Journal of Mathematics Research; Vol. 14, No. 2; April 2022
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Modified Extended Inverted Weibull Distribution with Application to Neck Cancer Data

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Received: January 11, 2022 Accepted: March 21, 2022 Online Published: March 24, 2022

doi:10.5539/jmr.v14n2p39 URL: https://doi.org/10.5539/jmr.v14n2p39

Abstract

This work introduces a new three-parameter modified extended inverted Weibull (MEIW) distribution which is a hybrid of the one-parameter inverted Weibull distribution. The density function of the MEIW can be expressed as a linear combination of the inverted Weibull densities. Some mathematical properties of the proposed MEIW model such as ordinary and incomplete moments, mean residual life, and mean waiting time, Tsallis entropy, moment generating function and order statistics are investigated. The maximum likelihood estimation method is considered to estimate the parameters of the MEIW model. The relevance of the MEIW model is studied via an application to neck cancer data.

Keywords: Tsallis entropy, maximum likelihood estimation, mean waiting time, modified extended inverted Weibull distribution

1. Introduction

The Inverted Weibull distribution (IW) distribution is also known as one-parameter inverse Weibull distribution, which can be applied in biological and reliability studies with some monotone failure rates. Several modifications of the IW distribution have been studied, which includes: Flaih et al. (2012) studied the properties of Exponentiated Inverted Weibull distribution and applied it to model the breaking strength of carbon fibers; Ogunde et al. (2017a, 2017b) investigated the properties of the transmuted inverted Weibull distribution and Exponentiated transmuted inverted Weibull respectively; Ogunde et al. (2020) proposed and studied the properties of Alpha Power Extended Inverted Weibull Distribution.

The cumulative distribution function (CDF) of the IW distribution is given by

$$G_{IW}(x;\varphi) = e^{-x^{-\varphi}}, \qquad x > 0, \varphi > 0 \tag{1}$$

Its corresponding probability density function (PDF) has the following form:

$$g_{IW}(x;\varphi) = \varphi x^{-\varphi-1} e^{-x^{-\varphi}}, \qquad x > 0, \varphi > 0 \tag{2}$$

Several methods have been developed in recent decades to improve the fit of a statistical distribution. Namely: Marshal–Olkin generated family of distributions developed by Marshall and Olkin (1997), the beta-G family of distributions was developed and studied by Eugene, Lee and Famoye (2002), gamma-G family was proposed by Zografos and Balakrishnan (2009) and generalized distributions by Cordeiro and de Castro (2011). With the present advancement in technology, a number of efficient techniques have been developed to generate a new family of distributions. The most recently developed families of distributions include the transmuted geometric family by Afify et al. (2016), the beta transmuted family was developed by Afify et al. (2017), Alizadeh et al. (2017) proposed and studied the generalized transmuted family of distributions, exponentiated logarithmic generated family was developed by Marinho et al. (2018), Alizadeh et al. (2018) developed and studied the transmuted Weibull-G family of distributions. Ali Al-Shomrani et al. (2016) proposed the Topp–Leone Family of Distributions, Exponentiated generalized Topp Leone-G family of distributions was developed by Reyad et al. (2019). Sine Topp-Leone-G family of distributions was studied by Al-Babtain et al. (2020), and many more others.

The main motivation of this paper is to extend the standard inverted Weibull distribution to the standard modified extended inverted Weibull distribution by adding two extra shape parameters; the shape parameter are to be address the lack of fit of the inverted Weibull distribution for modeling lifetime data which exhibited non-monotone failure rates.

1.1 Modified Extended Inverted Weibull Model

The inverted Weibull (IW) distribution is extended using the Exponentiated-G family of distributions given by Cordeiro et al. (1998), to obtain

$$F^{EW}(x;\varphi,\rho) = 1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}, \qquad x>0 \tag{3}$$

Where φ and ρ are positive shape parameters. The associated PDF is given by

$$f^{EW}(x;\varphi,\rho) = \rho\varphi x^{-\varphi-1}e^{-x^{-\varphi}}\left(1-e^{-x^{-\varphi}}\right)^{\rho-1}, \quad x>0 \tag{4}$$

Suppose that the random variable X has the Extended Inverted Weibull distribution where its CDF and PDF are given in equations (3) and (4). Given N, let $X_1,\dots,X_N$ be independent and identically distributed random variables with Extended Inverted Weibull distribution. Let N be distributed according to the zero truncated Poisson distribution by Cohen (1960) PDF

$$P(N=n) = \frac{\theta^n e^{-\theta}}{n!\,(1-e^{-\theta})}, \qquad n=1,2,\dots\theta>0$$

Let $Y=\max(Y_1,\dots Y_N)$, then the marginal CDF of $X/N=n$ is given by

$$G_{X/N=n}(x;\varphi,\rho,\theta) = \left\{1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right\}^{n}, \quad x>0;\varphi,\rho,\theta>0 \tag{5}$$

which is the modified exponentiated Inverted Weibull distribution denoted by MEIW(φ,ρ,θ) is defined by the marginal cdf of X, that is,

$$F(x;\varphi,\rho,\theta) = \frac{1-e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]}}{(1-e^{-\theta})}, \quad x>0;\varphi,\rho,\theta>0 \tag{6}$$

The $MEIW$ density function is given by

$$f(x;\varphi,\rho,\theta) = \frac{1}{(1-e^{-\theta})}\rho\varphi\theta x^{-\varphi-1}e^{-x^{-\varphi}}\left(1-e^{-x^{-\varphi}}\right)^{\rho-1} e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]} \tag{7}$$

The expression for the survival and the hazard function is, respectively, given by

$$S(x;\varphi,\rho,\theta) = \frac{e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]}-e^{-\theta}}{(1-e^{-\theta})} \tag{8}$$

and

$$h(x;\varphi,\rho,\theta) = \frac{\rho\varphi\theta x^{-\varphi-1}e^{-x^{-\varphi}}\left(1-e^{-x^{-\varphi}}\right)^{\rho-1} e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]}}{e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]}-e^{-\theta}} \tag{9}$$

Figures 1, 2 and 3 are, respectively, the graph of the density, survival and hazard functions of MEIW distribution for different arbitrary values of the parameters of the distribution.

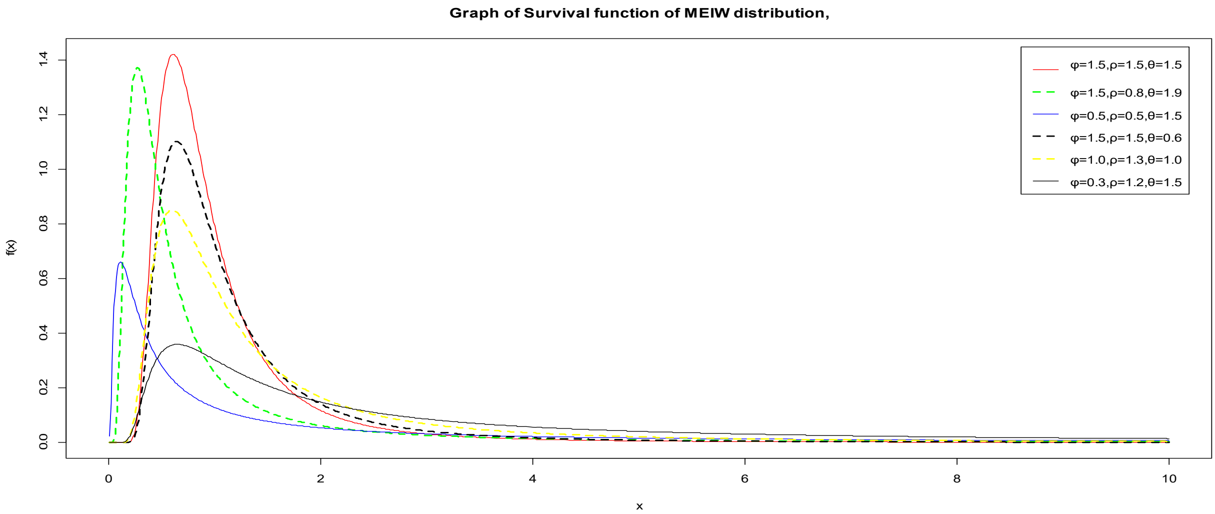

Figure 1. Plot of density of MEIW distribution

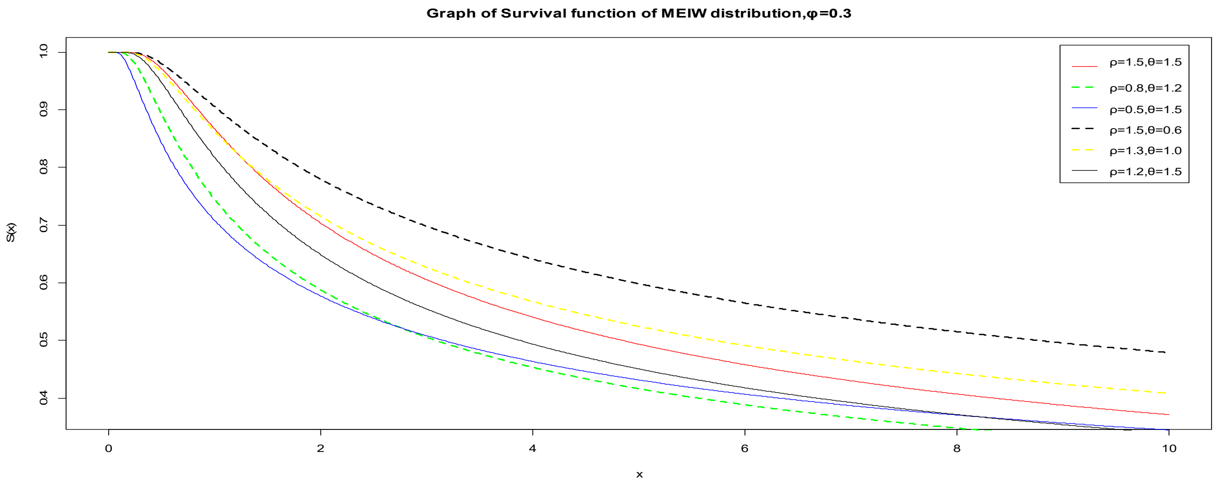

Figure 2. Plot of survival function of MEIW distribution

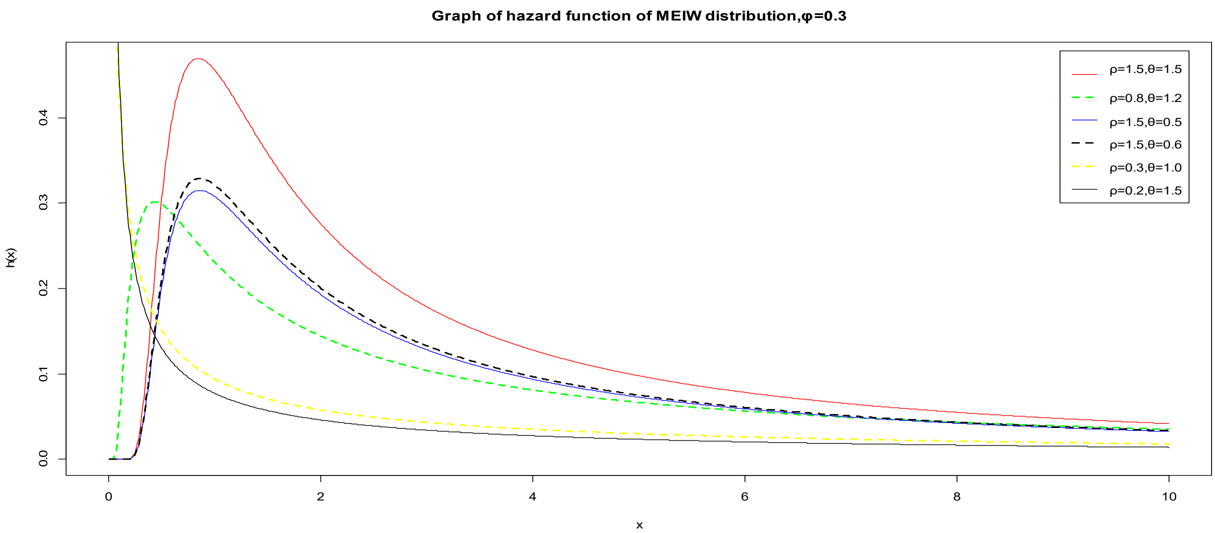

Figure 3. Plot of hazard function of MEIW distribution

The MEIW distribution is a flexible model that contains some distributions as special models:

(i) If $\theta = 1$, then the MEIW distribution reduces to the Extended Inverted Weibull (EIW) distribution.

(ii) If $\rho = 1$, then the MEIW distribution reduces to the Poisson Inverted Weibull (PIW) distribution.

(iii) If $\theta = \rho = 1$, then the MEIW distribution reduces to the Inverted Weibull (IW) distribution.

2. Linear Representation

Here, we present the MEIW PDF as a mixture linear representation of IW density. Consider the following power series:

$$(1-w)^f = \sum_{n=0}^{\infty} (-1)^n \binom{f}{n} w^n \tag{10}$$

$$e^v = \sum_{r=0}^{\infty} \frac{v^r}{r!} \tag{11}$$

Using equation (11) to simplify $e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]}$, the resulting equation can be written as

$$f(x) = \frac{\rho\varphi\theta}{(1-e^{-\theta})} x^{-\varphi-1} e^{-x^{-\varphi}} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{\rho(j+1)i!} \theta^i \binom{j}{i} \rho(j+1)\left(1-e^{-x^{-\varphi}}\right)^{\rho(j+1)-1}, \tag{12}$$

It should be noted that (12) is an Exponentiated Generalized Inverted Weibull distribution with parameters φ and $\rho(j+1)$

Further, applying (10) in (12), after some algebra, the MEIW distribution PDF reduces to

$$f(x) = \frac{\rho\varphi\theta}{(1-e^{-\theta})} x^{-\varphi-1} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j}}{i!} \theta^i \binom{j}{i} \binom{\rho(j+1)-1}{k} e^{-(k+1)x^{-\varphi}}, \tag{13}$$

$$= \sum_{k=0}^{\infty} \lambda_k f(x; (k+1), \varphi)$$

Where,

$$\lambda_k = \frac{\rho\theta}{(1-e^{-\theta})} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j}}{i!} \theta^i \binom{j}{i} \binom{\rho(j+1)-1}{k}$$

Where $f(x; (k+1), \varphi)$ is the IW PDF with parameters $(k+1)$ and φ. This implies that (13) is a linear combination of the IW densities and some mathematical properties related to MEIW distribution can be obtained from those of the IW distribution.

Consider a random variable (RV) $Z \sim IW(\varphi)$ in (1). For $s < \varphi$, the s^{th} ordinary and incomplete moments of Z are given by

$$\mu'_{s,Z} = \Gamma\left(1 - {}^{s}/_{\varphi}\right), \quad s < \varphi \tag{14}$$

$$\mu'_{s,Z}(t) = \gamma\left(1 - {}^{s}/_{\varphi}, t^{\varphi}\right), \tag{15}$$

Respectively, where $\Gamma(f) = \int_0^{\infty} m^{f-1} e^{-m} dm$ is the complete gamma function and $\gamma(f,z) = \int_0^{z} m^{f-1} e^{-m} dm$ is the lower incomplete gamma function.

3. Some Statistical Properties

In this section, we examine some statistical properties of the MEIW distribution, such as ordinary moments, incomplete moments, moment generating function, mean deviation, and quantile function.

3.1 s^{th} Ordinary Moments of MEIW Distribution

The s^{th} ordinary moment of RV X is

$$\mu_s' = E(X^s) = \sum_{k=0}^{\infty} \lambda_k \int_{-\infty}^{\infty} x^s f(x; k+1, \varphi) dx \tag{16}$$

For $(s < \varphi)$, we obtain

$$\mu_s' = \sum_{k=0}^{\infty} \lambda_k (k+1)^{s-1/\varphi} \Gamma\left(1 - {s}/{\varphi}\right) \tag{17}$$

The coefficient of variation (Q_v)), Skewness (Q_s), and kurtosis (Q_k) can easily be obtained. The variance (σ^2), coefficient of variation (Q_v), coefficient of *skewness* (Q_s) and coefficient of kurtosis (Q_k) are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad Q_v = \frac{\sigma}{\mu} = \frac{(\mu_2' - \mu^2)^{1/2}}{\mu} = \left(\frac{\mu_2'}{\mu^2} - 1\right)^{1/2}$$

$$Q_s = \frac{E[(X-\mu)^3]}{\sqrt[2/3]{E(X-\mu)^2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{\sqrt[2/3]{(\mu_2' - \mu^2)}}$$

And

$$Q_k = \frac{E[(X-\mu)^4]}{(E[(X-\mu)^2])^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{\sqrt[2/3]{(\mu_2' - \mu^2)}}$$

respectively. Table 1 present the moments of MEIW distribution with some selected values of the parameters and for a fixed value of $\rho = 6.5$.

Table 1. Table of moments for $MEIW$ Distribution, Q_s and Q_k

	(θ, φ)					
Moments	(0.1,0.2)	(0.5,0.5)	(1.0,1.2)	(1.5,1.5)	(2.0,1.8)	(2.5,3.5)
(EX)	0.0986	0.2334	0.5229	0.6109	0.6809	1.2570
$E(X^2)$	0.0127	0.0710	0.3556	0.4826	0.6045	1.9964
$E(X^3)$	0.0022	0.0297	0.3351	0.5308	0.7404	4.3888
$E(X^4)$	0.0006	0.0186	0.4812	0.9048	1.4273	15.3749
$E(X^5)$	0.0003	0.0206	1.2524	2.8610	5.2386	105.3201
$E(X^6)$	0.0004	0.0727	10.4961	29.6747	64.4645	2491.8360
σ^2	0.0030	0.0165	0.0822	0.1094	0.1409	0.4164
Q_v	0.5555	0.5504	0.5483	0.5414	0.5513	0.5134
Q_s	2.2183	2.5492	2.6836	2.8275	2.9708	3.0993
Q_k	21.3767	18.9554	20.6433	22.6041	24.8778	27.3716

Hence, for $s = 1$ in (16), we obtain an expression for the mean of X.

Thus, the s^{th} incomplete moment of RV X is

$$\xi_s' = \sum_{k=0}^{\infty} \lambda_k \int_{-\infty}^{t} x^s f(x; (k+1, \varphi) dx \tag{18}$$

Additionally, we obtain an expression for the s^{th} incomplete moment of MEIW distribution given by

$$\xi_s(t) = \sum_{k=0}^{\infty} \lambda_k (k+1)^{s-1/\varphi} \gamma\left(1 - {s}/{\varphi}, (k+1)t^{\varphi}\right) \tag{19}$$

An expression for the incomplete moment of MEIW distribution given in (19) can be used to obtain an expression for

Bonferroni and Lorenz curves which have applications in medicine, insurance, reliability, demography, and insurance, which is also useful in economics for the study of income and poverty analysis

3.2 Mean Residual Life and Mean Waiting Time

The mean residual life (MRL) can be applied in economics, biomedical sciences, product quality control, insurance, and demography. It can be define as the expected additional life length for a unit that is alive at age t, and it is represented mathematically by $m_x(t) = E(X - t/X > t),\ t > 0$.

The MRL of X can be obtained by using the formula:

$$m_x(t) = \frac{[1 - \xi_1(t)]}{S(t)} - t, \tag{20}$$

Where $S(t)$ is the survival function of X. By inserting (19) in (20), the value of the MEIW distribution is given as

$$m_x(t) = \frac{1}{S(t)} \sum_{k=0}^{\infty} \lambda_k \gamma\left(1 - {}^{1}/_{\varphi}, (k+1)t^{\varphi}\right) - t \tag{21}$$

The mean inactivity time (MIT) (mean waiting time) is defined by $M_x(t) = E(tX/X \le t),\ t > 0$, and it can be obtained by the formula:

$$M_x(t) = t - \left[\frac{\xi_1(t)}{F(t)}\right] \tag{22}$$

By substituting (19) in (22), the MIT of the MEIW distribution is given as

$$M_x(t) = t - \frac{1}{F(t)} \left[\sum_{k=0}^{\infty} \lambda_k \gamma\left(1 - {}^{1}/_{\varphi}, (k+1)t^{\varphi}\right)\right] \tag{23}$$

4. Order Statistics

Consider a random sample from the MEIW (ρ, φ, θ) denoted by $\underline{X}$ of size m and have the following order statistics denoted by $X_{1:m} < X_{2:m} <\ldots< X_{m:m}$. Then, the PDF of the r^{th}order statistics is given by

$$f_{r:m}(x) = \frac{m!}{(r-1)!\,(m-r)!} F^{r-1}(x)[1 - F(x)]^{m-r} f(x)$$

$$f_{r:m}(x) = \frac{m!}{(r-1)!\,(m-r)!} \sum_{i=0}^{m-r} (-1)^i \binom{m-r}{i} F^{i+r-1}(x) f(x) \tag{24}$$

Considering $F^{j+r-1}(x)f(x)$ and further applying Taylor series given in (10), we have

$$F^{i+r-1}(x)f(x) = \rho\varphi\theta \sum_{j,k,l,p=0}^{\infty} \frac{(-1)^{j+k+l+p}}{j!} \binom{k}{l} \binom{\rho(l+1)-1}{p} [\theta(j+1)]^k$$

$$\times \binom{i+r-1}{j} \left(\frac{1}{1-e^{-\theta}}\right)^{i+r} x^{-\varphi-1} e^{-(l+1)x^{-\varphi}} \tag{25}$$

Combining (24) and (25), we have an expression for the r^{th}order statistics of MEIW distribution as

$$f_{r:m}(x) = \frac{\rho\varphi\theta m!}{(r-1)!\,(m-r)!} \sum_{i=0}^{m-r} \sum_{j,k,l,p=0}^{\infty} \frac{(-1)^{j+k+l+p}}{j!} \binom{k}{l} \binom{\rho(l+1)-1}{p} [\theta(j+1)]^k$$

$$\times \binom{i+r-1}{j} \left(\frac{1}{1-e^{-\theta}}\right)^{i+r} x^{-\varphi-1} e^{-(l+1)x^{-\varphi}} \tag{26}$$

5. Entropy

Measure of entropy plays a vital role in reliability analysis. It is used to determine the amount of variation and uncertainty in the dataset. If the value of entropy is large, it indicates large uncertainty in the data. Hence, for measuring the amount of uncertainty of a random variable x following the MEIW distribution, Renyi (1960) and Havarda and Charvat (1967) entropies are considered. The entropies are as follows:

$$R_x(v) = \frac{1}{v-1} log\left[\left(\frac{\rho\theta}{1-e^{-\theta}}\right)^v \varphi^{v-1}\mathcal{W}_{ijk}(v)\right] \quad (27)$$

where

$$\mathcal{W}_{ijk}(v) = \sum_{i,j,k}^{\infty}(-1)^{i+j+k}\binom{j}{i}\binom{\rho j+v(\rho+1)}{k}\frac{(\theta v)^i}{i!}(k+1)^{\frac{1-v(\varphi+1)}{\varphi}}\Gamma\left(1+\frac{(\varphi-1)(v-1)}{\varphi}\right)$$

$$R_x(q) = \frac{1}{q-1}\left[1-\left(\frac{\rho\theta}{1-e^{-\theta}}\right)^q \varphi^{q-1}\mathcal{W}_{ijk}(q)\right] \quad (28)$$

Where

$$\mathcal{W}_{ijk}(q) = \sum_{i,j,k}^{\infty}(-1)^{i+j+k}\binom{j}{i}\binom{\rho j+q(\rho+1)}{k}\frac{(\theta q)^i}{i!}(k+1)^{\frac{1-q(\varphi+1)}{\varphi}}\Gamma\left(1+\frac{(\varphi-1)(q-1)}{\varphi}\right)$$

Proof. By definition, Renyi and q-entropy are:

$$R_x(v) = \frac{1}{v-1} log\left\{\int_{-\infty}^{\infty} f^v(x)dx\right\}, \quad (29)$$

$$R_x(q) = \frac{1}{q-1}\left\{1-\int_{-\infty}^{\infty} f^v(x)dx\right\}, \quad (30)$$

Then

$$R_x(v) = \frac{1}{v-1} log\left\{\int_{-\infty}^{\infty}\left\{\frac{1}{(1-e^{-\theta})}\rho\varphi\theta x^{-\varphi-1}e^{-x^{-\varphi}}\left(1-e^{-x^{-\varphi}}\right)^{\rho-1} e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]}\right\}^v dx\right\} \quad (31)$$

By applying equations (10) and (11) in the equation above, we have

$$R_x(v = \frac{1}{v-1} log\left\{\left(\frac{\rho\varphi\theta}{(1-e^{-\theta})}\right)^v \sum_{i,j,k}^{\infty}(-1)^{i+j+k}\binom{\rho j+v(\rho+1)}{k}\frac{(\theta v)^i}{i!}\int_{-\infty}^{\infty} x^{-v(\varphi+1)}e^{-(k+v)x^{-\varphi}}\right\}$$

Finally,

$$R_x(v) = \frac{1}{v-1} log\left[\left(\frac{\rho\theta}{1-e^{-\theta}}\right)^v \varphi^{v-1}\mathcal{W}_{ijk}(v)\right]$$

The q-entropy, introduced by Havarda and Charvat (1967), is defined as

$$R_x(q) = \frac{1}{q-1}\left\{1-\int_{-\infty}^{\infty}\left(\frac{1}{(1-e^{-\theta})}\rho\varphi\theta x^{-\varphi-1}e^{-x^{-\varphi}}\left(1-e^{-x^{-\varphi}}\right)^{\rho-1} e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]}\right)^q dx\right\}$$

Moreover, by applying equations (10) and (11) in the equations above, we have

$$R_x(q) = \frac{1}{q-1}\left\{1-\left(\frac{\rho\varphi\theta}{(1-e^{-\theta})}\right)^q \sum_{i,j,k}^{\infty}(-1)^{i+j+k}\binom{\rho j+q(\rho+1)}{k}\frac{(\theta q)^i}{i!}\int_{-\infty}^{\infty} x^{-q(\varphi+1)}e^{-(k+v)x^{-\varphi}}dx\right\}$$

Finally,

$$R_x(q) = \frac{1}{q-1}\left[1-\left(\frac{\rho\theta}{1-e^{-\theta}}\right)^q \varphi^{q-1}\mathcal{W}_{ijk}(q)\right]$$

6. Stochastic Ordering

The MEIW distribution is ordered with respect to the strongest "likelihood ratio" ordering as demonstrated in the following theorem.

Theorem 1 : Let $X_1 \sim MEIW(\varphi_1, \rho_1\theta_1)$ and $X_2 \sim MEIW(\varphi_2, \rho_2\theta_2)$

Proof : The likelihood ratio is

$$\frac{f_{x_1}(x)}{f_{x_2}(x)} = \frac{\varphi_1\rho_1\theta_1 x^{-\varphi_1-1}e^{-x^{-\varphi_1}}\left(1-e^{-x^{-\varphi_1}}\right)^{\rho_1-1} e^{\left[-\theta_1\left(1-\left(1-e^{-x^{-\varphi_1}}\right)^{\rho_1}\right)\right]}(1-e^{-\theta_2})}{\varphi_2\rho_2\theta_2 x^{-\varphi_2-1}e^{-x^{-\varphi_2}}(1-e^{-x^{-\varphi_2}})^{\rho_2-1} e^{\left[-\theta_2\left(1-\left(1-e^{-x^{-\varphi_2}}\right)^{\rho_2}\right)\right]}(1-e^{-\theta_1})} \quad (32)$$

thus,

$$\frac{\partial}{\partial x} log\left(\frac{f_{x_1}(x)}{f_{x_2}(x)}\right) = \frac{\varphi_2-\varphi_1}{x} + (\varphi_2-\varphi_1)x^{\varphi_2-\varphi_1-1} - (\rho_1-1)\frac{\varphi_1 x^{-\varphi_1-1}e^{-x^{-\varphi_1}}}{(1-e^{-x^{-\varphi_1}})}$$
$$+(\rho_2-1)\frac{\varphi_2 x^{-\varphi_2-1}e^{-x^{-\varphi_2}}}{(1-e^{-x^{-\varphi_2}})} + \theta_1 x^{-\varphi_1-1}e^{-x^{-\varphi_1}}\left(1-e^{-x^{-\varphi_1}}\right)^{\rho_1-1}\left[\left(1-\left(1-e^{-x^{-\varphi_1}}\right)^{\rho_1}\right)\right]$$
$$-\theta_2 x^{-\varphi_2-1}e^{-x^{-\varphi_2}}\left(1-e^{-x^{-\varphi_2}}\right)^{\rho_2-1}\left[\left(1-\left(1-e^{-x^{-\varphi_2}}\right)^{\rho_2}\right)\right]$$

Case i: if $\varphi_1 = \varphi_2 = \varphi,\ \rho_1 = \rho_2 = \rho$ and $\theta_1 \geq \theta_2$ then $\frac{\partial}{\partial x} log\left(\frac{f_{x_1}(x)}{f_{x_2}(x)}\right) \leq 0$, which implies that $X_1 \leq_{lr} X_2$ and hence $X_1 \leq_{hr} X_2,\ X_1 \leq_{mrl} X_2$ and $X_1 \leq_{st} X_2$.

Case ii: if $\theta_1 = \theta_2 = \theta,\ \varphi_1 = \varphi_2 = \varphi$ and $\rho_1 \geq \rho_2$ then $\frac{\partial}{\partial x} log\left(\frac{f_{x_1}(x)}{f_{x_2}(x)}\right) \leq 0$, which implies that $X_1 \leq_{lr} X_2$ and hence $X_1 \leq_{hr} X_2,\ X_1 \leq_{mrl} X_2$ and $X_1 \leq_{st} X_2$.

7. Quantile Function

Let X follow the MEIW distribution with the CDF given in (6). Then, the quantile function of X, denoted by $Q\ (u)$, is as follows:

$$X = Q(u) = F^{-1}(u),$$

It then follows that the quantile function of MEIW distribution is given by

$$Q(u; \varphi, \theta, \rho) = \left[-log\left(1-\left[1-\frac{1}{\theta}\left(1+log\left[u\left(1-e^{-\theta}\right)\right]\right)\right]^{1/\rho}\right)\right]^{-1/\varphi} \quad (33)$$

where u follows the uniform distribution over the interval [0,1]. In particular, when the value of u is taken as 0.5, we obtain the median (second quartile) given by

$$M = Q(u; \varphi, \theta, \rho) = \left[-log\left(1-\left[1-\frac{1}{\theta}\left(1+log\left[0.5\left(1-e^{-\theta}\right)\right]\right)\right]^{1/\rho}\right)\right]^{-1/\varphi} \quad (34)$$

Table 2 presents various numerical values of the median of the MEIW distribution for some values of the parameters. It can be observed that various values can be obtained, that is, $M \in (0, 0.50)$.

Table 2. The numerical values of the median of the $MEIW$ distribution for some values of the parameters

(θ, ρ, φ)	M
(1.5,0.8,1.8)	0.4096
(1.8,,0.8,1.0)	0.5000
(1.8,1.5,1.0)	0.3277
(1.8,2.5,2.0)	0.0792
(2.8,2.5,2.0)	0.0902

(2.8,3.5,2.5)	0.0390
(3.5,3.5,3.5	0.0097
(4.5,3.5,5.5)	0.0082
(5.5,5.5,3.5)	0.0049
(5.5,5.5,5.5)	0.0002

7.1 Stress-Strength Reliability

In engineering practice, it is of common interest that the life of a component is subjected to a random stress. The random strength can be modeled by a random variable X_1 and the random stress can be modeled by a random variable X_2. The probability that the component functions satisfactorily is given by $R = P(X_1 < X_2)$ and is regarded as a measure of component reliability in several applications. Let X_1 be a random variable following the MEIW distribution with parameters φ, ρ_1 and θ_1 with CDF $F_1(x)$ and X_2 be a random variable following the MEIW distribution with parameters φ, ρ_2 and θ_2 with PDF $f_1(x)$. Then an expression for Stress-strength reliability of MEIW distribution can be obtained using the relation given by

$$R = P(X_1 < X_2) = \int_{-\infty}^{\infty} F_1(x; \varphi, \rho_1, \theta_1) f_2(x; \varphi, \rho_2, \theta_2) dx \tag{35}$$

Then we have,

$$R = \int_{-\infty}^{\infty} \frac{1-e^{\left[-\theta_1\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho_1}\right)\right]}}{(1-e^{-\theta_1})} \frac{\varphi\rho_2\theta_2}{(1-e^{-\theta_2})} x^{-\varphi-1} e^{-x^{-\varphi}} \left(1-e^{-x^{-\varphi}}\right)^{\rho_2-1} e^{\left[-\theta_2\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho_2}\right)\right]} dx \tag{36}$$

Using (10) and (11) in (36) and subsequently carrying out a simple algebraic manipulation, we obtain an expression for stress-strength reliability for MEIW distribution as

$$= \frac{\rho_2\theta_2}{(1-e^{-\theta_1})(1-e^{-\theta_2})} \sum_{p,q,l,m,n,r=0}^{\infty} \frac{(-1)^{p+q+l+m+n+r}}{p!\,q!} \binom{p}{q} \binom{l}{m}$$
$$\times \binom{\rho_2(1+m)+\rho_2 q - 1}{r} \theta_1^p \theta_2^l (1+r)^{-1} \tag{37}$$

8. Maximum Likelihood

This method is the most widely used method for parameter estimation (Casella and Berger, 1990), because it has several flexible properties including asymptotic efficiency, consistency, and invariance property, and many others. Suppose $(x_1, x_2 \dots, x_n)$ is a random sample of size n from MEIW(ω), where $\omega = (\varphi, \rho, \theta)$. Then, the likelihood function for ω is of the form

$$L(\omega) = \prod_{i=1}^{n} \frac{1}{(1-e^{-\theta})} \rho\varphi\theta x^{-\varphi-1} e^{-x^{-\varphi}} \left(1-e^{-x^{-\varphi}}\right)^{\rho-1} e^{\left[-\theta\left(1-\left(1-e^{-x^{-\varphi}}\right)^{\rho}\right)\right]} \tag{38}$$

Correspondingly, the log-likelihood function, $log[L(\omega)] = l$ is given as

$$l = log\left(\frac{\rho\varphi\theta}{1-e^{-\theta}}\right) - (\varphi+1)\sum_{i=1}^{n} log(x_i) - \sum_{i=1}^{\infty} {x_i}^{-\varphi} + (\rho-1)\sum_{i=1}^{n}\left(1-e^{-{x_i}^{-\varphi}}\right)$$
$$-\sum_{i=1}^{n} \theta\left(1-\left(1-e^{-{x_i}^{-\varphi}}\right)^{\rho}\right) \tag{39}$$

The maximum likelihood estimators of $\hat{\varphi}_{MLE}$, $\hat{\rho}_{MLE}$ and $\hat{\theta}_{MLE}$ of the parameters φ, ρ, and θ can be obtained numerically by optimizing the log-likelihood function in (39). Alternatively, the log-likelihood equation in (39) can be maximized by differentiating the nonlinear equation. The component of the score vector

$$V_n(\omega) = \left(\frac{\partial l}{\partial\varphi}, \frac{\partial l}{\partial\rho}, \frac{\partial l}{\partial\theta}\right)^T,$$

are given by

$$\frac{\partial l}{\partial\varphi} = \frac{n}{\varphi} - \sum_{i=1}^{n} log(x_i) - \sum_{i=1}^{\infty} x_i{}^{-\varphi} \, log(x_i) + (\rho - 1)\sum_{i=1}^{n} \frac{log(x_i)x_i{}^{-\varphi}e^{-x_i{}^{-\varphi}}}{(1 - e^{-x_i{}^{-\varphi}})} \tag{40}$$

$$\frac{\partial l}{\partial\rho} = \frac{n}{\rho} + \sum_{i=1}^{n}\left(1 - e^{-x_i{}^{-\varphi}}\right) + \theta\sum_{i=1}^{n} \frac{\left(1 - e^{-x^{-\varphi}}\right)^{\rho} log\left(1 - e^{-x_i{}^{-\varphi}}\right)}{(1 - (1 - e^{-x_i{}^{-\varphi}})^{\rho})} \tag{41}$$

and

$$\frac{\partial l}{\partial\theta} = \frac{ne^{-\theta}}{1 - e^{-\theta}} - \sum_{i=1}^{n}\left(1 - \left(1 - e^{-x_i{}^{-\varphi}}\right)^{\rho}\right) \tag{42}$$

respectively. The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters, ρ φ and θ must be found by using iterative methods. The maximum likelihood estimate of the parameters, denoted by $\hat{\Delta}$ are obtained by solving the nonlinear equation $\left(\frac{\partial l}{\partial\varphi}, \frac{\partial l}{\partial\rho}, \frac{\partial l}{\partial\theta}\right)^T = 0$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $I(\Delta) = \left[I_{V_i}, I_{V_j}\right]_{3\times3} = E\left(-\frac{\partial^2 l}{\partial_{V_i}\partial_{V_j}}\right)$, $i, j = 1, 2, 3,$ can be numerically obtained by R or MATLAB software. The total Fisher information matrix $nI(\Delta)$ can be approximated by

$$J(\hat{\Delta}) \approx \left[-\frac{\partial^2 l}{\partial_{V_i}\partial_{V_j}} \mid_{\Delta=\hat{\Delta}}\right]_{3\times3}, \quad i, j = 1,2,3 \tag{43}$$

For a given set of observations, the matrix given in equation (44) is obtained after convergence of the Newton-Raphson procedure in R or MATLAB software.

8.1 Asymptotic Confidence Intervals

The asymptotic confidence intervals for the parameters of the MEIW distribution are presented. Similar results can be obtained for any other models under the same class of distributions. Numerically, expectations in the Fisher Information Matrix (FIM) can be obtained. Let $\hat{\Delta} = \left(\hat{\rho}, \hat{\varphi}, \hat{\theta}\right)$ be the maximum likelihood estimate of $\Delta = (\rho, \varphi, \theta)$. Under satisfying conditions for parameters in the interior parameter space, but not on the boundary, we have: $\sqrt{\hat{\Delta} - \Delta} \overset{d}{\rightarrow} N_3\left(\underline{0}, I^{-1}(\Delta)\right)$ with $I(\Delta)$ as the expected Fisher information matrix. Replacing the expected Fisher information matrix with the observed information matrix, the asymptotic behavior remains valid. The multivariate normal distribution $N_3\left(\underline{0}, J(\Delta)^{-1}\right)$ with mean vector $0 = (0, 0, 0)^T$, can be used to construct the confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1 - \eta)\%$ two-sided confidence intervals for $\rho, \varphi,$ and θ are given by:

$$\hat{\rho} \pm V_{\eta/2}\sqrt{I^{-1}{}_{\rho\rho}(\hat{\Delta})}, \hat{\varphi} \pm V_{\eta/2}\sqrt{I^{-1}{}_{\varphi\varphi}(\hat{\Delta})} \; and \; \hat{\theta} \pm V_{\eta/2}\sqrt{I^{-1}{}_{\theta\theta}(\hat{\Delta})}$$

respectively, where $I^{-1}{}_{\rho\rho}(\hat{\Delta})$, $I^{-1}{}_{\varphi\varphi}(\hat{\Delta})$ and $I^{-1}{}_{\theta\theta}$ are the diagonal elements of $I^{-1}{}_{n}(\hat{\Delta})$ and $V_{\eta/2}$ is the upper $V_{\eta/2}{}^{th}$ percentile of the distribution of the standard normal.

8.2 The Likelihood Ratio Test

The likelihood ratio (LR) test statistic can be used to compare the MEIW distribution with its sub models. The unrestricted estimates, $\hat{\rho}$, $\hat{\varphi}, and \, \hat{\theta}$ and restricted estimates $\breve{\rho}, \breve{\varphi}$ and $\breve{\theta}$ can be computed to construct the LR statistics for testing hypotheses concerning the sub models of the MEIW distribution. For example, to test $\theta = 1$, (MEIW against EIW) the LR statistic reduces to $\omega = 2[ln(L(\hat{\varphi}, \hat{\rho}, \hat{\theta})) - ln(L(\breve{\varphi}, \breve{\rho}, 1))]$, and the LR test rejects the null hypothesis, when $> \chi^2_{\epsilon}$, where χ^2_{ϵ} denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom.

9. Applications

In this section, we present an application of the Modified Exponentiated Inverted Weibull distribution on real data set. We shall compare the fit of the MEIW with the Exponentiated Inverted Weibull (EIW), Exponentiated Frechet (EF), Exponentiated Weibull distributions with corresponding PDFs (for):

$$f_{EF}(x;\varphi,\rho,\theta) = \rho\varphi\theta x^{-\varphi-1}e^{-\theta x^{-\varphi}}\left(1-e^{-\theta x^{-\varphi}}\right)^{\rho-1}, x>0;\ \rho,\varphi,\theta>0$$

$$f_{EIW}(x;\varphi,\rho) = \rho\varphi x^{-\varphi-1}e^{-x^{-\varphi}}\left(1-e^{-x^{-\varphi}}\right)^{\rho-1}, x>0;\ \rho,\varphi\ >0$$

$$f_{EW}(x;\varphi,\rho,\theta) = \rho\varphi\theta x^{\varphi-1}e^{-\theta x^{\varphi}}\left(1-e^{-\theta x^{\varphi}}\right)^{\rho-1}, x>0;\ \rho,\varphi,\theta>0$$

The real data represents the survival times of patients suffering from neck cancer disease. The patients in this group were treated with combined radiotherapy and chemotherapy (CT & RT). The data are 12.2 23.56 23.74 25.78 31.98 37 41.35 47.38 55.46 58.36 63.47 68.46 78.26 74.47 81.43 84 92 94 110 112 119 127 130 133 140 146 155 159 173 179 194 195 209 249 281 319 339 432 469 519 633 725 817 776.

Kumar et al. (2015) fitted this data to the inverse Lindley distribution, Saeed and Ali (2020) fitted the data set to the Sinh Inverted Exponential distribution. The Exploratory data analysis for the neck cancer data is given in Table 3 which shows that the data is over-dispersed with an excess kurtosis of 1.87 (leptokurtic). The results of the estimated values of the parameters (2*Log-likelihood, AIC, BIC, CAIC, and HQIC) are listed in Table 4. The Total time on Test (TTT) which shows that the neck cancer data exhibits non-monotone failure rate and the empirical density plot which clearly indicates that the data is moderately positively skewed is given in Figure 5 , the fitted PDF and estimated CDF of the MEIW curve to this data are given in Figures 6. The selection criterion is that the lowest 2*Log-likelihood and AIC correspond to the best model fitted. The MLEs, AIC, BIC, CAIC and HQIC are shown in Table 4. From the Table, we can observe that the MEIW model shows the smallest 2*Log-likelihood, AIC, BIC, CAIC and HQIC than other competing distributions.

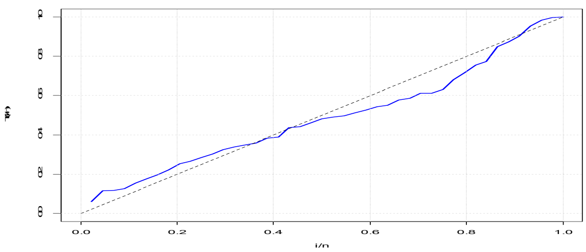

Diagram I

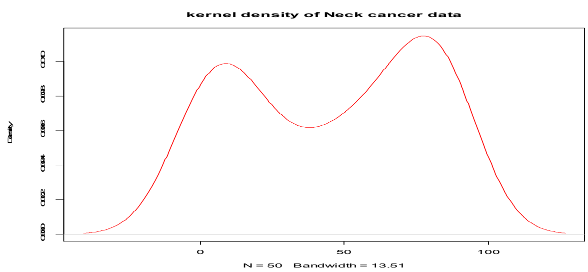

Diagram II

Figure 4. Graph of TTT plot (Diagram I) and Kernel density Plot (Diagram II)

Table 3. Exploratory Data Analysis of the neck cancer data

Min.	Lower quartile	median	Mean	Upper quartile	Max.	range	variance	Kurtosis	Skewness
12.20	67.21	128.50	200.75	219.0	817.0	188.55	43804.05	4.87	1.68

Table 4. Analysis results of the MEIW model and other competing models for neck cancer data

$Model$	ρ	φ	θ	$-2l$	AIC	$CAIC$	BIC	$HQIC$
$MEIW$	0.018 (0.003)	5.461 (0.003)	9.715 (1.715)	516.912	522.912	523.512	528.265	524.897
EIW	0.078 (0.250)	2.663 (0.569)	− (−)	625.424	655.425	655.717	658.993	656.748
EW	0.403 (0.207)	0.401 (0.636)	11.057 (1.755)	549.728	555.729	556.329	561.081	0.056
EF	7.422 (1.232)	1.026 (0.112)	11.3759 (2.478)	555.910	561.911	562.511	567.263	0.097
IW	− (−)	0.278 (0.034)	− (−)	325.712	681.534	681.629	683.318	682.196

The LR statistic was obtained for testing the hypotheses H_0: θ=1 $versus$ $H_1 = H_0$ $is\ not\ true$, that is to compare the

MEIW model with the EIW model. The LR statistic $w = 2\{655.717\text{-}523.512\} = 132.205(p - value < 0.01)$, sufficing to show that the $MEIW$ model is a better model that can be used in fitting the data.

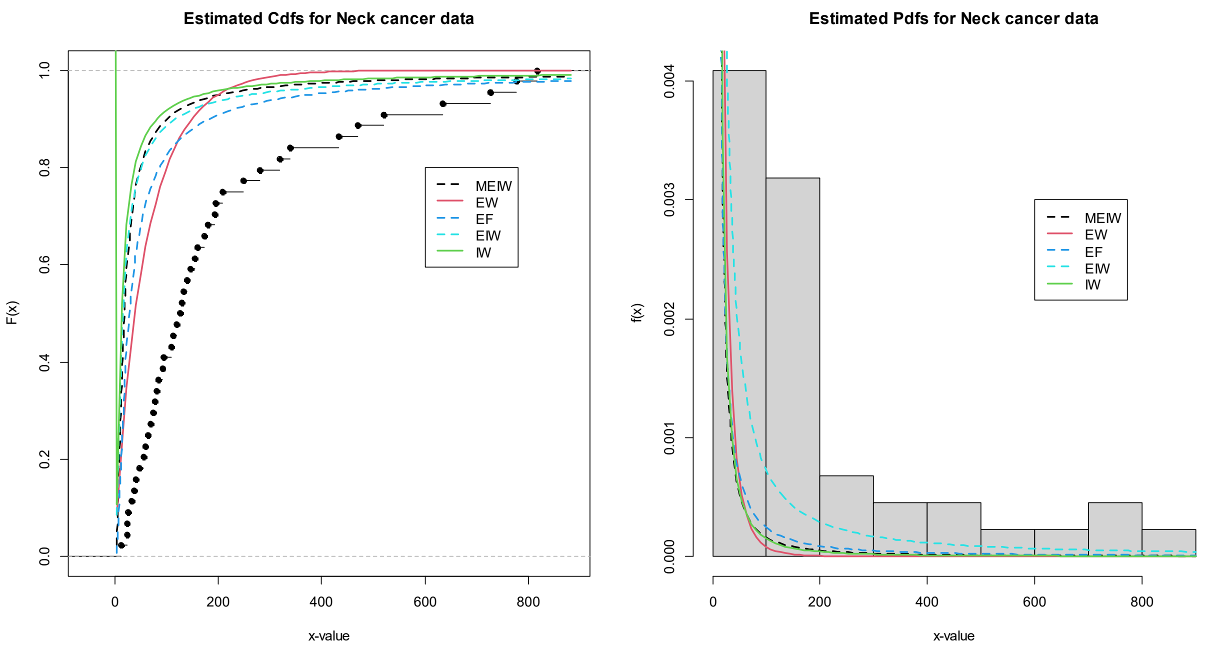

Figure 5. Plots of estimated CDF and histogram fitted PDF of the fitted models for the neck cancer disease data from left to right

10. Conclusion

In this study, a three-parameter model called Modified Exponentiated Inverted Weibull distribution is proposed and studied. Various properties of MEIW model are derived. Maximum likelihood estimation method is used to estimate the parameters of the model. The application of MEIW model is demonstrated by using a neck cancer data and it provides a reasonable good parametric fit to the data set than some other competitive models considered in this work because the MEIW model contains the minimum information loss as a result of its smallest AIC, BIC CAIC and HQIC values.

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