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Generalized Bur X Lomax Distribution: Properties, Inference and Application to Aircraft Data

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Abstract

We proposed and studied a flexible distribution with wider applications called Generalized Burr X Lomax (GBX-L) distribution. Some well-known mathematical properties such as ordinary moments, incomplete moment probability weighted moments, stress-strength model, mean residual lifetime, characteristic function, quantile function, order statistics and Renyi entropy of GBX-L distribution are investigated. The expressions of order statistics are derived. Parameters of the derived distribution are obtained using the maximum likelihood method and simulation studied is carried out to examine the validity of the method of estimation. The applicability of the proposed distribution is exemplified using aircraft data.

Keywords: Quantile function, Probability Weighted Moments, Characteristics function, Generalized Bur X Lomax distribution

1. Introduction

The Lomax or Pareto type II distribution was developed by Lomax (1954). It has several areas of applications such as determining the size of cities, flood, internet traffic control,, income and wealth inequality, reliability analysis, actuarial science, biological and medical sciences, queue theory, wind speed, life testing, sea waves and many others. A random variable X is said to be distributed according to Lomax distribution, if it distribution function is given by

$$G(x;a,b) = 1-\left(1+\frac{x}{a}\right)^{b}, \qquad x>0; a,b>0 \tag{1}$$

And its associated pdf is given by

$$g(x;a,b) = 1-\left(1+\frac{x}{a}\right)^{b}, \qquad x>0; a,b>0 \tag{2}$$

b is a shape parameter and a is a scale parameter. It should be noted that the pdf of Lomax distribution is naturally a special case of some well-known distributions, and this includes Feller-Pareto, Pareto type II, Pareto type IV, Fisher distribution, and many others. However, the Lomax distribution is limited in applications as a result of some of its limitations, and this includes: Lack of flexibility, heavy tailed features, poor fits etc. when used to model real life data which exhibits non-monotonic, bathtub failure rate. Based on the afore-mentioned reasons various efforts have been made to generalize the Lomax distribution in other to induce flexibility into the distribution and also improve its fits for a better modeling capability. Among these are: exponentiated Lomax (El-Bassiouny et al., 2015), beta-Lomax, Kumaraswamy-Lomax, McDonald-Lomax (Lemonte and Cordeiro, 2013), gamma-Lomax (Cordeiro et al., 2013), Marshall-Olkin Extended Lomax distribution by Ghitany et al. (2007), Al-Zahrani and Sagor (2014) developed and studied the Poisson Lomax distribution , Weibull Lomax was developed by Tahir et al. (2015), half logistic Lomax by Anwar and Zahoor (2018)., Abdul-Moniem and Abdel-Hameed (2012) developed and studied the Exponentiated Lomax distribution, the transformed-transformer Lomax distribution was studied by Alzaghal and Hamed (2019), Kilany (2016) studied the weighted Lomax distribution, and the Power Lomax distribution was developed and studied by Rady et al. (2016) and Harris Extended Power Lomax Distribution was developed by Ogunde et al. (2021).

We are motivated by the advantages offer by the generalized distribution when use to extend the baseline distribution.

Such advantage includes the ability of the new distribution to model data of any shape of the hazard function which further extend its scope of applications. Based on this, we are proposing a new extension of the Lomax distribution developed by using Generalized Bur X family of distributions which added two extra shape parameters to the Lomax distribution that induces flexibility into the Lomax distribution and also improve its fits by controlling its skewness and kurtosis for a better modeling capability. The new model is called Generalized Burr X Lomax (GBX-L) distribution.

The rest of the paper is organized as follows. In Section 2, we introduce the GBX-L model. Section 3 focuses on its properties. In Section 4, we carried out the estimation procedures and also simulation of data set and finally in Section 5, we concluded.

2. Generalized Burr X Lomax (GBX-L) Distribution

Let $f(x;\ \varphi)$ and $F(x;\ \varphi)$ represents the PDF and CDF of the baseline distribution with parameter vector φ. Then, according to Aldahlan and Khalil (2021), the CDF of the $GBX-G$ family is given by

$$G(x)=\left[1-exp\left(-\left[\frac{F(x;)^{\alpha}}{1-F(x)^{\alpha}}\right]^{2}\right)\right]^{\eta} \tag{3}$$

And its associated PDF is given by

$$g(x)=\frac{2\alpha\eta f(x)F(x;)^{2\alpha-1}}{[1-F(x)^{\alpha}]^{3}}exp\left(-\left[\frac{F(x;)^{\alpha}}{1-F(x)^{\alpha}}\right]^{2}\right)\left[1-exp\left(-\left[\frac{F(x;)^{\alpha}}{1-F(x)^{\alpha}}\right]^{2}\right)\right]^{\eta-1} \tag{4}$$

By taking F(x) as the CDF of the Lomax distribution in (1), we obtain the CDF OF GBX-L distribution with a wide range of applications including ecology, medicine, and reliability. The four-parameter GBX-L distribution is given by

$$G(x)=\left[1-exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^{2}\right)\right]^{\eta}\ , \tag{5}$$

for $x>0; a,b,\alpha,\eta>0.$ The corresponding PDF is given by

$$g(x)=\frac{2b\alpha\eta}{a}\left(1+\frac{x}{a}\right)^{-b-1}\left[1-\left(1+\frac{x}{a}\right)^{-b}\right]^{2\alpha-1}exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^{2}\right)$$
$$\times\left[1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}\right]^{-3}\left[1-exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^{2}\right)\right]^{\eta-1} \tag{6}$$

for $x>0, a,b,\alpha,\eta>0.$

Plots of the CDF and PDF are given in Figure 1 for several values of $a,b,\alpha,$ and η. The plots of the CDF show that the GBX-L has a proper PDF and the plots of the PDF show that the GBX-L can both be symmetric and asymmetric shape.

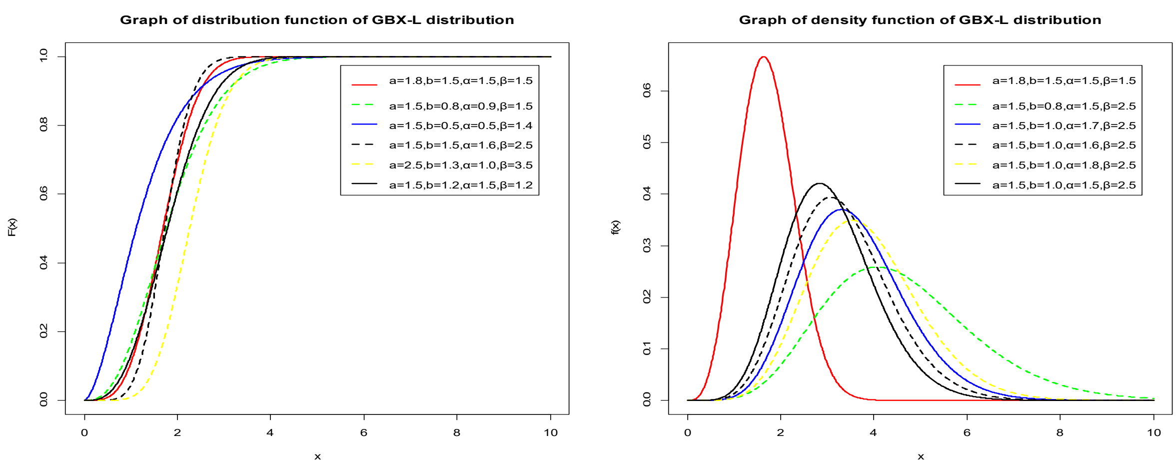

Figure 1. Plot of the CDF and the PDF of GBX-L distribution for different values of $a,b,\alpha,$ and η

2.1 Reliability, Hazard, Cumulative Hazard Function of GBX-L Model

The Reliability function $[R(x)]$ and the hazard function $[h(x)]$ for the GBX-L will be obtained in this sub-section. Using some values of $a, b, \alpha,$ and η, some plots of the survival and the hazard function are presented. The Reliability and the hazard function of the GBX-L are respectively given by

$$R(x) = 1 - G(x)$$

$$R(x) = 1 - \left[1 - exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^{2}\right)\right]^{\eta}, \tag{7}$$

$$h(x) = \frac{H^{*}\frac{2b\alpha\eta}{a}\left(1+\frac{x}{a}\right)^{-b-1}\left[1-\left(1+\frac{x}{a}\right)^{-b}\right]^{2\alpha-1} exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^{2}\right)}{1-\left[1 - exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^{2}\right)\right]^{\eta}} \tag{8}$$

for $x > 0, a > 0, b > 0, \alpha > 0,$ and $\eta > 0$**,** where,

$$H^{*} = \left[1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}\right]^{-3}\left[1 - exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^{2}\right)\right]^{\eta-1}$$

The graph the hazard function is drawn below in figure 2 for various values of $a, b, \alpha,$ and η. This graph indicates that new GBX-L model is capable of modeling upside-down bathtub (unimodal), increasing, decreasing hazard rate functions which are widely used in engineering for repairable systems.

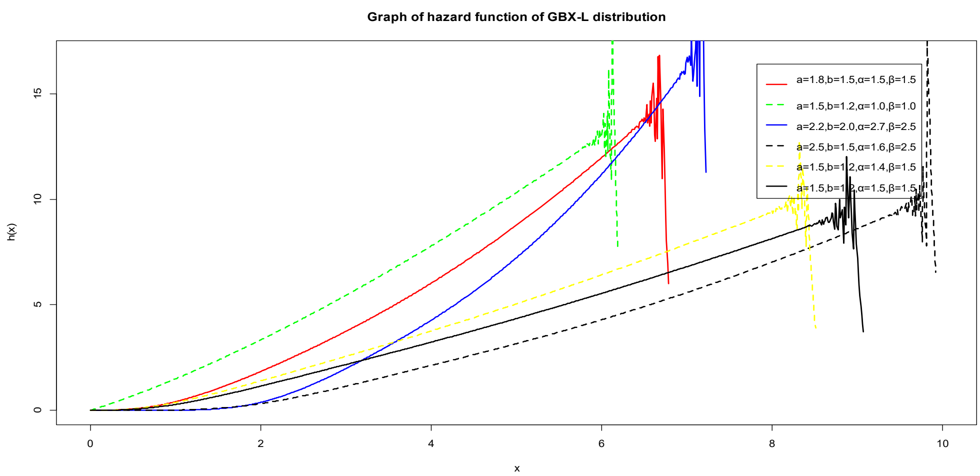

Figure 2. Plot of the hazard function of GBX-L model

3. Some Statistical Properties

This section provides some statistical properties of GBX-L distribution.

3.1 Quantile Function GBX-L Distribution

Let X denotes a random variable has the pdf (6), the quantile function, say $Q(u)$ of X is given by

$$Q(u) = a\left(\left[1-\left\{\frac{-log\left(1-u^{1/\eta}\right)}{1-log\left(1-u^{1/\eta}\right)}\right\}^{1/\alpha}\right]^{-1/b}-1\right) \tag{9}$$

where, u is a uniform distribution on the interval $(0,1)$ and In particular, the first quartile, the median, and the third quartile are obtained by putting $u = 0.25, 0.5$ an 0.75, respectively, in (9).

3.2 Important Representation

In this subsection, a useful expansion of the pdf for GBX-L is provided.

Since the generalized binomial series is

$$(1-z)^m = \sum_{i=0}^{\infty}(-1)^i\binom{m}{i}z^i \tag{10}$$

And

$$(1-z)^{-m} = \sum_{i=0}^{\infty}\binom{m+i-1}{i}z^i \tag{11}$$

$|z| < 1$ and m is a positive real non-integer. Then, by applying the binomial theorem (10) in (11), the density function of GBX-L distribution becomes

$$g(x) = \frac{2b\alpha\eta}{a}\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+k+l}}{j!\,(i+1)^{-j}}\binom{\eta-1}{i}\binom{-2j-3}{k}\binom{\alpha[2j+k+2]-1}{l}\left(1+\frac{x}{a}\right)^{-(b[l+1]+1)}$$

$$g(x) = \sum_{l=0}^{\infty}\xi_{i,j,k}\,f(x;a,(b[l+1]+1))$$

where

$$\xi_{i,j,k} = \frac{2\alpha\eta b}{(b[l+1]+1)}\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+k+l}}{j!\,(i+1)^{-j}}\binom{\eta-1}{i}\binom{-2j-3}{k}\binom{\alpha[2j+k+2]-1}{l}$$

and $g(x;a,(b[l+1]+1))$ is the Lomax PDF with parameters with positive shape parameter $(b[l+1]+1)$ and positive scale parameter a. This shows that the GBX-L model can be written as a linear combination of Lomas density functions. Hence mathematical properties of the GBX-L can be obtained from the Lomax properties.

Hence, the pdf (6) can be written as

$$g(x) = \frac{1}{a}\sum_{l=0}^{\infty}\xi_{i,j,k}(b[l+1]+1)\left(1+\frac{x}{a}\right)^{-(b[l+1]+1)} \tag{12}$$

3.3 Moments of GBX-L Distribution

Moments are very important in carrying out any statistical analysis, especially in applications. Therefore, we derive the v^{th} moment for the GBX-L distribution. If X has the pdf (6), then v^{th} moment is obtained as follows:

$$\mu'_v = \int_{-\infty}^{\infty}x^v f(x)dx \tag{13}$$

Putting (12) in (13), we have

$$\mu_v' = \frac{1}{a}\sum_{l=0}^{\infty} \xi_{i,j,k}(b[l+1]+1) \int_{-\infty}^{\infty} x^v \left(1+\frac{x}{a}\right)^{-(b[l+1]+1)} dx \tag{14}$$

Consequently, taking $u = {}^{x}/_{a}, adu = dx$, then (14) we transform to

$$\mu_v' = a^v \sum_{l=0}^{\infty} \xi_{i,j,k}\,(b[l+1]+1) \int_{-\infty}^{\infty} u^v (1+u)^{-(b[l+1]+1)} dx \tag{15}$$

By letting, $u = \frac{z}{1-z}, 1+m = (1-z)^{-1}, dm = (1-z)^{-2}dx$ and putting it in (15), we have

$$\mu_v' = a^v \sum_{l=0}^{\infty} \xi_{i,j,k}\,(b[l+1]+1) \int_{-\infty}^{\infty} z^v (1-u)^{-(b[l+1]+1)} dz$$

Since,

$$\int_{-\infty}^{\infty} z^v (1-u)^{-(b[l+1]+1)} dz = B[(v+1),\{b(l+1)-v\}]$$

where, $B(p,q)$ represents the standard beta function defined by $B(p,q) = \int_0^1 j^{p-1}(1-j)^{q-1}dj$ with $u > 0$ and $v > 0$. Finally, we obtained an expression for the v^{th} moment of $GBX-L$ given as

$$\mu_v' = a^v b \sum_{l=0}^{\infty} \xi_{i,j,k}(b[l+1]+1)\, B[(v+1),\{b(l+1)-v\}] \tag{16}$$

Table 1 drawn below give various values for the first six moment, coefficient of variation (CV) skewness(σ_s) and kurtosis (σ_k) for fixed values of $a = 3.0$ and $b = 5.5$ for $GBX-L$ distribution varying the values of α and η.

Table 1. The first six moments, CV σ_s and σ_k of $GBX-L$ distribution

$moments$	$\alpha = 0.5$, $\eta = 1.0$	$\alpha = 1.5$, $\eta = 2.0$	$\alpha = 1.5$, $\eta = 2.5$	$\alpha = 2.0$, $\eta = 2.5$	$\alpha = 2.5$, $\eta = 3.5$
μ_1'	0.2589	1.2298	1.2895	1.6274	2.0217
μ_2'	0.0929	1.6066	1.7461	2.7553	4.1924
μ_3'	0.0399	2.2085	2.4675	4.8322	8.9015
μ_4'	0.0195	3.1717	3.6214	8.7474	19.3228
μ_5'	0.0105	4.7337	5.4986	16.2978	42.8263
μ_6'	0.0061	7.3115	8.6103	31.1718	96.7994
σ^2	0.0259	0.0942	0.0833	0.1069	0.1051
CV	0.6216	0.2496	0.2238	0.2009	0.1604
σ_s	24.4650	−18.0960	−40.0071	−95.1116	−245.002
σ_k	886.5888	−95.6387	−335.6256	−1093.98	−3415.411

3.4 Incomplete Moment of GBX-L Distribution

More importantly, the first incomplete moment can be used to obtain the mean and the Bonferroni and Lorenz curves. The curves are very important in reliability, economics, demography, medicine, insurance and many others. The $\boldsymbol{v^{th}}$ incomplete moment, say $\boldsymbol{\phi_v(t)}$ can be expressed from (12) as

$$\phi_v(t) = \frac{1}{a}\sum_{l=0}^{\infty} \xi_{i,j,k} \int_{0}^{t} x^v \left(1+\frac{x}{a}\right)^{-(b[l+1]+1)} dx \tag{17}$$

Taking $u = {}^{x}/_{a}, adu = dx$, then (17) we transform to

$$\phi_v(t) = a^v \sum_{l=0}^{\infty} \xi_{i,j,k} \int_0^t u^v (1+u)^{-(b[l+1]+1)} dx \tag{18}$$

By letting, $u = \frac{z}{1-z}, 1 + m = (1-z)^{-1}, dm = (1-z)^{-2} dx$ and putting it in (18), we have

$$\phi_v(t) = a^v \sum_{l=0}^{\infty} \xi_{i,j,k} \int_{-\infty}^{\infty} z^v (1-u)^{-(b[l+1]+1)} dz$$

Finally, we have

$$\phi_v(t) = a^v b \sum_{l=0}^{\infty} \xi_{i,j,k}\, B\left[(v+1), \{b(l+1)-v\}; \frac{t}{1-t}\right] \tag{19}$$

3.5 The Probability Weighted Moments

Considering the class of moments, called the probability-weighted moments (PWMs), has been proposed by (Greenwood et al. (1979)). This class is used to derive estimators of the parameters and quantiles of distributions expressible in inverse form. For a random variable X, the PWMs, denoted by $\varphi_{r,s}$, can be calculated through the following relation

$$\varphi_{r,s} = E[X^r (F(x))^s] = \int_{-\infty}^{\infty} x^r f(x) (F(x))^s dx \tag{19}$$

The PWMs of $GBX - L$ distribution is obtained by substituting (5) and (6) into (19), and using the binomial series given in (10) and (11), we have

$$\varphi_{r,s} = 2b\alpha\eta \sum_{m,p,q,v}^{\infty} \frac{a^v}{p!} (-1)^{m+p+v} \binom{\eta(s-1)-1}{m} \binom{4+q}{v} \binom{\alpha q + 2\alpha + 1}{v}$$
$$\times B(r+1, -[rb+1]) \tag{20}$$

3.6 Characteristics Function of GBX-L

The characteristic function of a distribution is always unique and is related to the moments of the distribution by

$$\phi_x(t) = E(e^{it}) = \int_{-\infty}^{\infty} e^{it} f(x) dx = \sum_{v=0}^{\infty} \frac{(it)^v E(X^v)}{v!} = \sum_{v=0}^{\infty} \frac{(it)^v}{v!} \mu_v' \tag{21}$$

Putting (18) in (21), we obtained the characteristics function of $GBL - X$ distribution as

$$\phi_x(t) = a^v b \sum_{l,v=0}^{\infty} \frac{(it)^v}{v!} \xi_{i,j,k} B[(v+1), \{b(l+1)-v\}] \tag{22}$$

3.7 Mean Residual Life of GBX-L Distribution

$$m(t) = E[X - t/X > t] = \frac{\int_t^{\infty} (x-t) f(x) dx}{R(t)} = \frac{\mu_1' - \int_0^t x f(x) dx}{R(t)} - t$$

The integral $\int_0^t x f(x) dx$ is the incomplete moment of GBX-L and μ_1' is the first moment obtained by taking $v = 1$ in (16) and $S(t)$ is given in (7).

$$m(t)=\frac{\mu_1'-a^v b\sum_{l=0}^{\infty}\xi_{i,j,k}\,B\left[(v+1),\{b(l+1)-v\};\frac{t}{1-t}\right]}{1-\left[1-exp\left(-\left[\frac{\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{a}\right)^{-b}\right)^{\alpha}}\right]^2\right)\right]^{\eta}}-t \tag{23}$$

3.8 Stress-Strength Model of $\boldsymbol{GBX-L}$ Distribution

Stress-Strength model is the most commonly used approach in reliability estimation. This model can be applied in engineering and physics as system collapse and strength failure. In stress-strength modeling, $\mathrm{R}=\boldsymbol{Pr(X_2<X_1)}$ is a measure used in determining the reliability of a system when it is subjected to random stress $\boldsymbol{X_2}$ and has strength $\boldsymbol{X_1}$. The system fails if the applied stress is greater than its strength and function satisfactorily whenever $\boldsymbol{X_1>X_2}$. R can be taken a measure of system performance is commonly encountered in electrical and electronic systems. Let $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ be two independent random variables having $\boldsymbol{GBX-L(\alpha_1,\eta_1,a,b)}$ and GBX-L$(\boldsymbol{\alpha_2,\eta_2,a,b})$ distribution. Then, we can write

$$\mathrm{R}=\int_{0}^{\infty}\boldsymbol{f_1(x;\alpha_1,\eta_1,a,b)F_1(x;\alpha_2,\eta_2,a,b)dx} \tag{24}$$

$$=\boldsymbol{b}\sum_{\boldsymbol{i,j,k,l,p,q,r=0}}^{\infty}\boldsymbol{(-1)^z(i+1)^k\binom{\eta_2}{i}\binom{\eta_1-1}{j}\binom{2k+p}{p}\binom{2l+q-1}{q}\binom{v}{r}B[1,-(b+r)]}$$

Where,

$\boldsymbol{z=i+j+k+l+r}$ and $\boldsymbol{v=\alpha_1(2k+p+2)+\alpha_2(2+q)-1}$

3.9 Order Statistics of $\boldsymbol{GBX-L}$ Distribution

Order statistics have been extensively studied and found applications in many applied fields of statistics, such as reliability and life testing. Let $X_1,X_2,\dots,X_n$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{1:n}<X_{2:n}<\cdots<X_{n:n}$ the corresponding ordered random sample from a population of size n. According to (David, 1981), the pdf of the s^{th} order statistic, is defined as

$$f_{s:n}=\frac{f(x)}{B(s,n-s+1)}\sum_{i=0}^{n-s}(-1)^s\binom{n-s}{i}F(x)^{i+s-1} \tag{25}$$

Putting (5) and (6) in (25), we have

$$f_{s:n}=\frac{f(x)}{B(s,n-s+1)}\sum_{i=0}^{n-s}(-1)^s\binom{n-s}{i}F(x)^{i+s-1} \tag{26}$$

Putting equation (5) and (6) in (26) and thereafter applying (10) and (11), the s^{th} order statistics of GBXL is given by

$$f_{s:n}=\frac{2b\alpha\eta}{aB(s,n-s+1)}\sum_{i=0}^{n-s}\sum_{j+k+l+m+p}^{\infty}\frac{(-1)^{s+j+k+m+p}}{k!}\binom{\eta(1+s)-1}{j}\binom{2k+l-1}{l}$$
$$*\binom{\alpha(l+2)-2}{m}\binom{m\alpha}{p}(j+1)^k\left(1+\frac{x}{a}\right)^{-[b(1+p)+1]} \tag{27}$$

3.10 Renyi Entropy of $\boldsymbol{GBX-L}$ Distribution

The entropy of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics. As mentioned by (Renyi 1961), the Renyi entropy is defined by

$$I_\rho(X)=\frac{1}{1-\rho}log\int_{-\infty}^{\infty}f(x)^{\rho}dx,\qquad \rho>0\ and\ \rho\neq 0 \tag{28}$$

By putting (6) in (28) and applying the binomial theory given (10) and (11), then the pdf $f(x)^{\rho}$ can be expressed as follows

$$f(x)^{\rho} = a\sum_{m,p,q,r}^{\infty}\frac{(-1)^{m+p+r}}{p!}(m+\rho)^{p}\binom{\rho(\eta-1)}{m}\binom{2p+q+2}{q}\binom{\alpha(2p+q+2\rho)-\rho}{r}$$

$$\times B[1,\rho b+\rho-r+3]$$

Finally, an expression for the entropy of $BGX-L$ distribution is given by

$$I_{\rho}(X) = \frac{1}{1-\rho}log\left[(2ba\eta)^{\rho}a^{1-\rho}\sum_{m,p,q,r}^{\infty}\frac{(-1)^{m+p+r}}{p!}(m+\rho)^{p}\binom{\rho(\eta-1)}{m}\binom{2p+q+2}{q}\binom{\alpha(2p+q+2\rho)-\rho}{r}B[1,\rho b+\rho-r+3]\right]$$

4. Simulation Study

To conduct a simulation study, (9) is used to generate random data from the Generalized Bur X-Lomax distribution. The simulation experiment is repeated for 1000 times each with sample of size n = 50, 100, 150 and 200 for parameter values of $a=0.5, b=0.5, \alpha=1.3,$ and $\eta=1.3$. Table 2 demonstrates the Mean estimate (ME), Absolute Bias (AB), Standard error (SE) and Mean square error (MSE).

Table 2. ME, AB, SE and MSE of distribution

$Par.$	Sample size	ME	AB	SE	MSE
$a=0.4$	50	0.1263	0.2737	0.4157	0.2477
	100	0.1001	0.2999	0.1771	0.1213
	150	0.0744	0.3256	0.0747	0.1116
	200	0.1305	0.2695	0.0955	0.0817
$b=0.5$	50	0.0461	0.4539	0.0885	0.2139
	100	0.0388	0.4612	0.0196	0.2131
	150	0.1291	0.3709	0.0742	0.1431
	200	0.1367	0.3633	0.0671	0.1365
$\alpha=1.3$	50	0.2628	1.0372	0.3941	1.2311
	100	0.1893	1.1107	0.0439	1.2356
	150	0.5745	0.7255	0.4126	0.6966
	200	0.8039	0.4961	0.3019	0.3372
$\eta=0.5$	50	1.4878	0.9878	1.3878	2.9017
	100	1.5215	1.0215	0.4258	1.2248
	150	0.6463	0.1463	0.3271	0.1284
	200	0.6466	0.1466	0.2934	0.1076

4.1 Maximum Likelihood Method

This section deals with the maximum likelihood estimators of the unknown parameters for the GBXL distributions based on the principle of complete samples. Let $X_1, X_2, \ldots, X_n$ represent the observed values from the GBXL distribution with set of parameter $\omega=(a,b,\alpha,\eta)^{T}$. The log-likelihood function $l=logL$ for parameter vector $\omega=(a,b,\alpha,\eta)^{T}$ is obtained as follows

$$l(\underline{x}) = log\left(\frac{b\alpha\beta}{a}\right)-(b-1)\sum_{i=1}^{n}Z_i+(2\alpha-1)\sum_{i=1}^{n}log\left[1-Z_i{}^{-b}\right]-\sum_{i=1}^{n}\left(\frac{\left[1-Z_i{}^{-b}\right]^{\alpha}}{1-\left[1-Z_i{}^{-b}\right]^{\alpha}}\right)^{2}$$

$$-\sum_{i=1}^{n}log\left(1-\left[1-Z_i{}^{-b}\right]^{\alpha}\right)+(\eta+1)\sum_{i=1}^{n}log\left(1-exp\left[-\left(\frac{\left[1-Z_i{}^{-b}\right]^{\alpha}}{1-\left[1-Z_i{}^{-b}\right]^{\alpha}}\right)\right]\right)$$

Where, $Z_i = \left(1 + \frac{x_i}{a}\right)$

The maximum likelihood (ML) method and its procedures exist in the literature with details.

4.2 Applications to Real Data

In this section, a real data set is employed to illustrate the importance of the developed GBXL. Recently, considerable extensions of Lomax distribution have been introduced, in the literature, by several authors, such as Harris Power Lomax (HPL) by Ogunde et al. (2021), Power Lomax (PL) by Rady et al.(2016) and Harris Lomax (HL) distribution. We fit GBX-L distribution to the two real data sets using MLEs and compared the suggested distribution with HPL, HL, PL and L distributions using different information criteria including: the $-2\hat{l}$(Maximized Log-likelihood), Akaike information criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian information criterion (BIC), and Hannan Quinn information criterion (HQIC). The mathematical form of these criteria are given by

$AIC = -2\hat{l} + 2z, \quad BIC = -2\hat{l} + zlog(n), \quad HQIC = -2\hat{l} + 2zlog[\log(n)]$ and

$$CAIC = -2\hat{l} + 2zn/(n - z - 1)$$

Where, l is the maximized likelihood function, z stands for the number of the model parameters and n is the sample size of the data considered. The model with minimum AIC (or CAIC, BIC, and HQIC) value is chosen as the best model to fit the data. Finally, we provide a representation of the histograms of the data sets and plot the fitted density functions to obtain a visual representation of the data set.

The data set represents the failure times of 84 Aircraft windshields recently studied by Ramos et al. (2013). The data set values are 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309,1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912,2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661,3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432,2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154,2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103,4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485,1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,2.324, 3.376, 4.663. Figure 3 represent the total test on test (TTT) plot and the Box plot for the aircraft data. The TTT plot indicates that the aircraft data exhibits an increasing failure rate and the Boxplots also indicates that the data is the skewed to the right (positively skewed). Figure 4 is shows the fitted densities of the distribution to the data. Table 2 represents the maximum likelihood estimates of the GBX-L model for aircraft data. Table 3 represents the goodness of fit criteria including AIC, CAIC, BIC, and HQIC. The numerical values in Table 3 indicate that the GBX-L model has the minimum value of the information criterion than all other models considered. Hence, we conclude that the GBX-L distribution perform better as compared to Harris Power Lomax, Power Lomax, Harris Lomax and Lomax distribution.

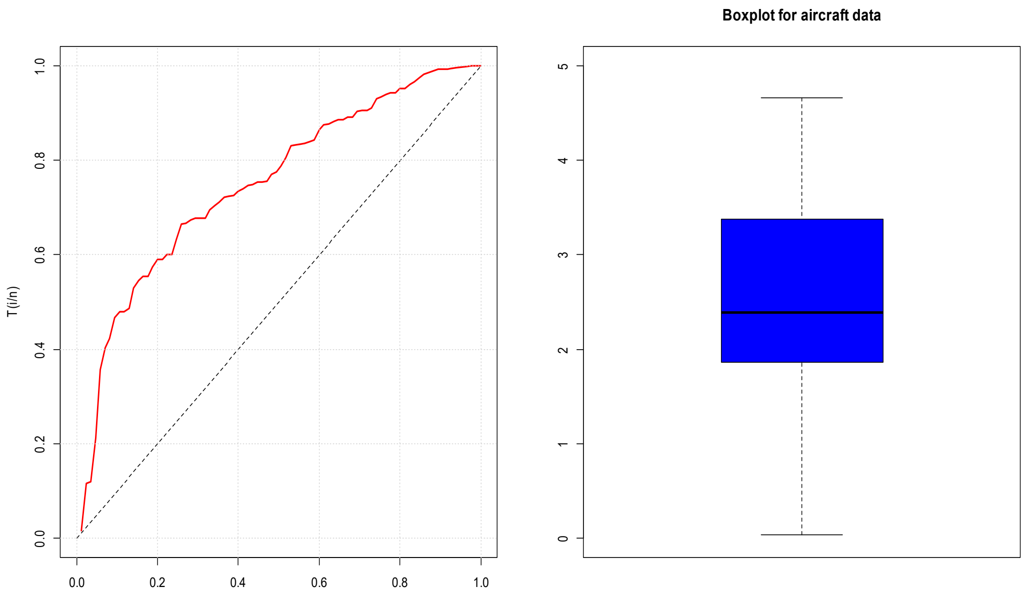

Diagram I Diagram II

Figure 3. Graph of TTT plot (Diagram I) and Box plot (Diagram II)

Table 6. the statistics AIC, BIC, HQIC, CAIC, K, P-value values for Aircraft data

$Model$	$Measures$				
	$-2l$	AIC	BIC	$CAIC$	$HQIC$
$GBX-L$	130.063	268.126	277.896	268.626	272.056
HPL	131.011	272.021	284.238	272.781	276.934
HL	149.558	307.115	317.563	307.537	311.333
PL	169.250	344.501	351.829	344.797	347.448
L	200.268	404.536	409.746	404.660	406.644

The Likelihood Ratio (LR) statistic was obtained for testing the hypotheses H_0: α=1 $versus$ H_1 = H_0 $is\ not\ true$, that is to compare the $GBX-L$ model with the L model. The LR statistic w = -2{130.063-200.268} = 140.41($p-value$ < 0.01), sufficient to show that the $GBX-L$ model is a better model that can be used to fit the data.

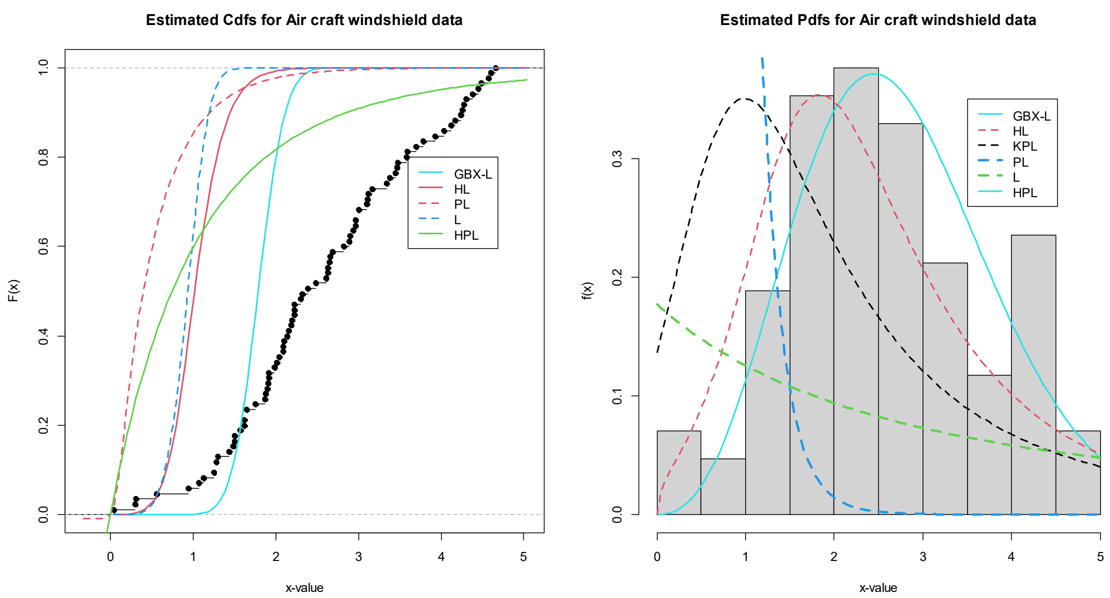

Figure 4. Plots of estimated CDF and histogram fitted PDF of the fitted models for the aircraft data

5. Conclusion

In the present paper, the new Generalized Burr X Lomax distribution is proposed and studied. Some characteristics of the GBX-L distribution, such as, expressions for the density function, moments, incomplete moment, probability weighted moments, characteristic function, quantile function, mean residual life, stress-strength model, orders statistics and Renyl entropy are discussed. The maximum likelihood estimation technique is employed for estimating the model parameters. Aircraft data is employed to validate the relevance of GBL-X Model when compared to other models such as Harris power Lomax, Harris Lomax, Power Lomax, and Lomax models. we also carried out data simulation to validate the method of estimation.

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