808 to 1810 through the works of Lagrange and Poisson on the trajectory of the planets of the solar system. In order to explain the variation of the orbital parameters, they introduced the symplectic structure associated to the manifold describing the states of the system and a fundamental operation on functions called Poisson's bracket. But, the latter also comes from the Hamiltonian formalism which does not automatically lead to a Poisson structure. Although contrary to the Riemannian case, not every manifold necessarily admits a symplectic structure including even dimensional manifolds. The aim of this paper is to show the interaction between the Kostant-Kirillov symplectic structure and quasi-Poisson structures coming from the Euler-Arnold systems. The Lie algebra theoretical approach based on the Kostant-Kirillov coadjoint action will allow us to obtain a class of the quasi-Poisson structures resulting from the characterization of the Hamiltonian system and to prove some results on the Kostant-Kirillov symplectic structure in the quasi-Poisson context.

Keywords: Symplectic structure, quasi-Poisson structures, Jacobiator, Euler-Arnold equation, Killing form, Kostant-Kirillov coadjoint action

1. Introduction

Introduced by Alekseev and Yvette Kosmann-Schwarzbach (Alekseev & al., 2000), the the quasi-Poisson structures appeared as a finite-dimensional alternative to infinite-dimensional constructions of Poisson structures on moduli spaces. These constructions have been proposed particularly in (Goldman , 1986), (Jeffrey & al., 1992) and (Huebschmann, 1995). However, examples of quasi-Poisson structures appear in the study of the equations motion in mechanics (see (Euler, 1765), (kowalewski, 1889) and (Appel'rot, 1894)). According to (Arnold, 1966), the Euler equations for a perfect fluid is related to the geodesic equations of a Lie group with an invariant metric. This is referred as the generalised Euler equations known as the Euler-Arnold equations. The Euler-Arnold systems thus govern the Hamiltonian dynamics on Lie groups. The prototype being the equation of motion of a rotating solid formulated by Euler in 1765 (Euler, 1765). The generalisation of this formalism to infinite dimension (groups of diffeomorphisms) was introduced in 1966 by Arnold. He showed that the equation of motion of perfect fluids can be reformulated as a geodesic flow over the group of diffeomorphisms. The general theory of Lie groups requires some properties of differentiable manifold. In this paper, we will focus on a rotation group in which the elements are orthogonal matrices with determinant 1. In the case of three-dimensional space, the rotation group is known as the special orthogonal group often denoted SO(3). The latter is used to describe the possible rotational symmetries of an object, as well as the possible orientations of an object in space and its representations are important in physics, where they give rise to the elementary particles of integer spin (i.e. an intrinsic form of angular momentum carried by elementary particles). A Lie group is a group that is also a differentiable manifold. Named after Norwegian mathematician Sophus Lie (1842-1899), who laid the foundations of the theory of continuous transformation groups, Lie groups play an enormous role in modern geometry, on several different levels. It is natural to associate any Lie group $\mathbf{G}$ to Lie algebra. There are two equivalent ways of introducing this Lie algebra. First is to introduce a space of vector fields on $\mathbf{G}$, the other is to provide the tangent space at the neutral element with a Lie bracket, derived from the local expression of the internal operation of $\mathbf{G}$. In the following, $\mathbf{G}$ denoted subgroup of SO(3) and the corresponding Lie algebra is $\mathfrak{g} = T_1\mathbf{G}$ consists of skew-symmeric 3×3 matrices where 1 is the neutral element of $\mathbf{G}$ and $T_1\mathbf{G}$ is the tangent space at 1. Example of basis of g is given by

$$e_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

We can explicitly describe the subgroup **G**. Namely, the exponential map exp permits to define the rotation around x-axis by the angle θ. We call it $g_\theta = exp(\theta e_x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$. Similarly, e_y, e_z generate rotations around y, z axes.
Now, we introduce the commutator of two elements of $\mathfrak{g}$ defined by

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (X, Y) \longmapsto [X, Y] = XY - YX. \tag{2}$$

Note that $\mathfrak{g}$ has a canonical structure of a Lie algebra with this commutator and we have

$$[e_x, e_y] = e_z, [e_x, e_z] = -e_y \quad and \quad [e_y, e_z] = e_x.$$

Let $I = \{x, y, z\}$ be an index set and let $X = \sum_{i\in I} j_i e_i$ be an element of $\mathfrak{g}$. Let's consider the following map :

$$\mathbb{C}^3 \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (\Lambda, X) \longmapsto \Lambda X = \sum_{i\in I} \Lambda_i j_i e_i, \quad \Lambda = (\Lambda_i)_{i\in I}. \tag{3}$$

We call the Euler-Arnold system the differential equation :

$$\dot{X} = [X, \Lambda X], \qquad X \in \mathfrak{g} \tag{4}$$

where $[-,-]$ is the commutator defined by (2). From the characterization of the Hamiltonian field of (4), there exists an antisymmetric matrix J_X and a differentiable function H (called Hamiltonian) such that

$$\dot{X} = J_X \frac{\partial H}{\partial X}.$$

According to (Weinstein, 1983), there exists a bivector field π_{J_X} associate to J_X. However, π_{J_X} is not always a Poisson bivector field. The Jacobi identity is obviously not satisfied. We will call it "quasi-Poisson structures of the Euler-Arnold systems". The question arises is : can we construct a symplectic structure on **G**?

Note that a symplectic structure or symplectic form on **G** is defined to be a differential 2-form ω on **G** that is closed and is non-degenerate. According to (Kirillov, 1976), the dual space $\mathfrak{g}^*$ of the corresponding Lie algebra $\mathfrak{g}$ plays an important role in the the Kirillov-Kostant bracket which is always degenerate at the origin in $\mathfrak{g}^*$. In this paper, we will establish the symplectic structure coincide with orbits of the coadjoint action of $\mathfrak{g}^*$, by extending the results contained in (Lesfari, 2009). We show that the Kostant-Kirillov symplectic structure is given by

$$\omega_f(\tau_1, \tau_2) = \langle f, j \wedge k \rangle,$$

with $j, k \in \mathbb{C}^3$ where $\tau_1 = f \wedge j$, $\tau_2 = f \wedge k$ and $\wedge$ is the usual vector product.
Some properties on the Kostant-Kirillov symplectic structure of **G** and quasi-Poisson structures of the Euler-Arnold systems are are described in section 2. the interaction between the Kostant-Kirillov symplectic structure and quasi-Poisson structures coming from the Euler-Arnold systems is detailed in the section 3 of this article.

2. Some Properties on the Kostant-Kirillov Symplectic Structure of the Lie Group G and Quasi-Poisson Structures of the Euler-Arnold Systems

In this section, we describe the Lie algebra theoretical approach based on the Kostant-Kirillov coadjoint action and we present a useful result on the Kostant-Kirillov symplectic structure in the quasi-Poisson context.

Let's consider $g_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ be an element of **G**, X belongs to $\mathfrak{g}$ and t be a real number. We have :

$$g_\theta exp(tX) g_\theta^{-1} = exp(t g_\theta X g_\theta^{-1}).$$

Therefore, $g_\theta X g_\theta^{-1}$ in an element of $\mathfrak{g}$ and considering the following automorphism

$$Ad(\mathfrak{g}) : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad X \longmapsto Ad(\mathfrak{g})X = g_\theta X g_\theta^{-1},$$

we have

$$Ad(\mathfrak{g})[X, Y] = [Ad(\mathfrak{g})X, Ad(\mathfrak{g})Y] \quad (X, Y \in \mathfrak{g}).$$

Since $(\mathfrak{g}, [-,-])$ is a Lie algebra, we have the following proprietes :

(i) $[X, Y] = -[Y, X]$,

(ii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Identity of Jacobi*).

Considering the above basis (e_x, e_y, e_z) of $\mathfrak{g}$, the bracket $[-, -]$ can be written as

$$[e_i, e_j] = \sum_{k \in I} j_{ij}^k e_k$$

Properties (i) and (ii) implies that

Proposition 2.1.

$$\begin{aligned} j_{ij}^k + j_{ji}^k &= 0, \\ \sum_{l \in I} (j_{ij}^l j_{lk}^m + j_{jk}^l j_{li}^m + j_{ki}^l j_{lj}^m) &= 0. \end{aligned}$$

Let $X \in \mathfrak{g}$, We call $Aut(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$ and adX the endomorphism of $\mathfrak{g}$ defined by

$$ad(X).Y = [X, Y].$$

The Jacobi identity shows that $ad(X)$ is a derivation and the space $Der(\mathfrak{g})$ of derivations of $\mathfrak{g}$ is a Lie algebra for the commutator defined by (2). The application $ad : \mathfrak{g} \longrightarrow Der(\mathfrak{g})$ is a homomorphism of Lie algebras,

$$ad[X, Y] = [ad(X), ad(Y)].$$

The application $ad : \mathfrak{g} \longrightarrow Der(\mathfrak{g}) \subset End(\mathfrak{g})$ is a representation of the Lie algebra $\mathfrak{g}$ called the *adjoint representation*.
For X, Y in $\mathfrak{g}$, let's consider

$$\langle \cdot, \cdot \rangle : (X, Y) \longmapsto \langle X, Y \rangle = Tr(ad(X)ad(Y)). \tag{5}$$

It's a symmetric bilinear form on $\mathfrak{g}$ which is associative,

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle,$$

i.e. the $ad(X)$ transformation is skew-symmetric with $\langle \cdot, \cdot \rangle$.

Definition 2.1. *The bilinear form $\langle \cdot, \cdot \rangle$ associated to the adjoint representation ad is called the* Killing form.

To illustrate, let us take a few examples.
- In the set $M(n, \mathbb{R})$ of $n \times n$ matrices with elements in $\mathbb{R}$, the Killing form is defined as

$$\langle X, Y \rangle = 2nTr(XY) - 2Tr(X)Tr(Y).$$

- In $\mathfrak{so}(\mathfrak{n})$, for $n \geq 2$, $\langle X, Y \rangle = (n - 2)Tr(XY)$. We can deduce that

$$\langle X, Y \rangle = Tr(XY) \quad for\ all\ X, Y\ in\ \mathfrak{g}.$$

Let's consider the Killing form of $\mathfrak{g}$. From the following identity

$$ad(Ad(g)X) = Ad(g)ad(X)Ad(g^{-1}) \quad (g \in \mathbf{G}, X \in \mathfrak{g}),$$

it follows that

$$\langle (Ad(g)X, Ad(g)Y \rangle = \langle X, Y \rangle, \quad (X, Y \in \mathfrak{g}),$$

i.e. $Ad(g)$ belongs to the orthogonal group of the Killing form. As a result

$$|\det(Ad(g))| = 1.$$

Let $\mathfrak{g}^*$ the dual space of the Lie algebra $\mathfrak{g}$. The coadjoint representation of a Lie group is the dual of the adjoint representation. The corresponding action of $\mathbf{G}$ on $\mathfrak{g}^*$ is called the coadjoint action. The orbits of that action are called coadjoint orbits, which are especially important in the orbit method of representation theory or, more generally, geometric quantization. An important class of symplectic stuctures consists of the coadjoints orbits by the coadjoint action (Kirillov, 1976). In the Kirillov method of orbits, representations of $\mathbf{G}$ are constructed geometrically starting from the coadjoint orbits. Specifically,

Definition 2.2. *The coadjoint representation Ad^* of $\boldsymbol{G}$ is the dual representation of the adjoint representation Ad, and is given by*

$$Ad^*(g) := Ad(g^{-1})^T$$

where $g \in \boldsymbol{G}$ and $(g^{-1})^T$ denotes the transpose of g^{-1}. The representation space of Ad^* is $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$.

In terms of Ad, we consider the following description of Ad^*,

$$\langle Ad^*(g)f, X\rangle = \langle f, Ad(g^{-1})X\rangle, \quad with \quad \langle f, X\rangle := f(X), \quad f \in \mathfrak{g}^*, g \in \mathbf{G}, X \in \mathfrak{g}.$$

In this paper, the description of the coadjoint representation simplifies to

$$f \longmapsto Ad(g^{-1})X_f$$

where X_f is defined by

$$f(Y) = \langle X_f, Y\rangle, \quad for\ all\ Y \in \mathfrak{g}.$$

According to (Lu & al., 1990), the Killing form $\langle , \rangle$ is non-degenerate and we can identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ via the map $f \longmapsto X_f$. Now, we will use the notation

$$g \cdot f := Ad^*(g)f \quad g \in \mathbf{G}, f \in \mathfrak{g}^*.$$

Definition 2.3. *Let $f \in \mathfrak{g}^*$. The coadjoint orbit O_f of f is defined by*

$$O_f = \{Ad(g)X_f : g \in \boldsymbol{G}\}.$$

Lemma 2.1. *(Kirillov,1976) Let $\mathfrak{g}_f$ be the Lie algebra of the stabilizing group $\boldsymbol{G}_f = \{g \in \boldsymbol{G} : g \cdot f = f\}$. The tangent space of the coadjoint orbit at f is*

$$T_f(O_f) \cong \mathfrak{g}/\mathfrak{g}_f.$$

Theorem 2.1. *Let $f \in \mathfrak{g}^*$ and let O_f be the coadjoint orbit at f. Consider the application*

$$\omega_f : \mathfrak{g} \times \mathfrak{g} \longrightarrow T_f(O_f), \quad (X, Y) \longmapsto \omega_f(X, Y) := f([X, Y]). \tag{6}$$

Then ω_f is a skew-symmetric bilinear form.

Proof. According to the Lemma 2.1, we can consider elements of $T_f(O_f)$ as elements on the form $X + Z$ for some $Z \in \mathfrak{g}_f$. Giving that $f([X + Z, Y]) = f([X, Y]) + f([Z, Y])$. We have : $[Z, Y] = 0$ for all $Z \in \mathfrak{g}_f$ and Y in $\mathfrak{g}$. We prove that ω_f is well-defined and it is obviously skew-symmetric and bilinear given that the Lie bracket is also.
The form ω_f defined by (6) is called ***Kostant-Kirillov structure***. Now, let's show that the coadjoint orbits are symplectic structures in $\mathbf{G}$. Recall that, giving a symplectic structure (or symplectic form) on $\mathbf{G}$ is to define a closed non-degenerate differential 2-form.

Theorem 2.2. *Let $f \in \mathfrak{g}^*$. The Konstant-Kirillov form ω_f is a symplectic structure on $T_f(O_f)$.*

Proof. According to the Theorem 2.1, ω_f is a well-defined 2-form. The Lie algebra $\mathfrak{g}$ has trivial center and thus ω_f is non-degenerate. It remains to be shown that ω_f is closed. Let $d(Ad^*)$ be the differential of Ad^* defined by $d(Ad^*) := ad^*$. This means that, the representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ corresponding to Ad^*. Let $\xi \in T_f(O_f)$ be the vector field. ξ is represented by $ad^*(X)f$ for X in $\mathfrak{g}$. Considering the value of f at X by $\langle f, X\rangle$, we have

$$Xf(Y) = \langle ad^*(X)f, Y\rangle.$$

Since,

$$\langle f, -ad_X Y\rangle = \langle ad^*(X)f, Y\rangle$$

and

$$\langle f, -ad_X Y\rangle = f([X, Y]),$$

we have,

$$Xf(Y) = f([X, Y]).$$

Before calculating the exterior derivative of ω_f, we need to recall the expression for the exterior derivative of an n-form ω which can be written explicitly as (Gengoux & al.,2013) :

$$d\omega(X_0,\dots,X_n)=\sum_{i=0}^{n}(-1)^i X_i\omega\Big(X_0,\dots,X_{i-1},X_{i+1},\dots,X_n\Big)+\sum_{0\le i<j\le n}(-1)^{i+j}\omega\Big([X_i,X_j],X_0,\dots,X_{i-1},X_{i+1},\dots,X_{j-1},X_{j+1},\dots,X_n\Big).$$

for all $X_0,\dots,X_n\in\mathfrak{g}$, $n\in\mathbb{N}$. In particular, for X,Y,Z in $\in\mathfrak{g}$, we have the formula for the exterior derivative of a 2-form :

$$d\omega(X,Y,Z)=X\omega(Y,Z)-Y\omega(X,Z)+Z\omega(X,Y)-\omega([X,Y],Z)+\omega([X,Z],Y)-\omega([Y,Z],X). \tag{7}$$

Applying the formula (7), we have :

$$d\omega_f(X,Y,Z)=X\omega_f(Y,Z)-Y\omega_f(X,Z)+Z\omega_f(X,Y)-\omega_f([X,Y],Z)+\omega_f([X,Z],Y)-\omega_f([Y,Z],X). \tag{8}$$

From (8), we obtain

$$d\omega_f(X,Y,Z)=Xf([Y,Z])-Yf([X,Z])+Zf([X,Y])-f([[X,Y],Z])+f([[X,Z],Y])-f([[Y,Z],X]). \tag{9}$$

the equality in (9) is equivalent to :

$$d\omega_f(X,Y,Z)=f([[Y,Z],X])-f([[X,Z],Y])+f([[X,Y],Z])-f([[X,Y],Z])+f([[X,Z],Y])-f([[Y,Z],X]). \tag{10}$$

Consequently, $d\omega_f(X,Y,Z)=0$. Thus, ω_f is closed. This concludes that ω_f is a symplectic structure on $T_f(O_f)$.

Note that, the symplectic Kostant-Kirillov structure ω_f is written (Berndt, 2007) as

$$\omega_f(\widetilde{f})(\xi_X(\widetilde{f}),\xi_X(\widetilde{f}))=\widetilde{f}([X,Y]),\quad for\ X,Y\in\mathfrak{g},\ \widetilde{f}\in O_f\quad where\quad (\xi_X g)(\widetilde{f}):=\frac{d}{dt}(g(Ad^*(exp(tX))\widetilde{f})). \tag{11}$$

Another natural way to define a symplectic structure is to consider the cotangent bundle $T^*(O_f)$. Let's consider the following map

$$\varpi: T_f^*(O_f)\longrightarrow T_f(O_f),\omega_f^{\varepsilon}\longmapsto\varepsilon\quad where\quad \omega_f^{\varepsilon}(\eta)=\omega_f(\eta,\varepsilon),\ for\ all\ \eta\ in\ T_f(O_f). \tag{12}$$

Lemma 2.2. *The map ϖ defined by (12) is an isomorphism generated by the symplectic structure ω_f.*

Proof. Let $\varpi^{-1}: T_f(O_f)\longrightarrow T_f^*(O_f)$ be the inverse map of ϖ. For all $\eta\in T_f(O_f)$, we have $\varpi^{-1}(\varepsilon)(\eta)=\omega_f(\eta,\varepsilon)$. Since, ω_f is bilinear, we have

$$\varpi^{-1}(\varepsilon+\varepsilon')(\eta)=\varpi^{-1}(\varepsilon+\varepsilon')(\eta)+\varpi^{-1}(\varepsilon+\varepsilon')(\eta),\quad for\ all\ \eta\in T_f(O_f).$$

Since, ω_f is a symplectic form, it follows that ω_f is non-degenerate. The non-degeneracy condition means that $\omega_f(\eta,\varepsilon)=0,\forall\eta\in T_f(O_f)$ implies that $\eta=0$. It follows that $Ker(\varpi^{-1})=\{0\}$. Hence ϖ^{-1} is injective. Furthermore ϖ^{-1} is an isomorphism because $dimT_f(O_f)=dimT_f^*(O_f)$. It suffices to conclude that, ϖ is an isomorphism.

According to (Lesfari, 2009), any symplectic structure induces a Hamiltonian vector field associated to a differentiable function H (called Hamiltonian) expressed in terms of a differential system. Let's consider the local coordinate system (x,y,z), the differential system can be expressed as follows :

$$\dot{X}=\frac{\partial H}{\partial x}\varpi(dx)+\frac{\partial H}{\partial y}\varpi(dx)+\frac{\partial H}{\partial z}\varpi(dx). \tag{13}$$

We have :

$$\varpi^{-1}=\begin{pmatrix}\omega_f(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) & \omega_f(\frac{\partial}{\partial x},\frac{\partial}{\partial y}) & \omega_f(\frac{\partial}{\partial x},\frac{\partial}{\partial z})\\ \omega_f(\frac{\partial}{\partial y},\frac{\partial}{\partial x}) & \omega_f(\frac{\partial}{\partial y},\frac{\partial}{\partial y}) & \omega_f(\frac{\partial}{\partial y},\frac{\partial}{\partial z})\\ \omega_f(\frac{\partial}{\partial z},\frac{\partial}{\partial x}) & \omega_f(\frac{\partial}{\partial z},\frac{\partial}{\partial y}) & \omega_f(\frac{\partial}{\partial z},\frac{\partial}{\partial z})\end{pmatrix}.$$

Let a_{ij} $(i,j=1,2,3)$ be the components of the matrix ϖ such that we have $\varpi(dx)=a_{11}\frac{\partial}{\partial x}+a_{21}\frac{\partial}{\partial y}+a_{31}\frac{\partial}{\partial z}$, $\varpi(dy)=a_{12}\frac{\partial}{\partial x}+a_{22}\frac{\partial}{\partial y}+a_{32}\frac{\partial}{\partial z}$ and $\varpi(dz)=a_{13}\frac{\partial}{\partial x}+a_{23}\frac{\partial}{\partial y}+a_{33}\frac{\partial}{\partial z}$.
Since ϖ is skew-symmetric, we have :

$$\dot{X}=(-a_{12}\frac{\partial}{\partial y}+a_{13}\frac{\partial}{\partial z})\frac{\partial H}{\partial x}+(a_{12}\frac{\partial}{\partial x}+a_{23}\frac{\partial}{\partial y})\frac{\partial H}{\partial y}+(-a_{13}\frac{\partial}{\partial x}-a_{23}\frac{\partial}{\partial y})\frac{\partial H}{\partial z},$$

It follows that

$$\dot{X} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})J_X \frac{\partial H}{\partial X} \quad where \quad J_X = \begin{pmatrix} 0 & -a_{12} & a_{13} \\ a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}, \tag{14}$$

which can be written in more compact form $\dot{X} = J_X \frac{\partial H}{\partial X}$. This is a complete characterization of hamiltonian vector field. The associated matrix J_X belongs to $\mathfrak{g}$ and determine a symplectic structure. Note that $J_X \in \mathfrak{g}$ is not necessarily a Poisson bivector. The vector field corresponding to J_X expressed in a coordinate system (x, y, z) is defined by

$$\pi_{J_X} = \sum_{i<j} \pi_{ij} \partial_i \wedge \partial_j$$

where

$$\pi_{ij} := \{x_i, x_j\} = \pi_{J_X}(dx_i, dx_j), \quad \partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y} \; and \; \partial_3 = \frac{\partial}{\partial z} \quad for \; all \; i, j = 1, 2, 3.$$

The bracket of two functions F and G being given by (Weinstein,1983)

$$\{F, G\} = \pi_{J_X}(dF, dG)$$

Either, locally $\{F, G\} = \sum_{i<j} \pi_{ij}(\partial_i F \partial_j G - \partial_i G \partial_j F)$. In this paper, $\pi_{12} = -2a_{12}, \pi_{13} = 2a_{13}$ and $\pi_{23} = 2a_{23}$ i.e.

$$\pi_{J_X} = -2a_{12}\partial_1 \wedge \partial_2 + 2a_{13}\partial_1 \wedge \partial_3 + 2a_{23}\partial_2 \wedge \partial_3. \tag{15}$$

In the following, the notation $J(\pi_{J_X})$ will refer to the Jacobiator associated to π_{J_X} and will correspond to the value of

$$\{\{F, G\}, H\} + \{\{F, G\}, H\} + \{\{F, G\}, H\}, \quad for \; all \; F, G, H \; in \; \mathbb{C}[x, y, z].$$

Definition 2.4. *The Jacobiator of* π_{J_X} *is defined by*

$$J(\pi_{J_X}) = 4a_{12}\partial_1(a_{13}) - 4a_{13}\partial_1(a_{12}) + 4a_{12}\partial_2(a_{23}) - 4a_{23}\partial_2(a_{12}) + 4a_{23}\partial_3(a_{13}) - 4a_{13}\partial_3(a_{23}). \tag{16}$$

Let's consider the partial differential equation

$$w\partial_1 v - v\partial_1 w + w\partial_2 u - u\partial_2 w + u\partial_3 v - v\partial_3 u = 0 \tag{17}$$

where u, v, w are the unknown functions.
There exists at least one solution. Indeed, (x, y, z) satisfies the equation (17).
π_{J_X} is called *Poisson bivector field* if and only if (a_{23}, a_{13}, a_{12}) is a solution of the partial differential equation (17). Otherwise, it will be called *Poisson quasi-bivector field*. As a result, π_{J_X} is a Poisson quasi-bivector field for

$$a_{23} \in \mathbb{C}[y, z], a_{13} \in \mathbb{C}[x, z] \; and \; a_{12} \in \mathbb{C}[x, y].$$

Let $n \geq 1$ be an integer and let $\pi_0 = -y^n\partial_1 \wedge \partial_2 + \frac{1}{n}x\partial_1 \wedge \partial_3 - \frac{1}{n}y\partial_2 \wedge \partial_3$. We have

$$J(\pi_0) = y^n.$$

Unless otherwise stated, y^n is not identically equal to zero. From this, π_0 is a Poisson quasi-bivector field. Let $(\mathcal{A}_0, ., \{., .\}_0)$ be the quasi-Poisson algebra defined by $\mathcal{A}_0 = \mathbb{C}[x, y, z]$ with quasi-Poisson structure $\{., .\}_0$ associated to π_0 defined by

$$\pi_0 = -y^n \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{1}{n}x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - \frac{1}{n}y\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}. \tag{18}$$

It follows

$$J_X = \begin{pmatrix} 0 & -\frac{1}{2}y^n & \frac{1}{2n}x \\ \frac{1}{2}y^n & 0 & -\frac{1}{2n}y \\ -\frac{1}{2n}x & \frac{1}{2n}y & 0 \end{pmatrix} \tag{19}$$

It is the matrix associated to π_0 on $\mathcal{A}_0 = \mathbb{C}[x, y, z]$.

The following section is devoted to the explicit calculation of a Kostant-Kirillov structure induced by π_0 in $\mathbf{J}_0^n$.

3. An Explicit Calculation of a Kostant-Kirillov Symplectic Structure on the Quasi-Poisson Tensor π_0

In this section, we calculate explicitly the coadjoint orbits of the Lie group **G** and their Kostant-Kirillow symplectic structures coadjoint orbits of the Lie group **G**. Let g_θ be the element of **G** defined above. Like any Lie group, there are two known operations on **G** called the left translation

$$L_{g_\theta} : \mathbf{G} \longrightarrow \mathbf{G}, \quad g_{\theta'} \longmapsto g_\theta g_{\theta'},$$

and the right translation

$$R_{g_\theta} : \mathbf{G} \longrightarrow \mathbf{G}, \quad g_{\theta'} \longmapsto g_{\theta'} g_\theta.$$

These applications are diffeomorphisms of **G** and we have

$$R_{g_\theta} \circ L_{g_\theta} = L_{g_\theta} \circ R_{g_\theta}.$$

We can define the automorphism of **G** as

$$R_{g_\theta^{-1}} L_{g_\theta} : \mathbf{G} \longrightarrow \mathbf{G}, \quad g_{\theta'} \longmapsto g_\theta g_{\theta'} g_\theta^{-1}.$$

Let 1 be the unit of **G**. Since $\mathfrak{g} = T_1\mathbf{G}$ is the Lie algebra of **G**, its adjoint representation Ad_{g_θ} is the derivative of $R_{g_\theta^{-1}} L_{g_\theta}$ in 1 defined by :

$$Ad_{g_\theta} : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \xi \longmapsto \frac{d}{dt} R_{g_\theta^{-1}} L_{g_\theta}(exp(t\xi)) \mid_{t=0} .$$

By a simple computation, we show that Ad_{g_θ} is an algebra homomorphism defined by

$$Ad_{g_\theta}[\xi, \eta] = [Ad_{g_\theta}(\xi), Ad_{g_\theta}(\eta)], \quad for\ all\ \xi, \eta \in \mathfrak{g}. \tag{20}$$

Now consider the notation

$$ad \equiv Ad_1^* : \mathfrak{g} \longrightarrow End(\mathfrak{g}), \quad \xi \longmapsto ad_\xi = \frac{d}{dt} Ad_{g_{\theta(t)}} \mid_{t=0},$$

with $\frac{d}{dt} g_\theta(t) \mid_{t=0} = \xi$ and $g_\theta(0) = 1$, where $End(\mathfrak{g})$ is the space of endomorphisms of $\mathfrak{g}$.
We have

$$ad_\xi(\eta) = [\xi, \eta] \quad for\ all\ \xi \in \mathfrak{g}\ and\ \eta \in End(\mathfrak{g}).$$

Indeed,

$$ad \equiv Ad_1^* = Ad_{*e}(\xi)(\eta) = \frac{d}{dt} Ad_{g_\theta(t)}(\eta) \mid_{t=0} = \frac{d}{dt}\Big(g_\theta(t)\eta g_\theta^{-1}(t)\Big) \mid_{t=0} .$$

Since $\frac{d}{dt}\Big(g_\theta(t)\eta g_\theta^{-1}(t)\Big) \mid_{t=0} = \dot{g_\theta}(t)(\eta) g_\theta^{-1}(t) \mid_{t=0} - g_\theta(t)\eta g_\theta^{-1}(t)\dot{g_\theta}(t) g_\theta^{-1}(t) \mid_{t=0} := \dot{g_\theta}(0)\eta - \eta\dot{g_\theta}(0)$, we have $ad_\xi(\eta) = \xi\eta - \xi\eta$ which completes the proof.

Considering the application

$$Ad^* : \mathbf{G} \longrightarrow End(\mathfrak{g}^*), \quad g_\theta \longmapsto Ad^*(g_\theta) \equiv Ad^*_{g_\theta}.$$

It is differentiable and its derivative in 1 can be expressed as

$$ad^* : \mathfrak{g} \longrightarrow End(\mathfrak{g}^*), \quad \xi \longmapsto ad^*_\xi.$$

Proposition 3.1. *Let $\xi, \eta \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$. Consider*

$$\{\cdot, \cdot\} : \mathfrak{g} \times \mathfrak{g}^*, \quad (\xi, f) \longmapsto \{\xi, f\} = ad^*_\xi(f), \tag{21}$$

Then

$$\langle\{\xi, f\}, \eta\rangle = \langle\{f, [\xi, \eta]. \tag{22}$$

Proof. Since $\{\xi, f\} = ad^*_\xi(f)$, $\langle\{\xi, f\}, \eta\rangle = \langle \frac{d}{dt}(Ad^*)_{exp(t\xi)}(f) \mid_{t=0}, \eta\rangle$ where $exp(t\xi) \mid_{t=0} = 1$ and $\frac{d}{dt} exp(t\xi) \mid_{t=0} = \xi$. Hence,

$$\langle\{\xi, f\}, \eta\rangle = \frac{d}{dt}\langle Ad^*_{exp(t\xi)}(f), \eta\rangle \mid_{t=0} .$$

Since $\langle Ad^*_{exp(t\xi)}(f), \eta\rangle = \langle f, Ad_{exp(t\xi)}(\eta)\rangle$ and $ad_\xi(\eta) = \frac{d}{dt}\langle Ad_{exp(t\xi)}(\eta)$, we have

$$\langle\{\xi, f\}, \eta\rangle = \langle f, ad_\xi(\eta)\rangle. \tag{23}$$

From (23), we have (22) which completes the proof.

Definition 3.1. *Let $f \in \mathfrak{g}^*$ and let $O_f^* = \{Ad^*_{g_\theta}(f) : g_\theta \in \mathbf{G}\} \subset \mathfrak{g}^*$, the dual of coadjoint orbit on the Lie group* $\mathbf{G}$. *A vector tangent τ to the orbit O_f^* of f is expressed as an element $A \in \mathfrak{g}$ by*

$$\tau = \{A, f\}, \quad A \in \mathfrak{g}. \tag{24}$$

We have to use the following lemma :

Lemma 3.1. *Let $\tau_1 = \{A_1, f\}$ and $\tau_2 = \{A_2, f\}$ be two vectors, tangent to the orbit O_f of f where $A_1, A_2 \in \mathfrak{g}$. Then,*

$$[\tau_1, \tau_2] \cong [A_1, A_2].$$

Since $\mathfrak{g}^* = \mathfrak{g}$, we can obviously remark that the symplectic Kostant-Kirillov structure ω_f defined on (6) is written as

$$\omega_f(\tau_1, \tau_2) = \langle f, [A_1, A_2] \rangle, \quad A_1, A_2 \in \mathfrak{g}, \quad f \in \mathfrak{g}^* := \mathfrak{g}. \tag{25}$$

Recall that $\mathbf{G}$ is a subgroup of SO(3) and the corresponding Lie algebra is $\mathfrak{g} = T_1\mathbf{G}$ consists of skew-symmetric 3×3 matrices where 1 is the neutral element of $\mathbf{G}$ and $T_1\mathbf{G}$ is the tangent space at 1. Consider the basis (e_x, e_y, e_y) defined on (1) with $e_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $e_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $e_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. From the Euler-Arnold system the differential systems defined on (4), we have

$$\dot{X} = (J_X)X \tag{26}$$

where

$$J_X = \frac{1}{2n} y e_x + \frac{1}{2n} x e_y + \frac{1}{2} y^n e_z \tag{27}$$

is the matrix associated to the Poisson quasi-bivector field defined on (18).

Proposition 3.2. *Let $n \geq 1$ be an integer and let θ be a given constant parameter. The adjoint orbit of the group $\mathbf{G}$ is*

$$O(\boldsymbol{J}_X) = \left\{ \frac{1}{2n} y e_x + (-\frac{1}{2}(\sin\theta) y^n + \frac{1}{2n}(\cos\theta) x) e_y + (\frac{1}{2}(\cos\theta) y^n + \frac{1}{2n}(\sin\theta) x) e_z \right\}$$

Proof. Consider the invertible matrix $g_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in \mathbf{G}$. Its inverse is $g_\theta^{-1} = com(g_\theta)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$. We have

$$g_\theta J_X g_\theta^{-1} = \begin{pmatrix} 0 & -\frac{(\cos\theta)}{2} y^n - \frac{(\sin\theta)}{2n} x & -\frac{(\sin\theta)}{2} y^n + \frac{(\cos\theta)}{2n} x \\ \frac{(\cos\theta)}{2} y^n + \frac{(\sin\theta)}{2n} x & 0 & -\frac{y}{2n} \\ \frac{(\sin\theta)}{2} y^n - \frac{(\cos\theta)}{2n} x & \frac{1}{2n} y & 0 \end{pmatrix}, \tag{28}$$

which completes the proof of $O(J_X)$.

Corollary 3.1. *For $n \neq 2$ and $\theta \neq \pi + 2k\pi$ (k be an integer), the adjoint orbit of the group $\mathbf{G}$ induces a Poisson quasi-bivector field*

$$\pi_{O(J_X)} = \left(-(\cos\theta) y^n - \frac{1}{n}(\sin\theta) x\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \left(-(\sin\theta) y^n + \frac{1}{n}(\cos\theta) x\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - \frac{1}{n} y \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

and the jacobiator of $\pi_{O(J_X)}$ is defined by :

$$J(\pi_{O(J_X)}) = \frac{1}{n}(n\cos\theta - \cos\theta + 1) y^n - \frac{1}{n^2}(\sin\theta) x. \tag{29}$$

From the above, we have the following classification :

Proposition 3.3.

	For $n = 1$, $\theta = 2k\pi$, $k \in \mathbb{Z}$	*For $n = 2$, $\theta = \pi + 2k\pi$, $k \in \mathbb{Z}$*	*For $n \geq 3$, $\theta = 2k\pi$, $k \in \mathbb{Z}$*
$O(J_X)$	$\left\{\frac{1}{2} y e_x + \frac{1}{2} x e_y + \frac{1}{2} y e_z\right\}$	$\left\{\frac{1}{4} y e_x - \frac{1}{4} x e_y - \frac{1}{2} y^2 e_z\right\}$	$\left\{J_X\right\}$
$J(O(J_X))$	y	$\boldsymbol{0}$	$J(\pi_0)$
$\pi_{J(O(J_X))}$	$-y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - y \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$	$y^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - \frac{y}{2} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$	π_0

The following proposition gives the Kostant-Kirillov orbit induced by π_0.

Proposition 3.4. *Let $n \geq 1$ be an integer. The coadjoint orbit $O^*(J_X)$ is isomorphic to*

$$\mathcal{V} = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + n^2 z_2^{2n} = 4n^2 \sum_{k\in I} j_k^2\Big\} \tag{30}$$

where $\sum_{k\in I} j_k e_k \in \mathfrak{g}$ and $I = \{x, y, z\}$ be an index set.

Proof. Note that $O^*(J_X)$ can be written as

$$O^*(J_X) = \Big\{A \in \mathfrak{g} : A = g_\theta^{-1}(J_X) g_\theta\Big\}$$

We have $det(A) = det(J_X)$ where $A \in \mathfrak{g}$. Hence, A and J_X have the same spectrum. Indeed, for every scalar λ :

$$det(A - \lambda I) = det\Big(g_\theta^{-1}(J_X - \lambda I)g_\theta\Big).$$

Since $det\Big(g_\theta^{-1}(J_X - \lambda I)g_\theta\Big) = det(g_\theta^{-1}.g_\theta).det(J_X - \lambda I)$, we have $det(A - \lambda I) = det(J_X - \lambda I)$. Therefore,

$$O^*(J_X) = \Big\{A \in \mathfrak{g} : A = g_\theta^{-1} J_X g_\theta,\ spectrum\ of\ A = spectrum\ of\ J_X\Big\}. \tag{31}$$

Let us determine the spectrum of J_X defined on (27). Since

$$J_X = \frac{1}{2n} y e_x + \frac{1}{2n} x e_y + \frac{1}{2} y^n e_z,$$

$$det(J_X - \lambda I)\} = -\lambda^3 - (\frac{1}{4n^2}y^2 + \frac{1}{4n^2}x^2 + \frac{1}{4}y^{2n})\lambda$$

Since $det(J_X - \lambda I)\} = 0$ is equivalent to $\lambda = 0$ and $\lambda^2 = -(\frac{1}{4n^2}y^2 + \frac{1}{4n^2}x^2 + \frac{1}{4}y^{2n})$, the spectrum of J_X is

$$spectrum(J_X) = \Big\{0, i\sqrt{\frac{1}{4n^2}y^2 + \frac{1}{4n^2}x^2 + \frac{1}{4}y^{2n}}, -i\sqrt{\frac{1}{4n^2}y^2 + \frac{1}{4n^2}x^2 + \frac{1}{4}y^{2n}}\Big\}.$$

Let's consider

$$A = \sum_{k\in I} j_k e_k \in \mathfrak{g}$$

where $I = \{x, y, z\}$ be an index set. Then

$$det(A - \lambda I)) = -\lambda^3 - \sum_{k\in I} j_k^2 \lambda.$$

The spectrum of A is

$$spectrum(A) = \Big\{0, i\sqrt{\sum_{k\in I} j_k^2}, -i\sqrt{\sum_{k\in I} j_k^2}\Big\}.$$

From (31), it follows

$$O^*(J_X) = \Big\{\sum_{k\in I} j_k e_k : \frac{1}{4n^2}y^2 + \frac{1}{4n^2}x^2 + \frac{1}{4}y^{2n} = \sum_{k\in I} j_k^2\Big\} \tag{32}$$

where $\sum_{k\in I} j_k e_k \in \mathfrak{g}$ and $I = \{x, y, z\}$ be an index set.

Let's consider

$$\mathcal{V} = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \frac{1}{4n^2} z_1^2 + \frac{1}{4n^2} z_2^2 + \frac{1}{4} z_2^{2n} = \sum_{k\in I} j_k^2\Big\}. \tag{33}$$

Therefore, the orbit $O^*(J_X)$ is isomorphic to $\mathcal{V}$ defined on (33).

Remark 3.1. *For $n \neq 2$ and $\theta \neq \pi + 2k\pi$ (k be an integer) and let $O^*(J_X)$ the coadjoint orbit (Kostant-kirillov orbit). We have*

$$g_\theta^{-1} J_X g_\theta = \frac{1}{2n} y e_x + \Big(\frac{1}{2}(\sin\theta) y^n + \frac{1}{2n}(\cos\theta) x\Big) e_y + \Big(\frac{1}{2}(\cos\theta) y^n - \frac{1}{2n}(\sin\theta) x\Big) e_z,$$

the Poisson quasi-bivector field associated is

$$\pi_{O^*(J_X)} = \Big(-(\cos\theta) y^n + \frac{1}{n}(\sin\theta) x\Big)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \Big((\sin\theta) y^n + \frac{1}{n}(\cos\theta) x\Big)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - \frac{1}{n} y \frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$$

and the jacobiator of $\pi_{O^*(J_X)}$ is defined by :

$$J(\pi_{O^*(J_X)} = \frac{1}{n}(n\cos\theta - \cos\theta + 1) y^n + \frac{1}{n^2}(\sin\theta) x. \tag{34}$$

We obtain a similar result as before about a classification of the Kostant-Kirillov orbit :

	For $n = 1$, $\theta = 2k\pi$, $k \in \mathbb{Z}$	For $n = 2$, $\theta = \pi + 2k\pi$, $k \in \mathbb{Z}$	For $n \geq 3$, $\theta = 2k\pi$, $k \in \mathbb{Z}$
$O^*(J_X)$	$\Big\{\frac{1}{2} y e_x + \frac{1}{2} x e_y + \frac{1}{2} y e_z\Big\}$	$\Big\{\frac{1}{4} y e_x - \frac{1}{4} x e_y - \frac{1}{2} y^2 e_z\Big\}$	$\Big\{J_X\Big\}$
$J(O^*(J_X))$	y	$\mathbf{0}$	$J(\pi_0)$
$\pi_{J(O^*(J_X))}$	$-y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + x\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - y\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$	$y^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - \frac{y}{2}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$	π_0

Note that, we have a Poisson structure through the action of $g_{\pi+2k\pi}$ with $n = 2$ and k an integer.

To determine the symplectic structure on $O^*(J_X)$, we will use the following result.

Theorem 3.1. *Let $f \in \mathfrak{g}^*$. The Kostant-Kirillov symplectic structure is given by*

$$\omega_f(\tau_1, \tau_2) = \langle f, j \wedge k\rangle, \tag{35}$$

with $j, k \in \mathbb{C}^3$ where $\tau_1 = f \wedge j$, $\tau_2 = f \wedge k$ and $\wedge$ is the usual vector product.

Proof. Let $f \in \mathfrak{g}^*$. Recall that the Kostant-Kirillov symplectic structure ω_f defined in (6) can be written as (25) :

$$\omega_f(\tau_1, \tau_2) = \langle f, [A_1, A_2]\rangle, \quad A_1, A_2 \in \mathfrak{g}.$$

with $\tau_1 = \{A_1, f\}$ and $\tau_2 = \{A_2, f\}$ where $\langle\cdot,\cdot\rangle$ is the Killing form, $[\cdot,\cdot]$ the commutator defined by (2) and $\{\cdot,\cdot\}$ defined by (21). Let $X = \sum_{i\in I} j_i e_i$ and $Y = \sum_{j\in I} k_j e_j$ be two elements of $\mathfrak{g}$ where $I = \{x, y, z\}$ be an index set. We have :

$$[X, Y] = \sum_{i,a,b\in I,(a\neq b)} \wedge_{a,b}\, e_i \tag{36}$$

where

$$\wedge_{a,b} = j_a k_b - j_b k_a \quad (a \neq b). \tag{37}$$

Let $j = (j_x, j_y, j_z)$ and $k = (k_x, k_y, k_z)$ be two elements of $\mathbb{C}^3$. The Killing form can be written as :

$$\langle X, Y\rangle = j \wedge k, \tag{38}$$

where $\wedge$ is the usual vector product.
Let's consider the isomorphism

$$j \wedge k \longmapsto [X, Y] \tag{39}$$

where $j, k \in \mathbb{C}^3$ and $X, Y \in \mathfrak{g}$. From (25), we also have :

$$\omega_f(\tau_1, \tau_2) = \langle f, j \wedge k\rangle, \quad \tau_1 = f \wedge j \quad and \quad \tau_2 = f \wedge k. \tag{40}$$

Moreover, for $\sum_{k\in I=\{x,y,z\}} j_k e_k \in \mathfrak{g}$ and according to Proposition 3.4, we have the isomorphism between the coadjoint orbit $O^*(J_X)$ and

$$\mathcal{V} = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + n^2 z_2^{2n} = 4n^2 \sum_{k\in I} j_k^2\Big\}.$$

It follows that, the tangent vectors to $O^*(J_X)$ at a given point are also the tangent vectors to $\mathcal{V}$ at this point.
In the following, (f_1, f_2, f_3) be the local coordinate system and $f = (f_1, f_2, f_3)$ be the given point. The main result is :

Theorem 3.2. *Let (f_1, f_2, f_3) be the local coordinate system, $j = (j_x, j_y, j_z)$, $k = (k_x, k_y, k_z)$ and $f = (f_1, f_2, f_3)$. The symplectic structures induced by the Poisson quasi-structure π_0 are defined by :*

$$\omega_f(\tau_1, \tau_2)\frac{(f_1f_2 - f_1^2)j_zk_z + (f_2 - f_1)j_z\tau_{22} - f_2j_z\tau_{22} - f_1j_z\tau_{21} + \tau_{11}\tau_{22} - \tau_{12}\tau_{21}}{f_3}$$

with $j, k \in \mathbb{C}^3$ where $\tau_1 = f \wedge j$, $\tau_2 = f \wedge k$ and $\wedge$ is the usual vector product.

Proof. Let (f_1, f_2, f_3) be the local coordinate system and $f = (f_1, f_2, f_3)$. We have :

$$T_f(\mathcal{V}) = \left\{\left(z_1, z_2, -\frac{1}{f_3}\left(f_1z_1 + f_2(z_2 + 2n^2z_2^{2n-1})\right)\right), f_3 \neq 0\right\}. \tag{41}$$

Let $\tau_1 = (\tau_{11}, \tau_{12}, \tau_{13}), \tau_2 = (\tau_{21}, \tau_{22}, \tau_{23}) \in T_f(\mathcal{V})$. Consider $j = (j_x, j_y, j_z)$, $k = (k_x, k_y, k_z)$. The equations $f \wedge j = \tau_1$ and $f \wedge k = \tau_2$ are respectively equivalent to the systems

$$\begin{cases} -f_3j_y + f_2j_z = \tau_{11} \\ f_3j_x - f_1j_z = \tau_{12} \\ -f_2j_x + f_1j_y = -\frac{1}{f_3}\left(f_1\tau_{11} + f_2(\tau_{12} + 2n^2\tau_{12}^{2n-1})\right) \end{cases} \tag{42}$$

and

$$\begin{cases} -f_3k_y + f_2k_z = \tau_{21} \\ f_3k_x - f_1k_z = \tau_{22} \\ -f_2k_x + f_1k_y = -\frac{1}{f_3}\left(f_1\tau_{21} + f_2(\tau_{22} + 2n^2\tau_{22}^{2n-1})\right) \end{cases} \tag{43}$$

By resolution of these systems, we now have :

$$j = \left(\frac{1}{f_3}(\tau_{12} + f_1j_z), -\frac{1}{f_3}(\tau_{11} - f_2j_z), j_z\right) \quad with \quad f_2\tau_{12}^{2n-1} = 0 \quad (f_3 \neq 0) \tag{44}$$

and

$$k = \left(\frac{1}{f_3}(\tau_{22} + f_1k_z), -\frac{1}{f_3}(\tau_{21} - f_2k_z), k_z\right) \quad with \quad f_2\tau_{22}^{2n-1} = 0 \quad (f_3 \neq 0). \tag{45}$$

Since ω_f is intrinsic, we can choose as local coordinates f_1, f_2. From this, we will deduce the cases f_2, f_3 and f_3, f_1. Consider the basis $(\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2})$ of $T_f(\mathcal{V})$. We have :

$$\frac{\partial}{\partial f_1} = \left(1, 0, -\frac{f_1}{f_3}\right) \quad and \quad \frac{\partial}{\partial f_2} = \left(0, 1, -(1 + 2n^2)\frac{f_2}{f_3}\right).$$

We have :

$$j_z = -\frac{f_1}{f_3}j_x - (1 + 2n^2)\frac{f_2}{f_3}j_y \tag{46}$$

$$k_z = -\frac{f_1}{f_3}k_x - (1 + 2n^2)\frac{f_2}{f_3}k_y. \tag{47}$$

Since

$$j \wedge k = (j_yk_z - j_zk_x, j_zk_x - j_xk_z, j_xk_y - j_yk_x),$$

it follows that

$$j \wedge k = \left(\frac{-k_z\tau_{11} - j_z\tau_{22} + f_2j_zk_z - f_1j_zk_z}{f_3}, \frac{j_z\tau_{22} - k_z\tau_{12}}{f_3}, \frac{\tau_{11}\tau_{22} - \tau_{12}\tau_{21} + f_1k_z\tau_{11} + f_2k_z\tau_{12} - f_2j_z\tau_{22} - f_1j_z\tau_{21}}{f_3^2}\right).$$

Therefore,

$$\langle f, j \wedge k\rangle = \frac{(f_1f_2 - f_1^2)j_zk_z + (f_2 - f_1)j_z\tau_{22} - f_2j_z\tau_{22} - f_1j_z\tau_{21} + \tau_{11}\tau_{22} - \tau_{12}\tau_{21}}{f_3}. \tag{48}$$

According to theorem 3.1, the Kostant-Kirillov symplectic structure is given by

$$\omega_f(\tau_1, \tau_2) = \langle f, j \wedge k\rangle,$$

with $j, k \in \mathbb{C}^3$ where $\tau_1 = f \wedge j$, $\tau_2 = f \wedge k$ and $\wedge$ is the usual vector product. This concludes the proof of the main result.

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