A Finite Difference Method for Positive Definite Fractional Damped String Vibration Equations

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Abstract

In this paper, the finite difference method is used to solve the positive fractional derivative damped string vibration e-quations, and the vibration attenuation phenomenon of the model is described by numerical simulation. In numerical examples, the effects of the order of the positive definite fractional derivative and the damping coefficient on vibration are studied and compared respectively. The results show that, on the one hand, when the damping coefficient c is fixed, the closer the order $p(0 < p < 1)$ is to 1, the faster the attenuation is. On the other hand, when the order p is fixed, the larger the damping coefficient c is, the faster the attenuation is.

Keywords: positive definite fractional derivative, damped string vibration equations, finite difference method, numerical simulation

AMS subject classification: 65N15, 65N30.

1. Introduction

Vibration is a common physical phenomenon. But in some cases, vibration will cause serious damage, such as mechanical structure destruction, building destruction, environmental pollution and so on. Viscoelastic material is a kind of material specially used as damping layers, which are widely used in the research and development of dampers. Viscoelastic dampers are not only used in aerospace, military industry and agriculture, but also play an important role in earthquake and wind array control of civil engineering structures, effectively reducing the harm caused by vibration (Wang, Zhou & Ding, 2006; He & Tan, 2008; Chen, 2008). Therefore, the study of damped vibration process in viscoelastic media is a very difficult and practical work.

The mechanical properties of viscoelastic materials are between ideal solid and Newtonian fluid. Their mechanical behavior has strong memory and their constitutive relation does not satisfy Newton shear rate. The dependence of viscoelastic damping on frequency is in the form of fractional power (Li & Xu, 2000; Xu & Tan, 2006; Tong & Wang, 2004). Since the integral time derivative is defined by the local limit and lacks the time accumulation of the system evolution, it cannot accurately describe the "abnormal" mechanical phenomena of viscoelastic materials, such as the memory, path dependence and so on. Therefore, the use of fractional derivative to describe the viscoelastic constitutive relation has been widely studied since the 1980s (Torvik & Bagley, 1984; Koeller, 1984). The time fractional derivative contains integral convolution operators, and the integral term fully reflects the historical dependence of the mechanical process, which has great practical significance in describing the memorization process (Tong & Wang, 2004; Zhu, 2006; Gorenflo, Mainardi, Moretti, Pagnini & Paradisi, 2002; Mainardi, 1996). For example, Bagley and Torvik et al. first used the fractional derivative model to describe vibrator damping in viscoelastic media and obtained the fractional derivative damped vibration equations (Adolfsson, Enelund & Olsson, 2005). However, because the fractional derivative is not positive, it can not accurately reflect the frequency-dependent dissipation of sound waves in dissipative media in the frequency domain. Chen and Holm first proposed the definition of positive definite fractional derivative (Chen & Holm, 2002). Different from the fractional derivative, the Fourier transform of the positive definite fractional derivative has positive definiteness and is more in line with the frequency-dependent characteristics of dissipation in the frequency domain. Zhang and Chen established the positive definite fractional derivative damped particle vibration model, studied the mechanical process of the model and solved the positive definite fractional derivative equations by using the finite difference method for the first time (Zhang & Chen, 2009).

Based on the study of the positive definite fractional derivative damped particle vibration model (Zhang & Chen, 2009), in this paper, the positive definite fractional derivative is introduced into the damped string vibration equations for the

first time. The finite difference method is used to solve the positive definite fractional derivative damped string vibration equations, and numerical examples are given to describe the attenuation phenomenon of the model.

The rest of this paper is organized as follows. In section 2, the positive definite fractional derivative damped string vibration equations are introduced. In section 3, the finite difference schemes of the positive definite fractional derivative damped string vibration equations are elaborated. In section 4, two numerical examples are given to analyze the attenuation of the model. In section 5, this work is summarized.

2. Positive Definite Fractional Damped String Vibration Equations

Acoustic waves in dissipative media exhibit abnormal energy dissipation with power law frequency-dependent characteristics (Szabo,1994; Chen & Holm, 2003)

$$P(x + \Delta x) = P(x)e^{-\alpha(\omega)\Delta x}, \alpha(\omega) = \alpha_0|\omega|^p,$$

where ω is the angular frequency, p is the pressure, x is the propagation direction of the wave, α is the dissipation coefficient, $\eta(0 \leq \eta \leq 2)$ is the non-negative constant and Δx is the propagation distance of the wave. Although the fractional derivative has become an effective modeling method to describe complex mechanical behavior, it can be seen from the definition that the fractional derivative is not positive and its Fourier transform is $(i\omega)^p$, which cannot accurately reflect the frequency-dependent dissipation of sound waves in the frequency domain. Szabo proposed the time domain convolution integral acoustic equations satisfying the causality law (Szabo, 1994). However, Szabo's equation contains supersingular integral, which makes numerical solutions very difficult. Therefore, Chen and Holm introduced the positive definite fractional derivative to eliminate the singularity of the model and obtained the modified Szabo's wave equation models (Chen & Holm, 2003).

The positive definite fractional derivative is defined as follows:

$$\frac{d^{|p|}u}{dt^{|p|}} = \begin{cases} \frac{-1}{pq(p)} \int_0^t \frac{u'(\tau)d\tau}{(t-\tau)^p}, 0 < p \leq 1, \\ \frac{1}{p(p-1)q(p)} \int_0^t \frac{u''(\tau)d\tau}{(t-\tau)^{p-1}}, 1 < p < 2, \end{cases}$$

where p is the order of the positive definite fractional derivative and the expression for the constant $q(p)$ is

$$q(p) = \frac{\pi}{2\Gamma(p+1)cos[(p+1)\pi/2]}.$$

Applying the positive definite fractional derivative mentioned above to the string vibration equations, now we consider the following initial and boundary value problems:

$$m\frac{\partial^2 u}{\partial t^2}(x,t) + (-1)^{[p]-1}c\frac{d^{|p|}u}{dt^{|p|}}(x,t) - a^2\frac{\partial^2 u}{\partial x^2}(x,t) + ku(x,t) = f(x,t), \tag{1}$$
$$x \in (0, L), t \in (0, T],$$

$$u(x, 0) = \varphi(x), \frac{\partial u}{\partial t}(x, 0) = \psi(x), x \in [0, L], \tag{2}$$

$$u(0, t) = u(L, t) = 0, t \in (0, T], \tag{3}$$

where m is mass, c is the damping coefficient, $a^2(> 0)$ is a constant, k is the elastic coefficient, f is the external load, $u(x,t)$ is the displacement, $\varphi(x)$ and $\psi(x)$ are both known smooth functions and $[p]$ is the smallest integer greater than p.

3. Finite Difference Schemes

Take the positive integers M and N, let $h = \frac{L}{M}$ be the spatial step, $\tau = \frac{T}{N}$ be the time step, then $x_i = ih(0 \leq i \leq M)$, $t_n = n\tau(0 \leq n \leq N)$. Define the grid function $u_i^n = u(x_i, t_n), 0 \leq i \leq M, 0 \leq n \leq N$ and introduce the following notations:

$$\overline{u}_i^n = \frac{1}{2}(u_i^{n+1} + u_i^{n-1}),$$

$$\delta_t^2 u_i^n = \frac{1}{\tau^2}(u_i^{n+1} - 2u_i^n + u_i^{n-1}),$$

$$\delta_x^2 u_i^n = \frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n),$$

$$\delta_x^2 u_i^n = \frac{1}{4h^2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + \frac{1}{2h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$+ \frac{1}{4h^2}(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}).$$

When $0 < p < 1$, at the node (x_i, t_n), the corresponding equation of the equation (1) is as follows:

$$m\frac{\partial^2 u}{\partial t^2}(x_i, t_n) + c\frac{d^{|p|}u}{dt^{|p|}}(x_i, t_n) - a^2\frac{\partial^2 u}{\partial x^2}(x_i, t_n) + ku(x_i, t_n) = f(x_i, t_n),$$
$$1 \le i \le M - 1, 1 \le n \le N. \tag{4}$$

The finite difference schemes of the equations (1)-(3) are given below.

3.1 Explicit Scheme

Both the second-order time derivative term and the second-order spatial derivative term are discretized by the central difference method. For the discrete method of the positive definite fractional derivative term, please refer to the reference (Zhang & Chen, 2009).

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_n) \approx \delta_t^2 u_i^n,$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_n) \approx \delta_x^2 u_i^n,$$

$$\frac{d^{|p|}u}{dt^{|p|}}(x_i, t_n) \approx \frac{-1}{p(1-p)q(p)} \sum_{j=0}^{n} \frac{u_i^{n+1-j} - u_i^{n-j}}{\tau^p}[(j+1)^{1-p} - j^{1-p}], 0 < p < 1.$$

Using U_i^n to represent the approximate solution of the differential discrete scheme, we can get

$$m\delta_t^2 U_i^n - \frac{c}{p(1-p)q(p)} \sum_{j=0}^{n} \frac{U_i^{n+1-j} - U_i^{n-j}}{\tau^p}[(j+1)^{1-p} - j^{1-p}]$$
$$- a^2\delta_x^2 U_i^n + kU_i^n = f_i^n, 1 \le i \le M - 1, 1 \le n \le N. \tag{5}$$

The initial value conditions (2) and the boundary value conditions (3) are discretized below.

The initial value condition $u(x, 0) = \varphi(x)$ can be directly discretized as

$$U_i^0 = u_i^0 = \varphi(x_i), 0 \le i \le M. \tag{6}$$

For the discretization of initial value condition $\frac{\partial u}{\partial t}(x, 0) = \psi(x)$, it can be obtained by Taylor expansion and (2)

$$u_i^1 \approx u(x_i, t_0) + \tau\frac{\partial u}{\partial t}(x_i, t_0) = \varphi(x_i) + \tau\psi(x_i), \tag{7}$$

then

$$U_i^1 = \varphi(x_i) + \tau\psi(x_i), 1 \le i \le M - 1. \tag{8}$$

The boundary value conditions $u(0, t) = u(L, t) = 0$ can be directly discretized as

$$U_0^n = U_M^n = 0, 0 \le n \le N. \tag{9}$$

Combining (5), (6), (8) and (9), the final explicit difference scheme of (1)-(3) is obtained as follows.

$$m\delta_t^2 U_i^n - \frac{c}{p(1-p)q(p)} \sum_{j=0}^{n} \frac{U_i^{n+1-j} - U_i^{n-j}}{\tau^p}[(j+1)^{1-p} - j^{1-p}]$$
$$- a^2\delta_x^2 U_i^n + kU_i^n = f_i^n, 1 \le i \le M - 1, 1 \le n \le N. \tag{10}$$

$$U_i^0 = \varphi(x_i), 0 \le i \le M. \tag{11}$$

$$U_i^1 = \varphi(x_i) + \tau\psi(x_i), 1 \le i \le M - 1. \tag{12}$$

$$U_0^N = U_M^N = 0, 0 \le n \le N. \tag{13}$$

Since certain conditions are required for the explicit scheme to achieve a stable condition, in order to obtain the steady-stable differential scheme, the implicit scheme is given below.

3.2 Implicit Scheme

For the equation (4), the weighted average of the central difference quotients of the $n - 1$th, the nth and the $n + 1$th layers is used to approximate the second-order spatial derivative term. The average of the $n - 1$th and the $n + 1$th layers is used to approximate the original function term as the following:

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_n) \approx \delta_{\bar{x}}^2 u_i^n,$$
$$u(x_i, t_n) \approx \overline{u}_i^n.$$

The discrete method for the remaining two items is the same as the discrete method in 3.1.

Using U_i^n to represent the approximate solution of the differential discrete scheme, we can get

$$m\delta_t^2 U_i^n - \frac{c}{p(1-p)q(p)} \sum_{j=0}^{n} \frac{U_i^{n+1-j} - U_i^{n-j}}{\tau^p}[(j+1)^{1-p} - j^{1-p}]$$
$$- a^2\delta_{\bar{x}}^2 U_i^n + k\overline{U}_i^n = f_i^n, 1 \le i \le M - 1, 1 \le n \le N. \tag{14}$$

Similar to 3.1, the initial value conditions and the boundary value conditions are discretized below.

The initial value condition $u(x, 0) = \varphi(x)$ can be directly discretized as

$$U_i^0 = u_i^0 = \varphi(x_i), 0 \le i \le M. \tag{15}$$

The initial value condition $\frac{\partial u}{\partial t}(x, 0) = \psi(x)$ can be directly discretized as

$$u_i^1 \approx u(x_i, t_0) + \tau\frac{\partial u}{\partial t}(x_i, t_0) = \varphi(x_i) + \tau\psi(x_i), \tag{16}$$

then

$$U_i^1 = \varphi(x_i) + \tau\psi(x_i), 1 \le i \le M - 1. \tag{17}$$

The boundary value conditions $u(0, t) = u(L, t) = 0$ can be directly discretized as

$$U_0^n = U_M^n = 0, 0 \le n \le N. \tag{18}$$

Combining (14), (15), (17) and (18), the final implicit difference scheme of (1)-(3) is obtained as follows.

$$m\delta_t^2 U_i^n - \frac{c}{p(1-p)q(p)} \sum_{j=0}^{n} \frac{U_i^{n+1-j} - U_i^{n-j}}{\tau^p}[(j+1)^{1-p} - j^{1-p}]$$
$$- a^2\delta_{\bar{x}}^2 U_i^n + k\overline{U}_i^n = f_i^n, 1 \le i \le M - 1, 1 \le n \le N. \tag{19}$$

$$U_i^0 = \varphi(x_i), 0 \le i \le M. \tag{20}$$

$$U_i^1 = \varphi(x_i) + \tau\psi(x_i), 1 \le i \le M - 1. \tag{21}$$

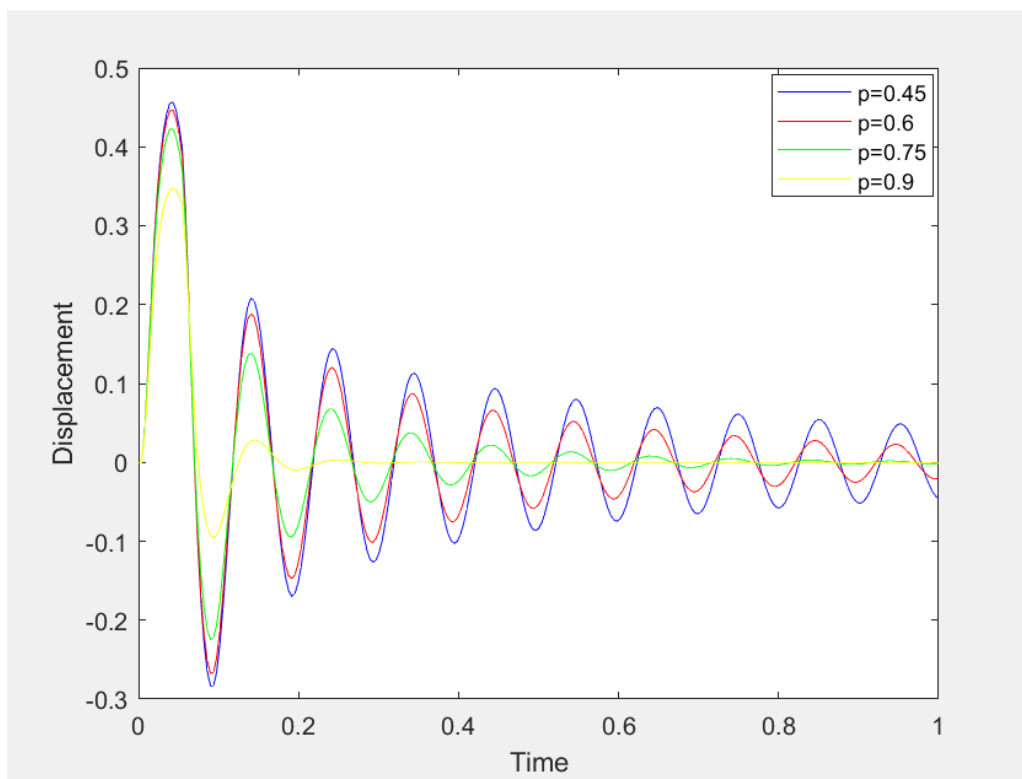Figure 1. Numerical solution of damped vibration equations at $c = 4.2 \times 10^6 N \cdot s/m$

$$U_0^N = U_M^N = 0, 0 \leq n \leq N. \tag{22}$$

4. Numerical Experiments

From the analytical solution of the positive definite fractional derivative equation, the decay exponent of the positive definite fractional derivative model is $-(cA/m)t^{2-p}(A > 1)$ (Zhang & Chen, 2009). The numerical simulation method is applied below to analyze the numerical results of the positive definite fractional derivative damped string vibration equation.

Equations (1)-(3) are solved numerically using the finite difference method. Two examples are given to study the influence of the damping coefficient c and the order of positive definite fractional derivative $p(0 < p < 1)$ on vibration attenuation.

4.1 Example1

Let the coefficients $a^2 = 1.44 \times 10^6 N \cdot m^2$, $m = 9.5 \times 10^5 kg$, $k = 8.9 \times 10^8 N/m$, the smooth functions $\varphi(x) = 0$, $\psi(x) = 0$, $x \in (0, 5)$, the external load

$$f(x, t) = \begin{cases} 6.8 \times 10^8, & 0 \leq t \leq 0.1, \\ 0, & t > 0.1. \end{cases}$$

Let $L = 5$, $M = 2^8$, then the space step $h = \frac{L}{M} = \frac{5}{2^8}$, the time step $\tau = \frac{2}{2^8}$.

On the one hand, when we take different values of p, the numerical results at $x = x_{M-1}$ are presented in Figure 1 with fixed $c = 4.2 \times 10^6 N \cdot s/m$. On the other hand, when we take different values of c (the unit of measure is N· s/m), the numerical results at $x = x_{M-1}$ are presented in Figures $2 - 4$ with fixed $p = 0.45$, $p = 0.75$ and $p = 0.9$, respectively.

It can be seen from Figure 1 that when c is fixed, the closer p is to 1, the faster the attenuation is. In Figures 2(a,b,c), when p is fixed, the bigger c is, the faster the attenuation is.
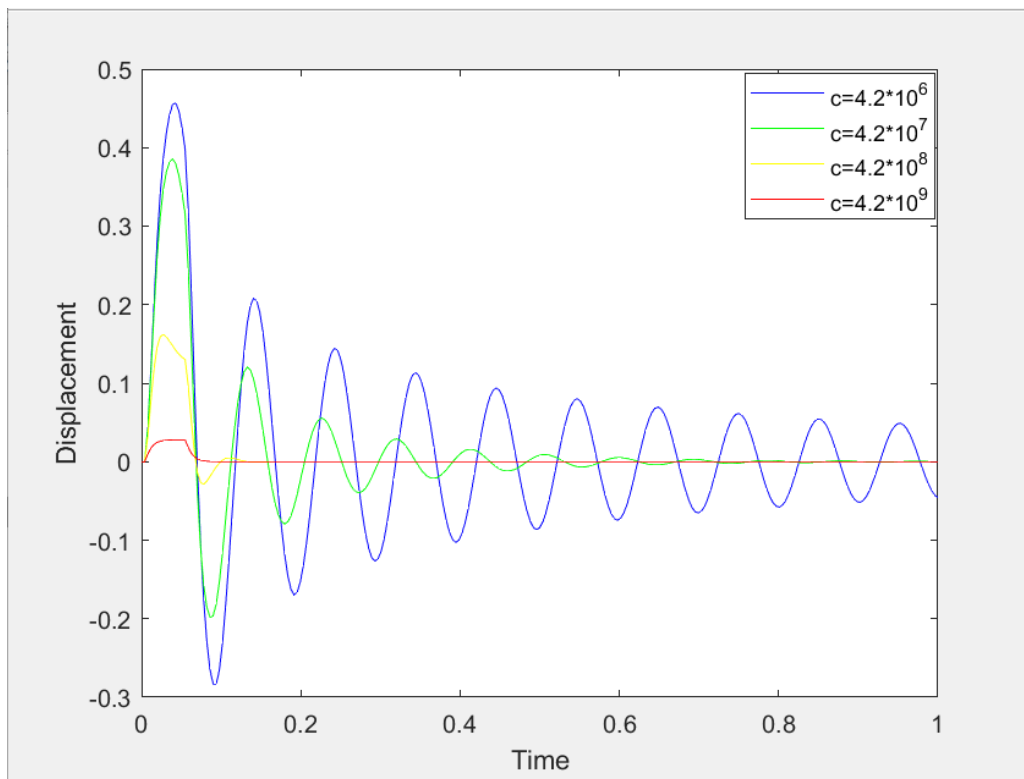
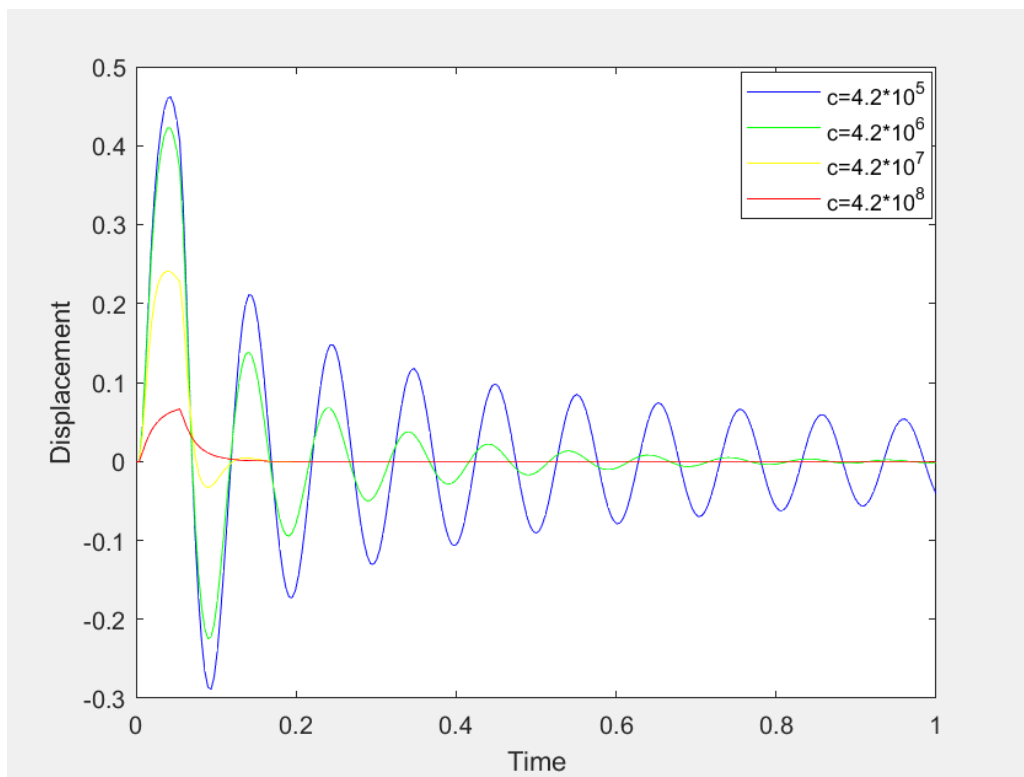Figure 2(a). Numerical solution of damped vibration equations at $p = 0.45$

Figure 2(b). Numerical solution of damped vibration equations at $p = 0.75$
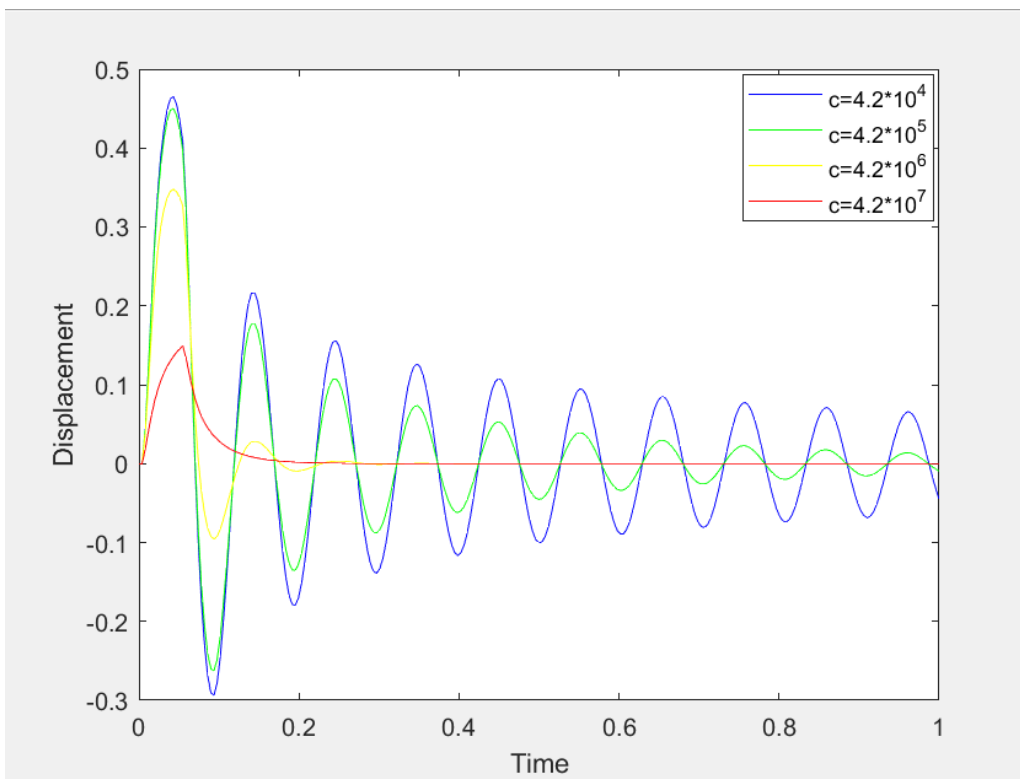
Figure 2(c). Numerical solution of damped vibration equations at $p = 0.9$

4.2 Example2

Let the coefficients $a^2 = 6.25 \times 10^6 N \cdot m^2$, $m = 4.5 \times 10^5 kg$, $k = 8.9 \times 10^8 N/m$, the smooth functions $\varphi(x) = 0$, $\psi(x) = 0$, $x \in (0, 5)$, the external load

$$f(x, t) = \begin{cases} 4.8 \times 10^8 t^2 sin\pi x, & 0 \leq t \leq 0.1, \\ 0, & t > 0.1. \end{cases}$$

Let $L = 5$, $M = 2^8$, then the space step $h = \frac{L}{M} = \frac{5}{2^8}$, the time step $\tau = \frac{1}{2^8}$.

Similar to example 1, when we take different values of p, the numerical results at $x = x_{M-1}$ are presented in Figure 5 with fixed $c = 6.2 \times 10^6 N \cdot s/m$. When we take different values of c (the unit of measure is N· s/m), the numerical results at $x = x_{M-1}$ are presented in Figures $6 - 8$ with fixed $p = 0.45$, $p = 0.75$ and $p = 0.9$, respectively.

It can be seen from Figure 3 that when c is fixed, the closer p is to 1, the faster the attenuation is. In Figures 4(a,b,c), when p is fixed, the bigger c is, the faster the attenuation is.

5. Conclusion

In this paper, we propose the positive definite fractional derivative damped string vibration equations. The explicit and implicit schemes are obtained by the finite difference method. Two numerical examples are given to study the effects of the order of the positive definite fractional derivative and the damping coefficient on the vibration attenuation of the model. In future work, we will have a more in-depth study on the positive definite fractional derivative damped string vibration equations.

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Conflict of Interests

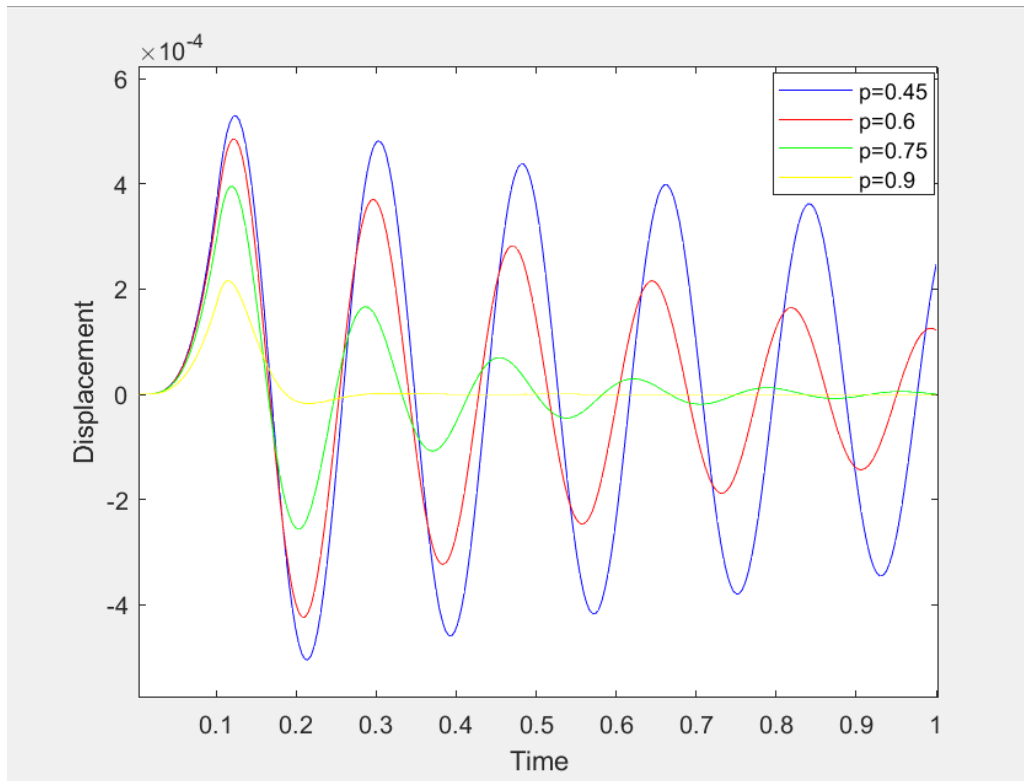The authors declare that there is no conflict of interests regarding the publication of this paper.

Figure 3. Numerical solution of damped vibration equations at $c = 6.2 \times 10^6 N \cdot s/m$
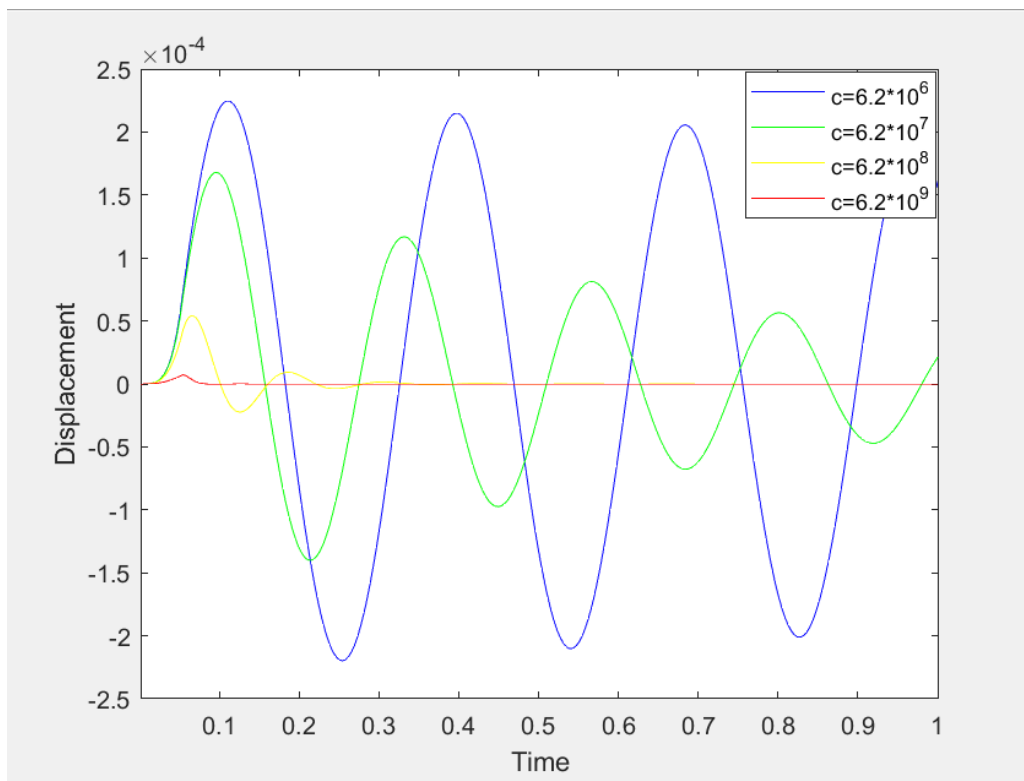

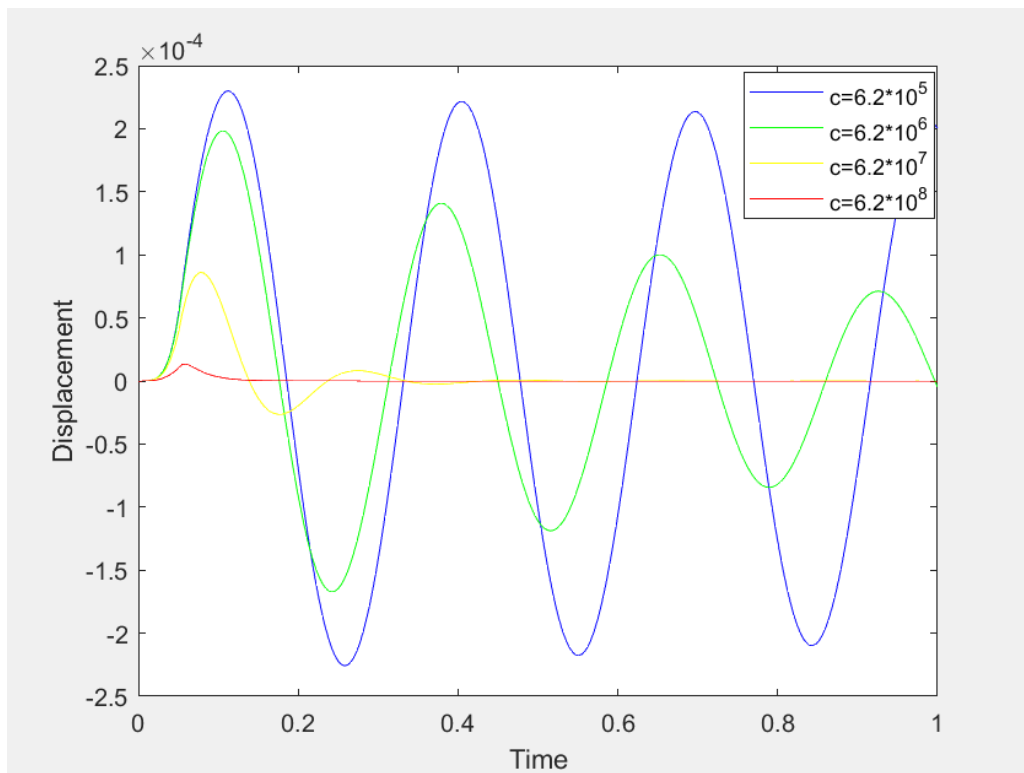Figure 4(a). Numerical solution of damped vibration equations at $p = 0.45$

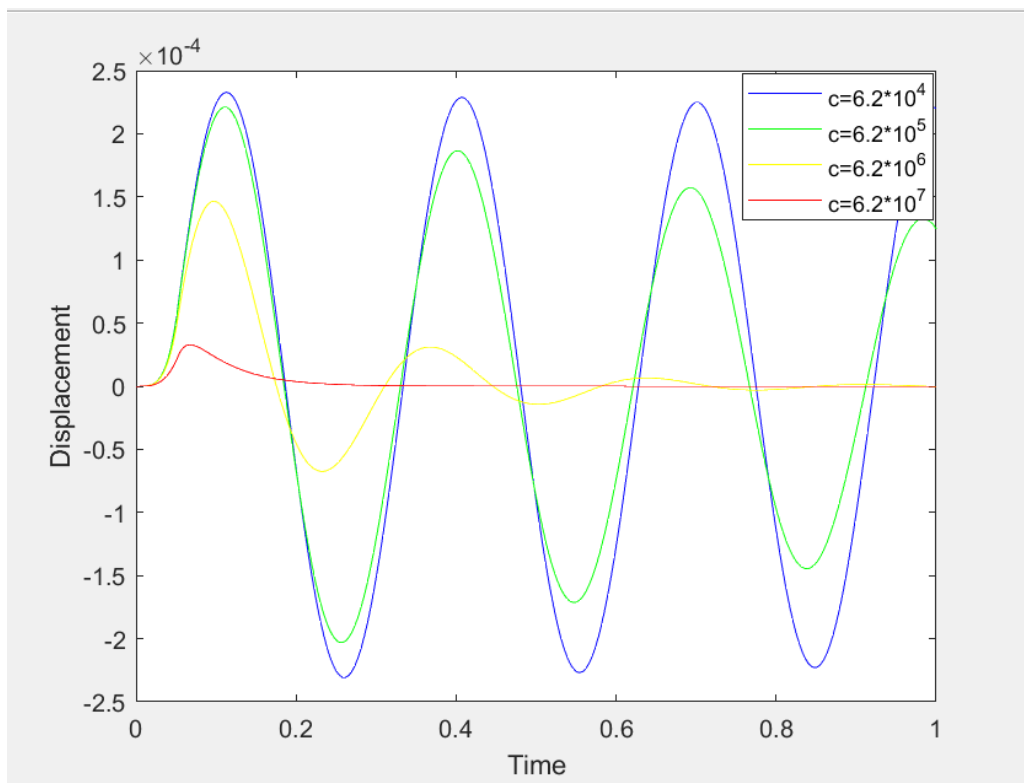Figure 4(b). Numerical solution of damped vibration equations at $p = 0.75$

Figure 4(c). Numerical solution of damped vibration equations at $p = 0.9$

Supplementary information

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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