A Hybrid Non-Local Model of Ontogenetic Growth Revealed a Phase Transition and Two Bifurcations

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Abstract

The problem this study deals with is the dynamics of growth of animals. In the study, some features of the growth of pigs were modelled. The research concerns the growth dynamics during a period of growth close to a bifurcation point. In the point, two bifurcations of the growth trajectory take place. The period of growth entails the weight in which an animal's growth stops when individual maximum weight is reached. In the study, methods of applied mathematics were used. The growth of animals was modelled by a hybrid and continuum methods as a dynamic system. In the hybrid model, time was considered as a discrete variable. In the study the factors, which control trajectories and the dynamics of growth were revealed. There are three results in this study. The first result suggests that in animals, the current weight M can be described by derivative of the average consumed feed. The second result gives the equation of the weight balance in an integral form. Third result implies that in ontogeny, growth of pigs has to be modelled as a dynamic system. The system has two bifurcations; one of the trajectory of the weight gain, and other of the trajectory of the growth invariant K. As a result, new growth trajectories emerge. In some instances, the findings can be translated to man in others they apply to animals.

Keywords: animal growth, ontogenetic trajectory, hybrid model, dynamic system, trajectory bifurcation, phase transition

1. Introduction

The problem this study deals with is the dynamics of growth of animals. The aim of the study was to build a model of growth of pigs and analyse some of its features. Our knowledge about determination of weight and size in animal species is insufficient (Conlon and Raff, 1999). We know little about how the species-specific size and form are created in ontogeny. The size and weight in animals have been determined by an interaction of genetic and environmental factors (Texada et al, 2020). The size of an individual animal affects its fitness in life history. In animals, ability to reproduce, survive and many other biological qualities are a function of growth and size (Boulan et al, 2015). The dynamics of growth of animals, the main factors that influence the growth remain hidden. In animals' development every so often the 'trait' that one is interested in is the growth process itself (Rice, 2008). In medicine and biology, we know how infants grow. It is we can say how they should normally grow on time and weight scales (Lui and Baron, 2011). However, it is not the answer; we need to know the dynamics of the growth in ontogeny. The descriptive knowledge of the process is insufficient. What is needed is an analytical model of growth. To comprehend a phenotype, the most direct way would be to use a mechanistic model for development of the trait (Rice, 2008). In this study methods of applied mathematics were used.

In animals' development, nutrition is a relevant factor; it contributes to the rate of growth and to the regulation of morphogenesis (Shingleton et al, 2008). Weight gain and feed conversion to animal weight were considered the main aspect in this study. Although there has been extensive research into the growth of animals, we know little about the quantitative details of the ontogenesis. In this study, we have built an analytical model of growth of pigs. The model is species-specific; it explains a few aspects of the dynamics of the growth of domestic pigs.

There are three main results in this study. The first result concerns the formalization of the current weight M. The second result presents the weight balance equation in an integral form. Third one shows that in the course of growth the two trajectories, of pig's weight M, and of the invariant of growth K, bifurcate simultaneously and in the same point. As a result, new growth trajectories emerge. In the course of the growth, the bifurcations were preceded by a first-order phase transition. It was shown that in ontogeny, this sequence of events models the systemic regulation of growth in animals.

1.1 Models of Growth of Animals

Attempts to build mathematical models of growth of animals have been continue for more than one hundred years. These attempts have been dominated by the use of so-called growth functions. The functions were obtained by solving equations in which a certain speculative proportionality between the growth rate, and animals' weight was specified without

reference to the biological species studied. The resulting models of growth were termed the general or universal growth functions (Boukal et al, 2014). The authors claimed that the growth functions were universal, this is to say applicable to all biological species. Despite the criticism of this claim (Agutter and Wheatley, 2004) the growth functions continue to be in use. In research, such functions have been used for interpolation purposes, in other words for the curve fitting to data; they have limited application and unclear methodology for identifying and reading parameters (Chirwa et al, 2014). Critically minded authors expressed the opinion that this approach deals with a retrospective curve fitting with little, if any biological insight. The possibility that no one of the growth functions is applicable to the modelling of growth of animals has to be more carefully considered (Agutter and Wheatley, 2004). In the life sciences, meaningful, analytical models of growth of animals or humans remain a problem. The initial problem is to formulate a concept of growth and implement it in a model. Conditionally, such a concept can be seen consisting of two parts. The both parts only conditionally distinct; in work they are mixed and merged. One part of the concept is mathematical. The task is to find a suitable mathematical method with the aim to model a biological concept. In a tight connection with this task stays the biological part of the concept. There are two problems to solve. The first problem is to formulate a physiologically correct relation between the traits. Second problem to solve is how to find a basic functional relation between the studied traits. In the life sciences, it is the norm of research to find a relation between the studied traits by analysing experimental data. In this study, an analytical, species-specific model of growth of pigs was built. The model was built based on the data obtained in the experiments.

1.2 A Hybrid Model

In the life sciences, there are two frequently used types of dynamic systems continuum, and discrete. In simple words, a discrete dynamic system is a difference equation, and a continuum dynamic system is a differential equation; both types are deterministic (Stewart, 2011). The hybrid systems entail both continuum and discrete variables and exhibit a mixture of continuum and discrete time dynamic; they are often used to capture the behaviour of systems that contain some sort of discontinuous events (Di Bernardo and Hogan, 2010).

2. Methods

In the study, methods of applied mathematics were used. A model of animal growth was built by extension of the model earlier introduced by Stass (2021). The model describes some aspects of the growth of pigs. To meet the purpose of the study both continuum and a non-local hybrid technique were used. The hybrid technique considers time as a discrete variable. The minimum discrete time $\Delta t = 1$ day corresponds to one cycle of the circadian rhythm, which oscillate with periods close to 24 hours (Goldbeter, 2018). During this time, the physiological functions of growth and development go through one cycle.

2.1 The Model's Variables

Let M denote an animal current weight, measured in kilograms.

$M = \{M \in R_+ \mid 30 \leq M \leq 600\}$, an animal individual maximum weight $M_x = 600$ kg.

m denotes an animal initial considered weight, measured in kilograms, $m \leq M$, $m_o = 30$ kg.

Let t denote the chronological discrete current time, measured in days from an animal's birth.

$t = \{t \in N \mid 0 \leq t < \infty\}$, $\Delta t = 1, 2, 3, ..., n$. $n \in N$. t_o denotes time corresponds to m_o, $t_o = 90$ days.

Let K denote the invariant of growth, nondimensional. $K = \{K \in R_+ \mid 1 \leq K < 11\}$, $K_o = 1$.

Let Z denote the current feed conversion coefficient, nondimensional.

$Z = \{Z \in R_+ \mid Z_o \leq Z \leq \infty\}$, $(Z = \infty) \rightarrow (M = M_x) \vee (M = M_{xx})$, where M_{xx} denotes species maximum weight.

Let F denote consumed food or feed, measured in kilograms. $F = \{F \in R_+ \mid 0 < F < \infty\}$.

3. Results

The output of the model was comparable with the experimental data; it was used to analyse nonlinear effects. In the study, new results as well as the analyses of the model provide an advanced insight into the growth process.

3.1 A Relation between Main Variables

In this study, the following concept of the model was applied. In humans and animals, the growth or weight gain has been considered as a transformation of feed consumed F to an organism weight M. The transformation of feed was modelled by the feed conversion coefficient Z. Food or feed consumed F has been transformed by an organism to body weight M with efficiency Z. The growth invariant K is a parameter, which makes the model of the growth a species-specific. In this section we analyse a relation between variables Z, M, and F. We begin by writing down a well-known formula. Let $\bar{Z}$ denote the average feed conversion coefficient.

$$M \cdot \bar{Z} = \int M \, d\bar{Z} + \int \bar{Z} \, dM . \tag{1}$$

Let $\bar{F}$ denote the average consumed food. In (1) let us give the following consideration.

$$\bar{F} = \int M d\bar{Z} \quad . \tag{2}$$

Or, what is the same

$$\bar{F} = M \cdot \bar{Z} - \int \bar{Z} \, dM . \tag{3}$$

By differentiation equation (2) in respect to $\bar{Z}$, or (3) in respect to M one can arrive at the same result, given below.

$$M = \frac{d\bar{F}}{d\bar{Z}} \quad . \tag{4}$$

Mathematically, equation (4) is not a unique solution; it is a feasible, the model-related solution. Biologically this is a meaningful result, which is consistent with the concept of the model. Under the model conditions equation (4) specifies a current weight.

In this section, we analyse a species-specific relation between variables M, F, Z, and K. I can remind the interested reader that in animals the growth invariant K, equation (9), is a species-specific parameter. Let us consider the following system.

$$\begin{cases} F \cdot Z = \int F \, dZ + \int Z \, dF \\ M \cdot Z = \int M \, dZ + \int Z \, dM \end{cases} \tag{5}$$

From the system (5) we get

$$\frac{1}{M} \cdot \int M \, dZ = \frac{1}{F} \cdot \int F \, dZ , \qquad M > 0 , \; F > 0. \tag{6}$$

Let us consider a system, given below.

$$\begin{cases} F \cdot M = \int F \, dM + \int M \, dF \\ Z \cdot K = \int K \, dZ + \int Z \, dK \end{cases} \tag{7}$$

From (5), (6) and (7) one can get the following equation.

$$\int (F - M) \, dZ = \frac{F}{M} \cdot \int M \, dZ - \frac{M}{K} \cdot \int K \, dZ . \tag{8}$$

Mathematically, equation (8) is not the unique solution it is a feasible solution. Biologically, this is a meaningful result, which is consistent with the concept of the model. Under the model conditions, equation (8) in biological sense is the weight balance. In equation (8), the term (F - M) denotes not utilised food, not absorbed food; it has to be considered as a variable. (F - M) = Y, where Y denotes not absorbed food. We can conclude that the weight balance is given by an integral equation. Equation (8) is applicable to both humans and animals; the growth invariant K makes (8) species-specific.

3.2 A Hybrid Non-Local Model of Animal Growth

In domestic pigs, the species-specific form of growth invariant K is as follows.

$$K = \frac{Mt}{m_o(2t - t_o)} \quad . \tag{9}$$

A relation between M, Z, K, and t given by system (10).

$$\begin{cases} \dfrac{M}{m_o} = 2K - 1 + \dfrac{(t - Kt_o)}{t} \\ \dfrac{(t - Kt_o)}{t} = \dfrac{(Z - 2K)(K - 1)}{ZK} \end{cases} \tag{10}$$

From (9) and (10) one can obtain the two following dynamic systems. One system is (11) and (12). This system by definition is nonlinear. Another system is equations (13) and (14). The both systems follow from the same set of equations. However, in this study, we concentrate on the system (13) and (14), which is more convenient for analyses.

$$\frac{1}{m_o} \cdot \frac{\Delta M}{\Delta t} = \frac{1}{t} \cdot \frac{m_o[Z(2K+1)-2K]}{m_o[Z(2K+1)-2K]-MZ} \tag{11}$$

$$\frac{\Delta K}{\Delta t} = \frac{1}{t} \cdot \frac{ZK^2}{Z(K+1)-2K} \quad , Z_o \neq 1 , Z > 0. \tag{12}$$

System (13) and (14) is as follows.

$$\frac{1}{m_o} \cdot \frac{\Delta M}{\Delta t} = \frac{1}{t} \cdot \frac{ZK(2K+1)-2K^2}{Z(K+1)-2K} \quad , Z_o \neq 1 , Z > 0 \tag{13}$$

$$\frac{\Delta K}{\Delta t} = \frac{1}{t} \cdot \frac{ZK^2}{Z(K+1)-2K} \quad , Z_o \neq 1 , Z > 0 . \tag{14}$$

From the dynamic system (13) and (14), by eliminating time t we get

$$\frac{1}{m_o} \cdot \frac{\Delta M}{\Delta K} = \frac{2K+1}{K} - \frac{2}{Z} \quad , Z > 0 . \tag{15}$$

It follows that weight gain in pigs, equation (15), is autonomous dynamic; it is not contingent on time explicitly. In this section, we concentrate on the growth dynamic close to a point in which an animal reaches its individual maximum weight. Under the model conditions, this weight is $M_x = 600$ kg. To define the point, we must find $K_x = K|_{M=M_x}$, and $Z_x = Z|_{M=M_x}$. It was proved in the experiment that the limit below holds.

$$\lim_{M \to M_x} \frac{2K-1}{K+1} = \sqrt{3} , where \ (K \to K_x)|_{M \to M_x} . \tag{16}$$

From the limit (16) one can find that numerically $K_x = 5 + 3\sqrt{3}$. From (15) and (16), under condition $(M=M_x) \wedge (K=K_x)$ we have $Z|_{(M=M_x) \wedge (K=K_x)} = Z_x$, $Z_x = 62, 5102$. It follows that the point in which an animal reaches its individual maximum weight is (M_x, K_x, Z_x). At this point, the following equations hold.

$$K_x^2 - K_x \cdot \frac{M_x}{2m_o} - 2 = 0 \ , \tag{17}$$

$$Z_x = \frac{2K_x(K_x-1)}{3} \ . \tag{18}$$

By analysing equations (17) and (18) K_x and M_x were considered as variables. On the trajectory $M \to M_x \to M_{xx}$, figure 1, there is an inflection point, M_{IP}. Numerically, $M_{IP} = 2m_oK_x = 611,769$ kg.

3.2.1 Bifurcation of the Weight Gain Trajectory

Bifurcation of the weight gain trajectory takes place in the point (M_x, t_x, K_x, Z_x), figure 1. Initially stable trajectory $(m_o \to M \to M_x)$ at the point (M_x, t_x, K_x, Z_x) loses its stability as Z_x grows into infinity $(Z_x \to \infty)$. As a result, two new trajectories emerge.
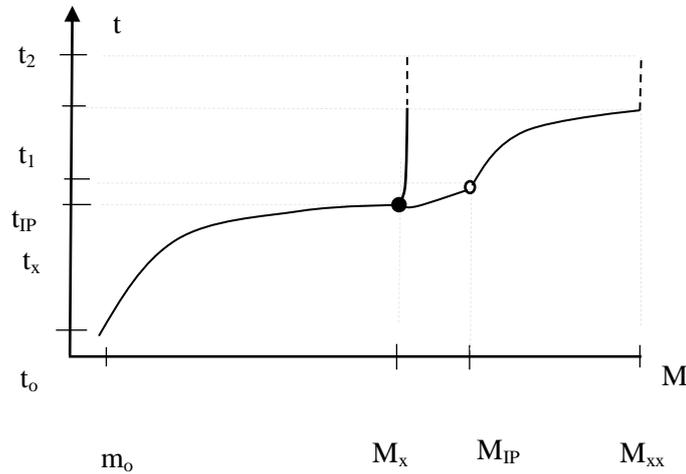
Figure 1. Bifurcation of the growth trajectory

● bifurcation point (M_x, t_x). **o** inflexion point (M_{IP}, t_{IP}). During bifurcation, two new trajectories emerged. One trajectory is $M_x|_{t=t_x} \to M_x|_{t=t_1}$, another trajectory is $M_x \to M_{xx}$. On trajectory $M_x|_{t=t_x} \to M_x|_{t=t_1}$ animals do not grow, the weight M_x remains constant. On trajectory $M_x \to M_{xx}$ animals continue to grow.

One trajectory is $M_x|_{t=t_x} \to M_x|_{t=t_1}$. Other trajectory is $M_x \to M_{xx}$, where time t_x corresponds to M_x. t_1 denotes obtainable life span, and M_{xx} denotes species maximum weight, figure 1. During a phase transition, the following passage occurs $Z_x \to \infty \to Z_{xv}$, where Z_{xv} denotes the feed conversion coefficient Z an instant after the transition. In this section we analyse a first-order phase transition $Z_x \to \infty \to Z_{xv}$.

$$Z_{xv} = \frac{2K_x^2}{3} \ , \ Z_o = \frac{2}{3} \ . \tag{19}$$

Let us consider equations (15), (17) and (18), we have

$$\frac{\Delta Z_x}{\Delta K_x} = \frac{2(K_x+1)}{3} - \frac{4K_x^2}{9Z_x} \ , \ Z_o = \frac{2}{3} \ . \tag{20}$$

Passage to the limit $(Z_x \to \infty)$ in (20) is as follows.

$$lim_{Z_x \to \infty} \frac{\Delta Z_x}{\Delta K_x} = \frac{2(K_x+1)}{3} \ . \tag{21}$$

From equation (21) we get

$$\frac{\Delta Z_{xv}}{\Delta K_x} = \frac{2(K_x+1)}{3} \ . \tag{22}$$

One can conclude that equation (22) is exactly (19). In (21), transition from Z_x to Z_{xv} was due to passage $Z_x \to \infty \to Z_{xv}$. In other words, during transition K_x remain constant while Z changes $Z_x \to \infty \to Z_{xv}$. This change of Z, a phase transition, provides a first-order or discontinuous transition to a new trajectory that emerges during bifurcation. The initial point on the emerged trajectory is (M_x, K_x, Z_{xv}). This result one can obtain in the following way. Let us consider (20) and (22) as a system, given below.

$$\begin{cases} \frac{\Delta Z_x}{\Delta K_x} = \frac{2(K_x+1)}{3} - \frac{4K_x^2}{9Z_x} \\ \frac{\Delta Z_{xv}}{\Delta K_x} = \frac{2(K_x+1)}{3} \end{cases} \tag{23}$$

From the system (23) it follows

$$\frac{\Delta Z_x}{\Delta Z_{xv}} = 1 - \frac{2K_x^2}{3Z_x(K_x+1)} \ . \tag{24}$$

Passage to the limit $(Z_x \to \infty)$ in (24) is given by

$$\lim_{Z_x \to \infty} \frac{\Delta Z_x}{\Delta Z_{xv}} = 1 \, . \tag{25}$$

From (25) one can conclude that in the instant of the phase transition, Z_x is indistinguishable from Z_{xv}. However, after the phase transition the following equation holds.

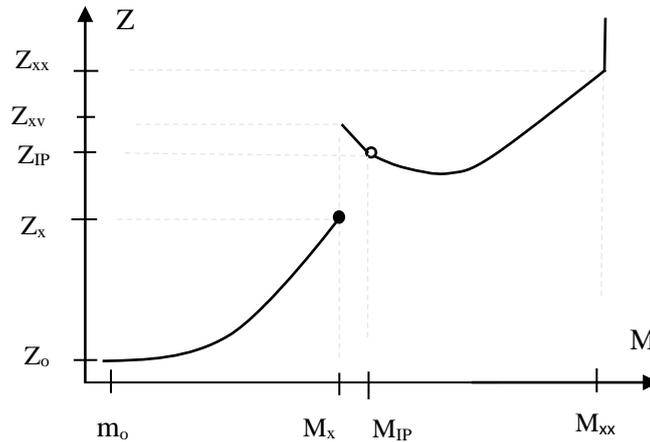$$Z_{xv} - Z_x = \frac{2K_x}{3} \, . \tag{26}$$

Figure 2. The trajectory of the feed conversion coefficient is a discontinuous function

● a first-order phase transition point. After the phase transition, the initial point on the trajectory is (M_x, Z_{xv}).

o inflection point (M_{IP}, Z_{IP}). In this inflection point, the rate of Z begins to increase.

The dynamic of the feed conversion coefficient Z is displayed in figure 2. Figure 2 shows a first-order phase transition (Heffern et al., 2021); there the order parameter Z displays a discontinuous transition from Z_x to Z_{xv}, at the critical value of M_x. After this phase transition the dynamics of growth in domain $M > M_x$ changes in essence.

On trajectory $M_x \to M_{xx}$, the feed conversion coefficient changes as follows $Z_x \to \infty \to Z_{xv} \to Z_{IP} \to Z_{xx} \to \infty$. In this inflection point Z_{IP}, the rate of Z begins to increase. After passing through the inflection point Z_{IP} the rate of Z reaches its local minimum. On this trajectory, in the point of the local minimum of Z the growth rate is expected to reach its local maximum. On this trajectory, the dynamic of growth is nonlinear and poorly understood.

On trajectory $M_x|_{t=t_x} \to M_x|_{t=t_1}$ the feed conversion coefficient changes as follows $Z_x \to \infty$ and remain infinity. The reason for that is M_x remains constant during time span $t_x \to t_1$. On this trajectory animals do not grow

3.2.2 Bifurcation of the Parameter K Trajectory

In the point (M_x, K_x, Z_x) bifurcation of the growth invariant K takes place. The cause for this bifurcation is the same as for the discussed one; the feed conversion coefficient at the point $Z = Z_x$ grows into infinity $Z_x \to \infty$. Both bifurcations show up simultaneously and in the same point. Both bifurcations cause dynamic of Z. In this dynamics, variable Z is the order parameter. On K trajectory, due to bifurcation, in the point (M_x, K_x, Z_x) two new trajectories emerge. One trajectory is $K_x \to K_{IP} \to K_{xx}$, this trajectory corresponds to $M_x \to M_{IP} \to M_{xx}$. Another trajectory is $K_x \to K_1 \to K_2$, it corresponds to $M_x|_{t=t_x} \to M_x|_{t=t_2}$. The dynamic of K is displayed in figure 3. During bifurcation in the point (M_x, K_x, Z_x), figure 3, trajectory $K_x \to K_1 \to K_2$ emerged. Let us consider two more points on this trajectory K_1, and K_2. On this trajectory, in all K points $M = M_x$, and $Z = \infty$. Consider (15), given below.

$$\frac{1}{m_o} \cdot \frac{\Delta M}{\Delta K} = \frac{2K+1}{K} - \frac{2}{Z} \, , \; Z > 0 \, .$$

In equation (15), under condition $M = M_x$, we take limit ($Z \to \infty$), given by

$$\lim_{Z \to \infty} \frac{\Delta M}{\Delta K} = m_o \cdot \frac{2K_1 + 1}{K_1} \, , where \; (K \to K_1)|_{Z \to \infty} \, . \tag{27}$$

In a more convenient form limit (27) is given by

$$\frac{1}{m_o} \cdot \frac{\Delta M_x}{\Delta K_1} = \frac{2K_1 + 1}{K_1} \; . \tag{28}$$
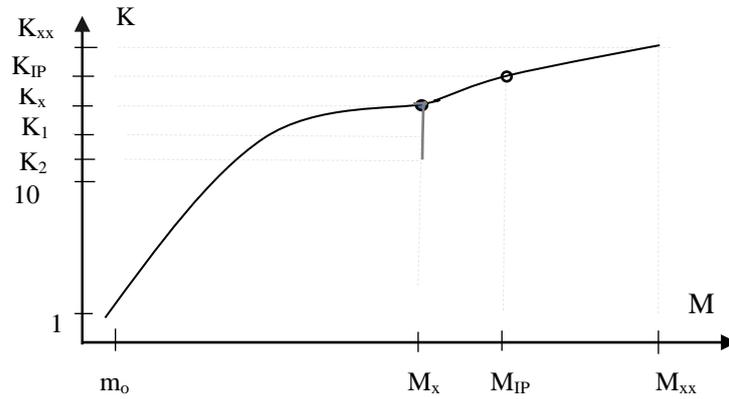

Figure 3. Bifurcation of the trajectory of parameter K

● bifurcation point (M_x, K_x). ○ inflection point (M_{IP}, K_{IP}). During bifurcation, the two trajectories emerge; one is $K_x \rightarrow K_1 \rightarrow K_2$, other $K_x \rightarrow K_{IP} \rightarrow K_{xx}$.

From (28) it follows $K_1 = K|_{(M=M_x) \wedge (Z=\infty)}$. Numerically, $K_1 = 10{,}04975$. Both K_x and K_1 are located on the same trajectory. Below it is shown that in both points, K_x, and K_1, $M = M_x$.

$$\frac{M_x}{m_o} = 2K_x - \frac{4}{K_x} \; . \tag{29}$$

$$\frac{M_x}{m_o} = 2K_1 - \frac{1}{K_1} \; . \tag{30}$$

From (18), (29) and (30) it follows

$$\frac{\Delta K_1}{\Delta K_x} = 1 - \frac{4K_1 - K_x}{3K_1 \cdot Z_x} \; . \tag{31}$$

In (31), the limit $(Z_x \rightarrow \infty)$ is given by

$$\lim_{Z_x \rightarrow \infty} \frac{\Delta K_1}{\Delta K_x} = 1 \quad . \tag{32}$$

The limit (32) means that in an instant of bifurcation, points K_x, and K_1 are indistinguishable from one another. However, after bifurcation, trajectory $K_x \rightarrow K_1 \rightarrow K_2$ emerges. K_2 is the last point on this trajectory. We can find K_2 by applying the following procedure. Let us consider a relation between M, and K.

$$MK = \int M \, dK + \int K \, dM \; . \tag{33}$$

Before the next step, consider the limit (t→∞) of (9).

$$\lim_{t \rightarrow \infty} K = \frac{M}{2m_o} \; . \tag{34}$$

The limit (34) in the form $dK = \frac{dM}{2m_o}$, after substitution into (33), we have $K_2 = K|_{(t=\infty) \wedge (M=M_x)}$, given below.

$$K_2 = \frac{M_x}{2m_o} + \frac{m_o}{2M_x} \; . \tag{35}$$

Numerically, $K_2 = 10{,}0250$. In the point $K = K_2$ condition $M = M_x$ holds, given by

$$\frac{M_x}{m_o} = K_2 + \sqrt{K_2^2 - 1} \; . \tag{36}$$

Equations (29), (30), and (36) confirm that points K_x, K_1, and K_2 are located on the same emerged trajectory. K_2 corresponds to t_2, figure 1. t_2 denotes theoretically maximum life span in the pig 49, 31 years. This follows from the calculation under condition $(M=M_x) \wedge (K=K_2)$. Maximum life span in the pig was modelled by the limit $(t \to \infty)$.

4. Discussion

The aim of this study was to build an analytical model of growth of animals. In the study, we have carried out analysis of the model. In this section, some of the results are discussed. In animals, nutrition is a vital factor in the development of body size. The nutritional information has been conveyed to the cellular and systemic levels to provide balanced growth (Boulan et al, 2015). If this logic is followed than food and food conversion are to consider as focal variables in the models of growth. Food or feed consumed has been transformed by an organism to human or animal weight. This concept was used in the model. One can conclude from the study that the three results have a novelty value. The first one implies that in animals, the current weight M is described by derivative of the average consumed feed. The second one says that in animals, the growth dynamics can be modelled by an integral equation of weight balance. Third result has proved that in pigs, the dynamic system of growth has two bifurcations. In the study, the bifurcations were preceded by a first-order phase transition. This sequence of events has a biological meaning. The study of how systems undergo transitions between states has been a highly important topic in physics and chemistry (Solé et al., 1996). In biology, phase transition is a less applied concept (Heffern et al., 2021). In this study we have used the concept of a first-order phase transition to analyse the dynamics of growth of animals. In this non-local hybrid model, the phase transition takes in the whole organism as an entity; with all its subsystems and scales. In a population of animals, the dynamic of growth trajectory with bifurcations in individual animals, creates three sets with growth phenotypes. The phenotypes differ in their longevity, growth rate, and growth cessation. It is plausible that this model, which links growth and longevity in animals, will be helpful in studying cancer. It is plausible that the weight balance equation (38) is applicable not only to the whole organism but also to its parts. The results are discussed below.

4.1 Biological Reading of the Model

At first, we discuss meaning of equation (4) that defines current weight M, given below.

$$M = \frac{d\bar{F}}{d\bar{Z}}$$

The equation suggests that the current weight M is a result of the process, which is described by the average food conversion coefficient $\bar{Z}$ derivative of the average consumed food $\bar{F}$. This is a nontrivial result. In animals and humans, this result has many implications. What one can infer first is that current weight M is not contingent on time; in mathematical sense the weight dynamics is an autonomous process. The same inference one can draw from (15) and (28). Looking at (4) one can wonder what a distribution of F, and Z should be to provide that the equation holds. It is a demanding task to find the distribution of variables F, and Z to explain the growth process. On the other hand, the interpretation is a standard one. Human or animal weight M is a function of the environmental factor F and the feed conversion coefficient Z, which contains genetic determinants. One can infer from the equation that since feed or food F is the environmental factor, the genetic determination of the growth is expressed through the food conversion coefficient Z. Indeed, since $\bar{Z} = f(F, M)$ and F is environmental factor, one can conclude that it is M, an organism that possesses the genetic information which is mediated by Z in the model. In other words, the result above is biologically acceptable. The equation (4) is not species-specific. It is applicable to both humans and animals. There are reasons to expect that the mechanisms of growth are nonlinear and with many interactions. As food is essential for growth regulation in animals (Lui and Baron, 2011) then Z is associated with systemic effects of metabolism. It is plausible to speculate that in animals, these systemic effects are a part of growth regulation systemic factors (Boulan et al, 2015). The mechanisms underlying feed conversion are complex. This does not mean that they are intractable. However, at present, the route from digestion to body weight remains unclear.

The general equation of weight balance of animals or humans is given below.

$$\int Y \, dZ = \frac{F}{M} \cdot \int M \, dZ - \frac{M}{K} \cdot \int K \, dZ$$

In biological sense, the equation is the weight balance. It turned out that in humans and animals, the weight balance is an integral equation. The equation describes a process of converting consumed food F to human or animal weight M. The process was modelled by an autonomous system; such a system is not contingent on time explicitly. The equation says that the weight balance is much more complicated than it was formerly expected. This is a species-specific equation; if in this equation specify invariant of growth K then it can serve as a species-specific model. For example, K can be specified as a species-specific function of M and Z, of the form $K_Q = f(M, Z)$.

Mathematically, equations (4) and (8) are not unique, they are feasible, the model-related equations. Biologically, under the model conditions the equations are meaningful and consistent with the concept of the model. By analysing (8), let us consider two theoretical options. For animals feed F is an environmental variable and may well be regarded as independent from Z. If in (8) F consider as a variable independent from Z, in other words consider the following relation

$$\int F \, dZ = FZ \,, \tag{37}$$

than (8) is wrong, it does not hold. Under the model conditions, assumption (37) is incorrect. This suggests that either (8) is incorrect and a relation between the analysed traits is much more complicated, or there is a functional relation between F, and Z. Such a relation one can conceive as a result of natural selection and adaptation to a certain feed. Since weight balance equation in the form (8) under assumption (37) is incorrect, we will consider it in the following form

$$\int Y \, dZ = \frac{F}{M} \cdot \int M \, dZ - \frac{M}{K} \cdot \int K \, dZ \,, \tag{38}$$

where Y denotes not absorbed feed or food, and Y is a function of F and Z. Below, we will show that (38) is biologically meaningful equation. In this section we will analyse (38). At first, we consider it in terms of integral average. One can rewrite (38) as follows

$$\bar{Y} = \frac{F}{M} \cdot \bar{M} - \frac{M}{K} \cdot \bar{K} \,, \tag{39}$$

where $\bar{Y}$, $\bar{M}$, and $\bar{K}$ denote the integral average for the corresponding variables. In the next step, we will test (39) for a boar in weight Mx = 600 kg. In (39) one can make the following substitution

$$\bar{Z} = \frac{F}{M} \,. \tag{40}$$

I can remind the interested reader numerical values of integral average variables $\bar{M}$ and $\bar{K}$ for a 600 kg heavy boar; they found elsewhere (Stass 2019). $\bar{K} = 6,9822$, $K_x = 10,1961$, $\bar{M} = 410,874$. From (39) and (40) it follows

$$\bar{Y} = \bar{M}(\bar{Z} - 1) \,. \tag{41}$$

Equation (41) is biased; numerically it is incorrect. Equation of the same form and comparable numerical output one can get from the algebraic equation F - M = Y. Second reason equation (41) is incorrect is its independence of K. It follows that although (41) can serve for numerical approximation it is biased. One can suggest that the above-mentioned move from (38) to (41) was hardly right. This result implies that under the model conditions equation (38) is feasible and meaningful. In other words, in this study a relation between variables holds in the form of the integral equation (38), and not in algebraic form. There is a reason to think that in animals the cumulative value of the studied variables is the integral sum.

Short insight into application of equation (4) is as follows. We will use the following function for the average value

$$\bar{Z} = \frac{M}{\int \frac{1}{Z} dM} \,, \qquad Z > 0 \,. \tag{42}$$

From (42) it follows

$$\frac{d\bar{Z}}{dM} = \frac{\bar{Z}(Z - \bar{Z})}{MZ} \,. \tag{43}$$

From the system (44) considering (4) and (43), given below

$$\begin{cases} \dfrac{d\bar{F}}{d\bar{Z}} = M \\ \dfrac{d\bar{Z}}{dM} = \dfrac{\bar{Z}(Z - \bar{Z})}{MZ} \end{cases} \,, \tag{44}$$

we get

$$\frac{d\bar{F}}{dM} = \frac{\bar{Z}(Z - \bar{Z})}{Z} \,. \tag{45}$$

I can remind the interested reader a result of the hybrid model. In the dynamics of M and K there is bifurcation point (M_x, K_x, Z_x). In this bifurcation point, Z grows into infinity ($Z_x \to \infty$). Considering this event, let us find the limit ($Z \to \infty$) of (45), given by

$$\lim_{Z \to \infty} \frac{d\bar{F}}{dM} = \bar{Z} \quad . \tag{46}$$

The limit (46) provides a link between the hybrid, and continuum model.

In the study, there was a problem which the continuum model solved. The problem concerns a relation between F, and Z when Z grows into infinity ($Z \to \infty$). Under this condition, both F and Z may well grow into infinity. A possible reason for that gives a phenomenological relation between F, and Z. When an animal reaches its maximum weight, it does not grow any more. In this point not any quantity of feed can increase its weight. In the point in which an animal has reached its individual maximum weight, abstraction ($Z \to \infty$) is reasonable. This abstraction means that no growth is possible regardless of a quantity of feed F consumed. The question is should we in the point in which an animal's individual maximum weight is reached apply abstraction ($F \to \infty$) as well? The following analyses shows that it is not the case; variable F grows up to a finite value. $F|_{Z \to \infty} \to C$, where C is a finite value. Let us consider (1), (3), and (40) it follows

$$F - \bar{F} = \int \bar{Z} \, dM . \tag{47}$$

Let us denote $F - \bar{F} = V$ then, considering (47) we get

$$\frac{dV}{dM} = \bar{Z}. \tag{48}$$

By comparing (48) with (46) one can arrive at the following relation

$$\frac{dV}{dM}|_{z \to \infty} = \lim_{Z \to \infty} \frac{d\bar{F}}{dM} = \bar{Z} \quad . \tag{49}$$

From equation (49) it follows that variable F grows up to a finite value. $F|_{Z \to \infty} \to C$, where C is a certain finite value. This is a meaningful outcome; it is biologically correct and supports the other results.

4.2 Bifurcation of the Growth Trajectory

In this study by analysing a non-local hybrid model, bifurcation of the growth trajectory in pigs was revealed. In the course of growth, the two trajectories, of weight gain M, figure 1, and growth invariant K, figure 3, have bifurcation. Both bifurcations take place in the same point (M_x, K_x, Z_x) and simultaneously as Z_x grows into infinity, $Z_x \to \infty$. In both cases it is the feed conversion coefficient Z, which causes bifurcations to show up, figure 2. In this model variable Z is the order parameter. The dynamics of Z displays a first-order phase transition (Heffern et al., 2021). After this phase transition the dynamics of growth in domain $M > M_x$ changes in essence. In the study, the bifurcations were preceded by a first-order phase transition. In the course of growth, the sequence of events at first the phase transition and then trajectory bifurcation has a biological meaning. This is a unique result for this kind of hybrid model. This result shows that in individual animals the growth trajectory is neither smooth nor continuous. On the trajectory $M_x \to M_{xx}$, the feed conversion coefficient changes as follows $Z_x \to \infty \to Z_{xv} \to Z_{IP} \to Z_{xx} \to \infty$. In this inflection point Z_{IP}, the rate of Z begins to increase. After passing the inflection point Z_{IP} the rate of Z reaches its local minimum, figure 2. On this trajectory, in the point of the local minimum of Z, denoted $Z_{xx\backslash min}$, the growth rate of an animal is expected to reach its local maximum. Close to the bifurcation point, the values of Z are the following. $(Z_x = 62,5102), (Z_{xv} = 69,3076), (Z_{IP} = 65,0237)$, see figure 2. By analysing figure 2 one can note that the part of the curve $Z_x \to Z_{xv} \to Z_{IP} \to Z_{xx\backslash min}$ to find in experiments is hardly possible. This part of the curve, figure 2, includes bifurcation of the growth trajectory; and, in experiments, Z_x and Z_{xv}, as well as Z_{IP} and $Z_{xx\backslash min}$ is indistinguishable from each other. At best, in experiments it is feasible to find the following curve $Z \to Z_x \to Z_{xx\backslash min} \to Z_{xx}$ thus leaving out the most essential part of the growth process. One can infer that in this study, methods complement each other. On the trajectory $M_x|_{t=t_x} \to M_x|_{t=t_1}$ animals do not grow. Due to bifurcation, the growth of animals has stopped; on this trajectory pigs do not grow any longer. On this trajectory, $Z_x = \infty$, and $M = M_x$. Through the dynamics of the feed conversion coefficient Z, the genetic information about the growth trajectory has been expressed. As a result, two bifurcations show up. The emerged trajectories are asymptotically stable; the animals, which can take on one of the emerged trajectories, can live a stable and predictable life stage. Bifurcations mentioned above are thought to be of the pitchfork form and of the supercritical type. It is a demanding task to describe the discussed bifurcations in the normal form. One can conclude that in this phase transition variable Z is the order parameter. Since we consider growth of individual animals, this phase transition takes in the whole organism as an entity; with all its

subsystems and scales. In the next stage bifurcation takes place. This sequence of events is to consider as a model of the systemic regulation of growth in individual animals.

4.3 Growth and Longevity in the Pig

In this study, three sets with asymptotic growth phenotypes were determined. In this section we will discuss life span of animals in each of the sets. This study supports the opinion that growth causes aging (Gems and Partridge, 2013). Under the model conditions, a simple case is with the phenotypes, which can reach their individual maximum weight M_x. When a pig reaches M_x it is 6,40 years of age. This follows from the calculation $(M=M_x)\wedge(K=K_x)$, and $t_x = 6,40$ years. Then two setups follow. The animals which growth has stopped in weight M_x can live $t_1= 24,90$ years, or potentially $t_2 = 49,31$ years. The animals, which continue to grow beyond the bifurcation point, have the same life span t_1, or t_2. However, longevity of pigs that cannot reach weight M_x is unclear. We can only say that their life span is less than 24,90 years. One can note regularity; the phenotypes which take on a stable growth trajectory can have the definite life span, either t_1, or t_2. In animals, which are not conditioned to take on a stable growth trajectory life span is less than 24,90 years and uncertain.

4.4 Continuum and Hybrid Methods

In this study, continuum and hybrid modelling techniques were used together; both methods complementing each other. While analysing longevity, as well as some other cases, one can infer that the same result is impossible to obtain by applying both methods. In pigs, the maximum theoretical longevity t_2 is possible to find by applying only a continuum technique, taking limit $(t \to \infty)$. By contrast, obtainable life span t_1 in the pig is possible to find only by applying the hybrid method, equations (27) and (28). In this case, limit $(Z \to \infty)$ was taken. In the model, each of the above results one can obtain only by one technique continuum, or hybrid. The reason for that is unclear. In one case the limit $(t \to \infty)$ in other case the limit $(Z \to \infty)$ was used. The procedure, passage to the limit was the same in both cases, differ only variables. One can conclude that in biology, the nature of a variable is a strong reason for choosing the modelling technique. It is the case in physical science. However, it is not at all obvious that the food conversion coefficient Z, which in this study is the order parameter, should be analysed by the hybrid method. One can conclude that in this field we have to go in essential biological particulars. One can infer that in this study, both methods complement each other. The aim for the further studies is to build on these models to acquire new knowledge in this field.

5. Conclusions

- In this non-local hybrid model, the main change in the system's dynamics was caused by a first-order phase transition. In animals' ontogeny the phase transition precedes bifurcation of the growth trajectory.

- In the model, variable Z is the order parameter. The dynamics of Z displays a first-order phase transition. After this phase transition the dynamics of growth in domain $M > M_x$ changes in essence. This phase transition takes in the whole organism as an entity; with all its subsystems and scales. In the next stage bifurcation take place. This process is to consider as a model of the systemic regulation of growth in individual animals.

- In the course of ontogenetic growth, bifurcation of the trajectory of growth is a qualitative change. As a result of bifurcation, new trajectories of growth emerged. In a certain stage of growth, bifurcation makes available for an animal to switch from a current trajectory to emerged trajectory of growth. This change one can consider as a transition to the asymptotic growth phenotype.

- In a population of animals, the growth dynamic with bifurcations in individual animals forms three sets with asymptotic phenotypes. The asymptotic growth phenotypes differ in their longevity, growth rate, and growth cessation.

- In animals, the weight gain is a dynamic process. Under the model conditions the process can be described by the average food conversion coefficient $\bar{Z}$ derivative of the average consumed food $\bar{F}$.

$$M = \frac{d\bar{F}}{d\bar{Z}}$$

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