# Relativity of $\pi$ as a Function of the Rotation of N-Sided Unit Polygons 

Luis Teia<br>Correspondence: Dr Luis Teia, von Karman Institute for Fluid Dynamics, Chaussée de Waterloo 72, 1640 Rhode-Saint-Genèse, Belgium. E-mail: luisteia@sapo.pt

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#### Abstract

This article presents a new way to determine the value of $\pi$, using as an approach the area formed by the interference pattern of several rotating unit squares. The same approach is then applied to other N -sided unit polygons (i.e., triangles, pentagons and hexagons) to investigate how they affect this proportionality between circularity and linearity to a degree other than orthogonal (i.e., when the system axes do not form a right-angle, expressible in the new method as an approach that uses squares). Applied examples involving the Earth's size and an orbiting satellite constellation are given.


Keywords: relativity, Pi, number theory, geometry, polygon, rotation

## 1. Introduction

The constant $\pi$ is an essential tool in mathematics that interconnects circularity to linearity (i.e., "the fact of being shaped like a circle" to "the fact of consisting of straight lines", according to the Cambridge online dictionary) in various aspects of modern society (e.g., a wheel rolling on a road, a turning engine pulling a convey belt, a pulley heaving a rope in a crane, etc). Records documenting the fundamental concept of this constant date back to the time of the Babylonians (Robson 2001, Swetz 2014). One definition presented in the $18^{\text {th }}$ century defines $\pi$ as the ratio of a circle's circumference to its diameter (Jones 1706). Various approaches are known to determine $\pi$, some being polygonal algorithms - for example, Archimedes approach and its modified version from Liu Hui increase the number of sides of a polygon as the means to achieve an estimation of $\pi$ - infinite series - such as in the work of Leibniz and Lambert - iterative algorithms - derived by John Machin's, and more recently by the Chudnovsky brothers and finally statistical approaches are also possible using Monte Carlo simulations (Ye 2016). The importance of $\pi$ in science is highlighted by the considerable effort employed in determining its ever-increasing accuracy. In 2021, an attempt to compute a new world record accuracy of $\pi$ to 62.8 trillion digits was achieved using the Chudnovsky algorithm, where the main venue was the testing of supercomputers and numerical analysis algorithms (Lu 2021, Thomas et al 2021). The importance of $\pi$ in education can never be understated (Canadian Ministry of Education 2020). The present article contributes to the pool of methods to compute $\pi$ by offering a new approach using the interference pattern caused by the rotation of N -sided unit regular polygons, while at the same time it presents a new perspective on the definition of $\pi$ beyond a constant, into the idea of a function.

## 2. Hypothesis

It is the hypothesis of this article that the relation between linearity and circularity - mathematically represented by $\pi$ - can be defined from the ratio between the area of a circle - computed from the rotational pattern of infinite N -sided unit revolving regular polygons - and a fraction of the area of the (respective) circumscribed polygon (sized by the number of sides of the polygon), where the constant value giving the proportion $\pi$ becomes a function dependent on the properties of the regular polygon used during this process (that encompasses within it the constant 3.1415...).

## 3. Theory

Consider a regular polygon of side unit. Rotating copies of this polygon about its centre at equidistant angles generates an interference pattern composed of small triangles which are interconnected to each other. As the number of polygons tends to infinity, their interference creates an endless number of triangles that when summed fill completely the area between two circles - external and internal formed by the edges and tangents of the square - creating a ring. The fractal sum of the rings inside the initial allows the entire area of the circle to be numerically computed. The area of the circle can then be related to express $\pi$. Usage of polygons to define $\pi$ is not uncommon - Archimedes approach and Liu Hui method are examples - creating a relation between the perimeter of a polygon with an ever-increasing number of sides (tending to a circle at infinity) that establishes the means to compute the circle's perimeter, as thus the value of $\pi$. The new method is different, in that a polygon (of number of sides $\boldsymbol{N}$ ) is rotated by $\boldsymbol{P}$ times around its center O, and it's
the interference pattern between these polygons that allows the computation of the circle's area, as thus of the constant $\pi$.

### 3.1 Square Approach

Figure 1a shows the fine interference pattern formed by the angular equidistant rotation of sixteen $(P=16)$ unit squares $(N=4)$, and Figure 1 b shows a zoom of the fundamental strip that tessellates it. For reference, the simpler approach of using only two squares (in black) is also shown for comparison. In both cases, it is possible to compute an approximation of the area of the circle, and thus of the value of the constant $\pi$. The greater the number of revolving squares, the finer the interference pattern, and thus the more accurate the prediction of $\pi$ will be. With the rotation of the squares, the number of outer points and inner tangent points tends to infinity, and both form an outer circle (here in blue) and an inner circle (here in red). Since the internal angle of the square is $\theta=180^{\circ}-360^{\circ} / \mathrm{N}=90^{\circ}$, this means that when one side of the square is tangent to the inner circle (at $T_{1}$ for instance), the other adjacent side is both perpendicular and equal to the diameter $T_{1} T_{2}$ (of the inner circle). The first phase of the method is to compute the area in between the two circles, here called a ring. A ring is composed of many triangle strips (visible in Figure 1a), which in turn is divided into a series of smaller triangles (that is, each parallelogram in the strip is divisible along the diagonal into two triangles - as shown in the zoom in Figure 1b). For increasing number of revolving polygons, the strips gradually become thinner and more numerous. The second phase comprises of filling the interior of the ring by repeating it inwards in a fractal manner, allowing for the completion of the circle's area. To illustrate this effect, Figure 2a shows two squares (clocked by 45 degrees) scaled down continuously.


Figure 1. (a) Ring formed by evenly spaced 16 revolving squares, and (b) zoom into interference strip AB'H
This is achieved by iteratively joining their midpoints, in a fractal convergence pattern towards the center. Then, by revolving this pair of squares in small steps creates a fine mesh that tends to occupy the whole center area (as seen in Figure 2 b ). The interference pattern in each fractal ring is composed of many copies of a single strip that also scales down - AB'H for the first (Figure 1b) and scaled down versions of $A B$ ' $H$ for the remainder infinite fractal levels (Figure 2b). In order to keep the method mathematically manageable, consider the simple rotation of four squares (instead of sixteen). As a major outcome, the mathematical process to be described leading to the calculation of $\pi$ is programmed (in a fully working manner) in the Octave open-source program (Eaton et al 2021) in annex. This program can run directly in the Octave's open-source software, when copied into an m-file in the "Editor" tab. Input of key parameters and initial formulas is done in line 1 to 19 . The first part of the present process implies finding an expression for the area of the strip - here present as $\mathrm{AB}^{\prime} \mathrm{H}$ in Figure 3 resulting from the interference of the squares - that in itself is the sum of a series of small triangles. As it will be shown, this series of triangles is interconnected by their sides and internal angles via the law of cosines. The method starts by finding the expression for the length of the longest side $z_{1}$ of the most outward triangle. Instead of relating the area of a circle to its radius, the present method will interconnect it to the side of the revolving square ABCD instead (which, for convenience and from our reference point view, will be assumed - for the purpose of the analysis - fixed with respect to all other), and this side will be defined as a function of the individual sides triangles composing the strip in the following manner. The length segment AD is assumed
stationary and constant $L=1$, and it is partitioned several times by the revolving squares.


Figure 2. Fractality of the approach shown by the infinite inward propagation of (a) 2 squares and (b) 16 squares


Figure 3. Interference strip created by four revolving squares: (a) overview and (b) key variables (lengths and angles)
For this particular case of four rotating squares (Figure 3a), the side of the square is expressible as the sum

$$
\begin{equation*}
2\left(x_{1}+x_{2}+x_{3}\right)+z_{4}=1 \equiv L \tag{1}
\end{equation*}
$$

This will later be generalized to an infinite number of squares $(N=\infty)$. The longest side of each successive triangle within the strip is denoted by the variable $z$, while the shorter side by the variable $x$. Symmetry mirrors the partition about the midpoint of AD , hence the multiplying factor two in some of the terms. One can simply scale the circle area by changing the value of L , but (for the purpose of determining $\pi$ ), it is a matter of convenience to set it as $L=A D=1$.

Zooming into the interference strip AB 'H (Figure 3b) allows the identification of all angles and lengths necessary for determining its area. The strip can be decomposed into pairs of adjacent outward and inward facing triangles, located by the index $n$ along the strip (starting outside and moving inwards). The outward facing angles of the triangles are given by

$$
\begin{equation*}
\gamma_{o, n}=\theta+(n) \delta \tag{2}
\end{equation*}
$$

where $\theta=180^{\circ}-360^{\circ} / \mathrm{N}$ is the polygon internal angle, the subscript $o$ denoting an obtuse angle facing outward away from the center of the circle (starting with the most outer triangle AB'E at $n=1$ ). In turn, the inward facing angles of the triangles are given as

$$
\begin{equation*}
\gamma_{i, n}=\theta+(n-1) \delta=\gamma_{o, n-1} \tag{3}
\end{equation*}
$$

and the subscript $i$ denotes an obtuse angle facing inward towards the center of the circle (starting with the most outer triangle AEF at $n=1$ ). Creating a relation between all sides of $x$ and $z$ is possible via the law of cosines. When applied between two adjacent triangles at the same level - like AB'E and AEF, both at $n=1$ - establishes a connection between two successive values of $z$. That is, for triangle $A B ' E$, the following relation is present

$$
\begin{equation*}
x_{1}^{2}-2 x_{1} x_{1} \cos \gamma_{o, 1}+x_{1}^{2}=z_{1}^{2} \tag{4}
\end{equation*}
$$

Herein, and throughout the remainder of this article, all trigonometric functions work directly on an argument specified in degrees. At the same time, for triangle AEF, the associated relation becomes

$$
\begin{equation*}
x_{1}^{2}-2 x_{1} x_{1} \cos \gamma_{i, 1}+x_{1}^{2}=z_{2}^{2} \tag{5}
\end{equation*}
$$

Both Eq.(4) and Eq.(5) have the common variable $x_{1}$. Equating both gives the (already simplified) relation

$$
\begin{equation*}
z_{2}=\frac{\sqrt{1-\cos \gamma_{i, 1}}}{\sqrt{1-\cos \gamma_{o, 1}}} z_{1} \tag{6}
\end{equation*}
$$

Further generalizing to a relation between two subsequent lengths $z_{n}$ and $z_{n+1}$ [while also expanding with Eq.(2) and Eq.(3)], results in

$$
\begin{equation*}
z_{n+1}=\frac{\sqrt{1-\cos \gamma_{i, n}}}{\sqrt{1-\cos \gamma_{o, n}}} z_{n}=\frac{\sqrt{1-\cos (\theta+(n-1) \delta)}}{\sqrt{1-\cos (\theta+(n) \delta)}} z_{n}=Q_{n} z_{n} \tag{7}
\end{equation*}
$$

In turn, when the law of cosines is applied between two adjacent triangles at different levels - like AEF at $n=1$ and GEF at $n=2$ - determines a connection between two successive values of $x$. For triangle GEF, the following relation is present

$$
\begin{equation*}
x_{2}^{2}-2 x_{2} x_{2} \cos \gamma_{o, 2}+x_{2}^{2}=z_{2}^{2} \tag{8}
\end{equation*}
$$

Equating to the formed relation for triangle AEF given by Eq.(5) - where both have the common variable $z_{2}$ results in the (already simplified) relation

$$
\begin{equation*}
x_{2}=\frac{\sqrt{1-\cos \gamma_{i, 1}}}{\sqrt{1-\cos \gamma_{o, 2}}} x_{1} \tag{9}
\end{equation*}
$$

Similarly, this can be generalized and further expanded using Eq.(2) and Eq.(3) as

$$
\begin{equation*}
x_{n+1}=\frac{\sqrt{1-\cos \gamma_{i, n}}}{\sqrt{1-\cos \gamma_{o, n+1}}} x_{n}=\frac{\sqrt{1-\cos (\theta+(n-1) \delta)}}{\sqrt{1-\cos (\theta+(n+1) \delta)}} x_{n}=K_{n} x_{n} \tag{10}
\end{equation*}
$$

We are now in a position to expand Eq.(1) using both Eq.(7) and Eq.(10), resulting in

$$
\begin{equation*}
2\left(1+K_{1}+K_{1} K_{2}\right) x_{1}+\left(Q_{3} Q_{2} Q_{1}\right) z_{1}=1 \tag{11}
\end{equation*}
$$

Substituting a re-arranged version of Eq.(4) as $z_{1}=x_{1} \sqrt{2\left(1-\cos \gamma_{o, 1}\right)}$ gives

$$
\begin{equation*}
x_{1}(P=4)=\frac{1}{2\left(1+K_{1}+K_{1} K_{2}\right)+\left(Q_{3} Q_{2} Q_{1}\right) \sqrt{2\left(1-\cos \gamma_{o, 1}\right)}} \tag{12}
\end{equation*}
$$

Consequentially, the expression for $z_{1}$ becomes

$$
\begin{equation*}
z_{1}(P=4)=\frac{\sqrt{2\left(1-\cos \gamma_{o, 1}\right)}}{2\left(1+K_{1}+K_{1} K_{2}\right)+\left(Q_{3} Q_{2} Q_{1}\right) \sqrt{2\left(1-\cos \gamma_{o, 1}\right)}} \tag{13}
\end{equation*}
$$

While Eq.(12) and Eq.(13) are applicable to the particular case of four revolving unit squares, the general case of $z_{1}$ for any number of rotating squares is extrapolated to

$$
\begin{equation*}
z_{1}(P)=\frac{\sqrt{2\left(1-\cos \gamma_{o, 1}\right)}}{2\left[1+\sum_{m=1}^{P-2} \prod_{n=1}^{m}\left(K_{n}\right)\right]+\prod_{n=1}^{P-1}\left(Q_{n}\right) \sqrt{2\left(1-\cos \gamma_{o, 1}\right)}} \tag{14}
\end{equation*}
$$

while the generalized function for $x_{1}$ being inherently

$$
\begin{equation*}
x_{1}(P)=\frac{1}{2\left[1+\sum_{m=1}^{P-2} \prod_{n=1}^{m}\left(K_{n}\right)\right]+\prod_{n=1}^{P-1}\left(Q_{n}\right) \sqrt{2\left(1-\cos \gamma_{o, 1}\right)}} \tag{15}
\end{equation*}
$$

These equations can be seen at work in the annexed Octave program, being composed of two for-loop cycles (lines 20 to 34 in the annexed program). Knowing first $x_{1}$ [from Eq.(12)] and $z_{1}$ [from Eq.(13)], it is possible - using Eq.(7) and Eq.(10) - gives values for $x_{n}$ and $z_{n}$ (these expression are programmed in lines 35 to 40 ). Now that expressions for the sides of the triangles have been determined, it is possible to compute their areas. The area $A_{o, 1}$ of the triangle $A B^{\prime} E$ along the strip $A B^{\prime} H$ is defined as

$$
\begin{equation*}
A_{o, 1}=h \times \frac{z_{1}}{2} \tag{16}
\end{equation*}
$$

where an expression for height $h$ is obtained by applying the Pythagoras' theorem

$$
\begin{equation*}
\frac{z_{1}^{2}}{2}+h^{2}=x_{1}^{2} \quad \Rightarrow \quad \mathrm{~h}=\sqrt{x_{1}{ }^{2}-\frac{z_{1}^{2}}{2}} \tag{17}
\end{equation*}
$$

Both areas $A_{o, 1}$ [triangle $\left.\mathrm{AB}^{\prime} \mathrm{E}\right]$ and $A_{i, 1}$ [triangle AEF$]$ (at $n=1$ ) result from replacing Eq.(17) into Eq.(16) giving

$$
\begin{equation*}
A_{o, 1}=\frac{z_{1}}{2} \sqrt{x_{1}^{2}-\frac{1}{4} z_{1}^{2}} \quad ; \quad A_{i, 1}=\frac{z_{2}}{2} \sqrt{x_{1}^{2}-\frac{1}{4} z_{2}^{2}} \tag{18}
\end{equation*}
$$

with variables $x_{1}, z_{1}$ and $z_{2}$ expressed by Eq.(12), Eq.(13) and Eq.(6), respectively. Calculation of areas of subsequent triangles is done by extrapolating Eq.(18). The computation of these areas is programmed in lines 45 to 50 of the Octave algorithm. In order to sum all the areas of the triangles within the strip, it is important to note that the fact that each radial position $n$ has an outward and inward facing triangle is accounted for mathematically by performing a sum of triangle pairs (along the circumferential direction). This is achieved by counting the index from 1 to 2 . There are $\mathrm{P}-1$ pairs of triangles (which in this case is $4-1=3$ ), thus a second sum is required counting another index $i$ from 1 to $\mathrm{P}-1$. This alternation is seen in the first two examples, where for the case of triangle AB ' E with indexes $(i, j)=$ $(1,1)$, the sides become $z_{i+j-1}=z_{1}$ and $x_{i}=x_{1}$ allowing the computation of $A_{o, 1}$, and for triangle AEF with indexes $(i, j)=(1,2)$ they become $z_{i+j-1}=z_{2}$ and $x_{i}=x_{1}$ allowing for the computation of $A_{i, 1}$ [given by Eq.(18)]. The same occurs for triangles GEF and KFG with indexes $(i, j)=(2,1)$ and $(i, j)=(2,2)$ respectively, resulting in sides $z_{i+j-1}=z_{2}$ and $x_{i}=x_{2}$ for $A_{o, 2}$, and $z_{i+j-1}=z_{3}$ and $x_{i}=x_{2}$ for $A_{i, 2}$, and so on and so forth. Thus, the area in the strip is given by a double sum that accounts for the contributions of index $i$ and $j$, resulting in

$$
\begin{equation*}
A_{\text {strip }}(N=4)=\sum_{i=1}^{P-1} \sum_{j=1}^{2} \frac{z_{i+j-1}}{2} \sqrt{x_{i}^{2}-\frac{1}{4} z_{i+j-1}^{2}} \tag{19}
\end{equation*}
$$

The area of the ring is formed by summing $P \times N$ identical strips (i.e., the product of the number of rotating polygons $P$ by its inherent number of sides $N$ ), which for the case of four revolving squares $(P=4)$ gives the expression

$$
\begin{equation*}
A_{\text {ring }}(N=4)=4 N\left\{\sum_{i=1}^{P-1} \sum_{j=1}^{2} \frac{z_{i+j-1}}{2} \sqrt{x_{i}^{2}-\frac{1}{4} z_{i+j-1}^{2}}\right\} \tag{20}
\end{equation*}
$$

The computation of the area of the strip and ring is programmed in lines 51 to 52 of the Octave algorithm. In order to account for the totality of the area in the circle, it is important to realize that there are an infinite number of rings propagating inwards in a fractal way - that converge towards the singularity at the center. In practice, this fractality is modeled in the following manner. The length of the side of the reference square is related to that of its immediate lower fractal by a factor of $\cos \left(45^{\circ}\right)=1 / \sqrt{2}$, which results in an area relation of $1 / \sqrt{2} \times 1 / \sqrt{2}=\cos ^{2}\left(45^{\circ}\right)=1 / 2$. That is, multiplying the ring area [given in Eq.(20)] by the factor $1 / 2$ transforms it into the area of the next inward adjacent ring (located inside the first). Applying this factor again and again provides the area of the next adjacent inward ring, and so on thereafter. Thus, the geometric area progression encompassing all the infinite rings (formed by considering the rotating squares at all levels) is modelled as the sum

$$
\begin{equation*}
\frac{1}{2^{1}}+\frac{1}{2^{2}}+\frac{1}{2^{3}}+\frac{1}{2^{4}}+\cdots=\sum_{k=1}^{\infty} \frac{1}{2^{k}}=\sum_{k=1}^{\infty} \cos ^{2 k}\left(45^{\circ}\right) \tag{21}
\end{equation*}
$$

It is important to note that the first term in the scaling series of Eq.(21) is $1 / 2^{1}$ (for $k=1$ ) which sizes the first ring of the series to be the one inside (not outside) the square of side 1 (i.e., the second ring counting from the outside in Figure 2 b - outer circle in red and inner circle in blue). This is important, as it sets the computed area of the circle to that inscribed within the square side $L=A I=1$ - a relation that is needed to determine $\pi$, where here the circle is inside the circumscribing polygon [as seen later in Eq.(24) and Eq.(25), where the area of the circle is smaller than of the square]. If desired, the area of the first outer ring (shown in Figure 2b circumscribing the unit square) can be included by adding the extra term $1 / 2^{0}=1$ (for $k=0$ ) in the scaling series Eq.(21). For now, we are only interested in the rings inside the unit square, hence multiplying the ring area Eq.(20) by the sequence of scaling factors given by Eq.(21) gives the resulting final expression of the approximated area of the circle (using in this example the interference pattern of four angularly equidistant squares having four sides each $N=4$ ) as being

$$
\begin{equation*}
A_{\odot}(N=4)=(4 \times 4)\left\{\sum_{i=1}^{P-1} \sum_{j=1}^{2} \frac{z_{i+j-1}}{2} \sqrt{x_{i}^{2}-\frac{1}{4} z_{i+j-1}^{2}}\right\} \times \sum_{k=1}^{\infty} \cos ^{2 k}\left(45^{\circ}\right) \tag{22}
\end{equation*}
$$

Computing the values of the sides z and x , angles and areas of the successive triangles within the strip (forming altogether the most outward ring) - from Eq.(2), Eq.(3), Eq.(7), Eq.(10) and Eq.(18), respectively - for this particular case gives Table 1. The computation of the area of the circle is programmed in lines 61 to 62 of the Octave algorithm.

Table 1. Properties of the triangles composing the strip in the most outward ring (square approach)

| Angle $\boldsymbol{\gamma}$ |  | Side z | Side x | Area A |
| :---: | :---: | :---: | :---: | :---: |
| Outward | Inward |  |  |  |
| $\gamma_{10}=112.5^{\circ}$ | $\gamma_{1 i}=90^{\circ}$ | $z_{1}=0.27589938$ | $x_{0}=0.16591068$ | $A_{o, 1}=0.01271552$ |
| $\gamma_{20}=135^{\circ}$ | $\gamma_{2 i}=112.5^{\circ}$ | $z_{2}=0.23463314$ | $x_{1}=0.12698254$ | $A_{i, 1}=0.01376318$ |
| $\gamma_{30}=157.5^{\circ}$ | $\gamma_{3 i}=135^{\circ}$ | $z_{3}=0.21116424$ | $x_{2}=0.10765060$ | $A_{o, 2}=0.00570089$ |
|  |  | $z_{4}=0.19891237$ |  | $A_{i, 2}=0.00744858$ |
|  |  |  | $A_{o, 3}=0.00221739$ |  |
|  |  |  | $A_{i, 3}=0.00409721$ |  |

Summing the areas of the triangles gives the area of the strip as 0.04594277 [same as solving Eq.(19)]. Knowing at the same time that the scaling series given by Eq.(21) tends to the value 1 at infinity, results in a total area for the approximated circle using only four revolved squares [Eq.(22)] to be

$$
\begin{equation*}
A_{\odot}(N=4)=(4 \times 4)\{0.04594277\} \times 1=0.73508432 \tag{23}
\end{equation*}
$$

The ratio between the approximate area of the circle and its circumscribed square becomes

$$
\begin{equation*}
\frac{A_{\odot}(N=4)}{A_{■}}=\frac{0.73508432}{1}=0.73508432 \tag{24}
\end{equation*}
$$

The computation of the area ratio and area of the circle is programmed in lines 63 to 66 of the Octave algorithm. It is found that by multiplying this ratio with the number of sides of a square $(N=4)$ gives

$$
\begin{equation*}
\pi_{90^{\circ}}=4 \times 0.73508432=2.94033728 \tag{25}
\end{equation*}
$$

which is - when using only four angularly equidistant revolved unit squares - a first approximation to the constant $\pi_{90^{\circ}}$ (where the $90^{\circ}$ relate the value of $\pi$ to the angle between opposing tangents and the respective diameter). Naturally, the prediction improves with increasing number of revolved squares (as shown in Figure 4), which at two million squares gives the approximated value of $\pi_{90^{\circ}}=3.141592653589(155)$ - an estimate valid up to twelve decimal places of the official value 3.141592653589793 (Bailey \& Borwein 2014). This computation takes around a minute on an average computer, using the annexed program (at the end of this article) that runs on the open-source software Octave (Eaton et al 2021). Greater accuracy is possible by increasing the number of squares in the method, at the expense of more computational time. This suggests that the ratio between areas (i.e., circle $A_{\odot}$ over circumscribed square $A_{■}$ ) is the same as the ratio between constants (i.e., $\pi_{90^{\circ}}$ over the number of sides of a square $N=4$ ), or

$$
\begin{equation*}
\frac{A_{\odot}(P=4)}{A_{■}}=\frac{\pi_{90^{\circ}}}{4} \quad \Rightarrow \quad A_{\odot}(P=4)=\pi_{90^{\circ}} \frac{D^{2}}{4} \tag{26}
\end{equation*}
$$

Simplification reduces this equality to the commonly known equation of area of a circle based on the side length diameter D of its circumscribed square. In this context, the constant $\pi_{90^{\circ}}$ is defined in a different way as traditionally, becoming equal to the ratio of the area of a circle $A_{\odot}$ to a fourth of the area of the circumscribing square $A_{\mathbf{\bullet}} / 4$.


Figure 4. Accuracy of $\pi_{90^{\circ}}$ for changing number of interfered rotating squares
It is interesting to note that polygons (like a triangle, a square, a hexagon, an octagon) are commonly sized by the length of their sides (being an explicit measure), not by the connection of their sides or corners to the center point (an implicit measure), being done mostly as a convenience (because relating the area and perimeter of a polygon to both implicit and explicit measures is possible). Since a circle is an infinitely-sided polygon, extension of the present argument suggests that the same reasoning can be applied to a circle when sizing it (that is, relating its properties to its infinitesimal tangential side, instead of its radius, which is possible but not convenient). The logic of this argument
indicates that the properties of a circle can be expressed as a function of either of the linear lengths - that is, its radius or to the side of the polygon describing the circle by rotation (as demonstrated above) - with its implied properties (area and perimeter, which include $\pi$ ), becoming consequentially dependent of this choice.

### 3.2 Triangle Approach

The same process is now applied using a different polygon - a unit regular triangle (setting three sides as $N=3$ ) [Figure 5a]. An important difference caused by this alteration is that while one side of the triangle is tangent to the inner circle (at $T_{1}$ for instance), the other adjacent side (i.e., parallel to the reference length $T_{1} T_{2}$ that represents linearity by interconnecting tangencies) is no longer perpendicular to the circle (i.e., the reference length that represents circularity), but it is instead at an internal angle $\theta=180^{\circ}-360^{\circ} / \mathrm{N}=60^{\circ}$. The difference in internal angles for different polygons - as in the example of $90^{\circ}$ for a square and $60^{\circ}$ for a triangle - will play an important role later during the investigation into the relativity of $\pi$ beyond an individual constant, and instead expressed as a function. The overlapping pattern of rotating triangles again forms an interference strip AB ' H (Figure 5b), that repeats itself angularly and has a higher resolution for increased number of rotating triangles. Continuing the process, the inner area of the circle is defined by an inward fractal propagation of two vertically mirrored triangles (Figure 6a), that when revolved in an angularly equidistant manner, forms altogether the interference pattern of the inner rings (Figure 6 b shows a construction formed by 16 triangles per ring, as an example). The areas of two subsequent rings are interconnected by the following rule - the length of the side of a triangle is related to its immediate lower fractal size by the factor $\cos \left(60^{\circ}\right)=1 / 2$, resulting in a fractal area relation of $1 / 2 \times 1 / 2=\cos ^{2}\left(60^{\circ}\right)=1 / 4$.


Figure 5. (a) Ring formed by 16 evenly spaced revolving triangles, and (b) zoom into interference strip AB'H
It is worth noting that the size of the circle found by rotation of the unit triangle (in Figure 6b) is different from that formed by the unit square (in Figure $2 b$ ). Their relative size is inconsequential as $\pi$ is a ratio (e.g., perimeter over diameter), and thus it is independent on the selected size of the circle used to define it. For convenience, the chosen side of the regular triangle is $L=A I=1$, and scaling to any circle requires only a different choice of $L$. Thus, the coefficient accounting for the area contribution of all infinite inward fractal levels (scaled from the first inside the red circle) is given by the series

$$
\begin{equation*}
\frac{1}{4^{1}}+\frac{1}{4^{2}}+\frac{1}{4^{3}}+\cdots=\sum_{k=1}^{\infty} \frac{1}{4^{k}}=\sum_{k=1}^{\infty} \cos ^{2 k}\left(60^{\circ}\right) \tag{27}
\end{equation*}
$$

It is important to note that the first term in the scaling series in Eq.(27) is $1 / 4^{1}$ (for $k=1$ ), which sizes the first ring of the series to be the one inside (not outside) the square of side 1 (i.e., the second counting from the outside in Figure $2 b)$. To account for the ring outside (between the blue and red circle) in the area of the circle, simply add to the series its respective term $1 / 4^{0}$ (for $k=0$ ).


Figure 6. Fractality of the approach shown by the infinite inward propagation of (a) 2 triangles and (b) 16 triangles
The total area of the (red) circle is approximated to the sum of the discrete areas formed by the interference pattern of (in this example) four angularly equidistant triangles (i.e., $P=4$ ) having three sides each (i.e., $N=3$ ), being described mathematically as

$$
\begin{equation*}
A_{\odot}(N=3)=(\mathrm{N} \times P)\left\{\sum_{i=1}^{P-1} \sum_{j=1}^{2} \frac{z_{i+j-1}}{2} \sqrt{x_{i}^{2}-\frac{1}{4} z_{i+j-1}^{2}}\right\} \times \sum_{k=1}^{\infty} \cos ^{2 k}\left(60^{\circ}\right) \tag{28}
\end{equation*}
$$

The various geometrical properties in this particular example - such as the angles $\gamma_{o, n}$ and $\gamma_{i, n}$, sides $z$ and $x$, and areas $A$ of the successive triangles within the strip (that together form the area of the most outward ring) - are computed from Eq.(2), Eq.(3), Eq.(7), Eq.(10) and Eq.(18) [respectively], being presented in Table 2. Altogether, the sum of the area of the triangles gives the area of the strip to be 0.06100423 . Replacing the area of the strip back into Eq.(28) - while knowing at the same time that the scaling series given by Eq.(27) tends at infinity to the value $1 / 3$ - results in a total area for the approximated circle using only a quantity of four $(P=4)$ revolved triangles ( $N=3$ ) to be

$$
\begin{equation*}
A_{\odot}(N=3)=(4 \times 3)\{0.06100423\} \times \frac{1}{3}=0.24401692 \mathrm{~mm}^{2} \tag{29}
\end{equation*}
$$

This means that the ratio between the approximate area of the circle and its circumscribed regular triangle is

$$
\begin{equation*}
\frac{A_{\odot}(N=3)}{A_{\Delta}}=\frac{0.24401692}{\sqrt{3} / 4}=0.5635329378 \tag{30}
\end{equation*}
$$

Following the same process (as done in the previous section 3.1), the multiplication of this ratio by the number of sides of a triangle $(N=3)$ gives

$$
\begin{equation*}
\pi_{60^{\circ}}=3 \times 0.5635329378=1.690598813 \tag{31}
\end{equation*}
$$

From this new perspective, when using a regular unit triangle during the process, the end result changes. It is important to note that this result does not substitute the traditional constant of $\pi\left(=\pi_{90^{\circ}}\right)=3.141592653589793$, but instead it complements as an extension (with a different $\pi_{60^{\circ}}$ ) into a possible function. More specifically, for a 60 -degree based system, the significance of $\pi_{60^{\circ}}$ becomes the area ratio of a circle to its circumscribed triangle times its number of sides (i.e., $N=3$ ). This suggests that (by replacing the four angularly equidistant revolved unit squares by triangles) the constant $\pi_{90^{\circ}}$ becomes $\pi_{60^{\circ}}$ having (as a first approximation) the value of 1.690598813. Since this approach using triangles is new, there is no official value for $\pi_{60^{\circ}}$. One way to ascertain the validity of Eq.(28) is by corroborating it against a numerically determined value using a Computer-Aided Design program - of which open-source software Geogebra (Feng 2014) and FreeCAD (Havre 2021) are examples.

Table 2. Properties of the triangles composing the strip in the most outward ring (triangle approach)

| Angle $\boldsymbol{\gamma}$ |  | Side z | Side x |
| :---: | :---: | :---: | :---: |
| Outward | Inward |  | Area A |
| $\gamma_{10}=90^{\circ}$ | $\gamma_{1 i}=60^{\circ}$ | $z_{1}=0.29885849$ | $x_{0}=0.21132487$ |
| $\gamma_{20}=120^{\circ}$ | $\gamma_{2 i}=90^{\circ}$ | $z_{2}=0.21132487$ | $x_{1}=0.12200847$ |
| $\gamma_{3 o}=150^{\circ}$ | $\gamma_{3 i}=120^{\circ}$ | $z_{3}=0.17254603$ | $x_{2}=0.08931640$ |
|  |  | $z_{4}=0.15470054$ |  |
|  |  |  | $A_{0,2}=0.022329100$ |
|  |  |  | $A_{i, 2}=0.0074430331$ |
|  |  |  | $A_{0,3}=0.0019943547$ |
|  |  |  |  |

The software Geogebra is used here, and this exercise is encouraged to students and experts to allow for an independent verification of the results. Draw a regular triangle of side unit with an inscribed circle, and measur the area of the circle and triangle 0.261799387799149 and 0.433012701892219 , respectively. According to Eq.(28) and Eq.(29), the value of $\pi_{60^{\circ}}$ is obtained by dividing the two areas, and multiplying the result by the number of sides of a triangle $(N=3)$, resulting in

$$
\begin{equation*}
\pi_{60^{\circ}}=3 \times \frac{A_{\odot}(N=3)}{A_{\Delta}}=3 \times \frac{0.261799387799149}{0.433012701892219}=1.813799364 \tag{32}
\end{equation*}
$$

This estimate is confirmed by measuring the perimeter of the circle, which gives in CAD a value with fourteen decimal places as 1.81379936423421 . Selecting any arbitrary circle circumscribed by a regular triangle of side $L=A I$ results in the same value of $\pi_{60^{\circ}}(L=A I=1$ is used as a matter of convenience). Now that an accurate reference value is determined, the prediction given by the method can be compared. More accurate estimates (than 1.690598813) can be computed from the annexed Octave program with the following settings: the number of sides is set to $N=3$; and the fractal area scaling factor set to $f=1 / 4$ (which essentially means, replacing - in the computation - the unit-side square by a unit-side regular triangle). As mentioned before, more time and/or more computational power allows for a greater number of triangles to be used, and thus achieving greater accuracy. By increasing the number of rotating triangles to two million, the approximation of the present method to $\pi_{60^{\circ}}$ reaches nine decimal places of the reference CAD value as 1.813799364 (099003) [Figure 7].


Figure 7. Accuracy of $\pi_{60^{\circ}}$ for changing number of interfered rotating triangles

This suggests that the ratio between areas (of circle over circumscribed triangle) is the same as the ratio between constants ( $\pi_{60^{\circ}}$ and the number of sides in a triangle), or

$$
\begin{equation*}
\frac{A_{\odot}(N=3)}{A_{\Delta}}=\frac{\pi_{60^{\circ}}}{3} \quad \Rightarrow \quad A_{\odot}(N=3)=\pi_{60^{\circ}} \frac{\sqrt{3}}{12} L^{2} \tag{33}
\end{equation*}
$$

This variant equation offers a new perspective to find the area of a circle using triangles - Eq.(31) - is just as valid as the classical Eq.(24) that uses squares, where the main different being the term composed of the fractional area of the polygon (i.e., that represents the linearity aspect within the ratio $\pi$ ). In this new view, the definition of $\pi_{60^{\circ}}$ (from an area perspective, instead of length) is the ratio of the area of a circle $A_{\odot}$ to a third of the area ABC of the circumscribed triangle $A_{\Delta} / 3$ (i.e., area CGOH) [Figure 8a], while the definition of $\pi_{90^{\circ}}$ is the ratio between the area of a circle $A_{\odot}$ to a fourth of the area ABCD of the circumscribed square $A_{■} / 4$ (i.e., area DGOH ) [Figure 8b]. Figure 8 shows in transparency the superimposed state of both the areas of the circle and of the fraction of the polygon. Note that a sector with the same area is achievable either as CGOH or COA for the triangle (Figure 8a), and DGOH or DOA for the square (Figure 8b).

(b)


Figure 8. Ratio of areas provides the value of (a) $\pi_{60^{\circ}}$ for triangles and (b) $\pi_{90^{\circ}}$ for squares

### 3.4 Higher polygons approach

This method can be applied to even higher polygons - like pentagons (Figure 9a) and hexagons (Figure 9b). The grid pattern becomes tighter as the polygon exhibits wider internal angles. In both Figures 9a and 9b, the side of the most outward polygon is $L=A I=1$, and are not to scale with respect to one another, which does not affect the associated estimates of $\pi_{108^{\circ}}$ and $\pi_{120^{\circ}}$. As shown before (for squares and triangles), the same process can be applied using other polygons, having only to change (in the annexed Octave program) the following key values: the number of sides of the new polygon (i.e., $N=5$ for the pentagon and $N=6$ for the hexagon); area of the polygon circumscribing the circle (i.e., 1.720477400588967 for the pentagon and 2.598076211353316 for the hexagon) and the area fractal scaling (i.e., $\cos ^{2}\left(36^{\circ}\right)$ for pentagon and $\cos ^{2}\left(30^{\circ}\right)$ for hexagon). In the pentagon approach (Figure 9a), the ratio of the area of the approximated circle and its circumscribed regular pentagon is

$$
\begin{equation*}
\frac{A_{\odot}(N=5)}{A_{\text {Pentagon }}}=\frac{(4 \times 5)\{0.03679145127946321\} \times \sum_{k=0}^{\infty} \cos ^{2 k}\left(36^{\circ}\right)}{1.720477400588967}=0.8102254139 \tag{34}
\end{equation*}
$$

And for the hexagon approach (Figure 9b), the ratio becomes

$$
\begin{equation*}
\frac{A_{\odot}(N=6)}{A_{\text {Hexagon }}}=\frac{(4 \times 6)\{0.03067014936071338\} \times \sum_{k=0}^{\infty} \cos ^{2 k}\left(30^{\circ}\right)}{2.598076211353316}=0.736083584657121 \tag{35}
\end{equation*}
$$



Figure 9. Area of the circle defined fractally by multiple rings, each formed by 16 (a) pentagons and (b) hexagons
Following the same procedure (as for the square and triangle), multiplying these ratios by the number of sides of the respective polygon gives

$$
\begin{equation*}
\pi_{108^{\circ}}=5 \times 0.8102254139=4.051127069515934 \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
\pi_{120^{\circ}}=6 \times 0.736083584657121=5.099736668974243 \tag{37}
\end{equation*}
$$

where Eq.(36) and Eq.(37) providing two extra constants for the function expression of $\pi_{\theta}$ (to de discussed later). As before, accuracy of the prediction increases with the number of rotating polygons (when comparing to the CAD measuring of the perimeter of the incircle with 15 decimal places). When the number of rotating pentagons and hexagons reaches two million, seven decimal places are achieved for $\pi_{108^{\circ}}=4.3240313$ (30148895) [Figure 10a] and nine decimal places for $\pi_{120^{\circ}}=5.441398092$ (964926) [Figure 10b].


Figure 10. Accuracy of $\pi_{\theta}$ for changing number of rotating (a) pentagons [for $\pi_{108^{\circ}}$ ] and (b) hexagons [for $\pi_{120^{\circ}}$ ]

Here, the generic circle area for any polygon is

$$
\begin{equation*}
A_{\odot}(N, P)=(N \times P)\left\{\sum_{i=1}^{P-1} \sum_{j=1}^{2} \frac{z_{i+j-1}}{2} \sqrt{x_{i}^{2}-\frac{1}{4} z_{i+j-1}^{2}}\right\} \times \sum_{k=1}^{\infty} \cos ^{2 k}\left(\frac{180}{N}\right) \tag{38}
\end{equation*}
$$

where $N$ is the number of sides of the polygon, $P$ the number of rotating elements, $z$ the circumferentially aligned length of the small triangles [given by Eq.(14)] and $x$ their radially aligned lengths [given by Eq.(15)]. Extrapolating from the triangular approach [Eq.(33)] and square approach [Eq.(26)] gives the resulting area expression for each corresponding circle as given by the product of relevant $\pi$ and the fractional area of that polygon (of side L), thus giving the corresponding expressions for polygons and hexagons respectively as

$$
\begin{equation*}
A_{\odot}(N=5)=\pi_{108^{\circ}} \frac{L^{2}}{4 \sqrt{5-2 \sqrt{5}}} \quad ; \quad A_{\odot}(N=6)=\pi_{120^{\circ}} \frac{\sqrt{3}}{4} L^{2} \tag{39}
\end{equation*}
$$

All together - Eq.(26), Eq.(33) and Eq.(39) - form (side-by-side) Table 3, highlighting the progression of the ratios.
Table 3. Ratio of areas of a circle over that of the corresponding circumscribed polygon

| Triangle | Square | Pentagon | Hexagon |
| :---: | :---: | :---: | :---: |
| $\frac{\pi_{60^{\circ}} \frac{\sqrt{3}}{12} L^{2}}{\frac{\sqrt{3}}{4} L^{2}}=\frac{\pi_{60^{\circ}}}{3}$ | $\frac{\pi_{90^{\circ}} D^{2} / 4}{D^{2}}=\frac{\pi_{90^{\circ}}}{4}$ | $\frac{\pi_{108^{\circ}} \frac{L^{2}}{4 \sqrt{5-2 \sqrt{5}}}}{\frac{5 L^{2}}{4 \sqrt{5-2 \sqrt{5}}}}=\frac{\pi_{108^{\circ}}}{5}$ | $\frac{\pi_{120^{\circ}} \frac{\sqrt{3}}{4} L^{2}}{\frac{3 \sqrt{3}}{2} L^{2}}=\frac{\pi_{120^{\circ}}}{6}$ |

### 3.3 Exercise Earth

A simple practical application of this method is the measurement of the projected area of the Earth (which, for the purpose of this exercise, is assumed to be a circle). Imagine taking a photograph of Earth from space (Figure 11) with its dimensions calibrated by a visible measurement (possibly a known distance between two points on the surface), allowing the conversion of any length in the photograph to a distance in kilometers. Circumscribing a square to the Earth (Figure 11a) and measuring its side $L=A B=D$ gives a value of $12,756.274 \mathrm{~km}$ (Williams 2021) - which is in fact the same as measuring the diameter of the Earth $A^{\prime} B^{\prime}$ (i.e., since the diameter is by definition the line within the circle that is perpendicular to two opposing tangents, then this perpendicularity denotes, for the purpose of this approach, implicitly the presence of a square). Applying the conventional area formula Eq.(26) gives

$$
\begin{equation*}
A_{\odot}(N=4)=\pi_{90^{\circ}} \frac{D^{2}}{4}=3.1415926536 \times \frac{12,756.274^{2}}{4}=127,801,973.3 \mathrm{~km}^{2} \tag{40}
\end{equation*}
$$



Figure 11. Measuring the projected area of Earth using revolving (a) squares and (b) triangles [not to scale]

Here, the projected area of the Earth was computed using an orthogonal axes system, where lengths are measured along linear paths (axes) at right angles to each other. Note that the interference pattern shown in Figure 11 (circling the Earth) is only to highlight the presence of the present method, in which only the overlapping fractal rings play a role in determining the Earth's projected area. While the application of radius/diameter (in Figure 11a) is one possibility, there are other approaches that provide the same result. Circumscribing a regular triangle to the Earth (Figure 11b) and measuring its side $L=A C$ gives a value of $14,539.008 \mathrm{~km}$. Alternatively, inscribing a triangle within the Earth provides a side $L / 2=A^{\prime} C^{\prime}=7,269.504 \mathrm{~km}$, which also provides the same end result $L$ above. Applying this value of $L$ to the area formula Eq.(33) for triangles gives the same area as Eq.(40) for squares or

$$
\begin{equation*}
A_{\odot}(N=3)=\pi_{60^{\circ}} \frac{\sqrt{3}}{12} L^{2}=1.8137993642 \times \frac{\sqrt{3}}{12} 22,094.51468^{2}=127,801,973.3 \mathrm{~km}^{2} \tag{41}
\end{equation*}
$$

Here, the projected area of the Earth was computed using a triangular axes system, where lengths are measured along linear paths (axes) at 60 degrees to each other. Since both equations give the same answer, they are both equally valid when determining the area of a circle, in any context, the main difference being a change in the adopted reference linear length, which is dependent on the chosen polygon. One equation does not replace the other, and they are both in fact complementary. Even higher polygons could be used to compute the same area via their respective equations. For instance, for pentagon ABCDE the side $L=E A$ (in Figure 12a) would measure 9,365.959212 km giving the area

$$
\begin{equation*}
A_{\odot}(N=5)=\pi_{108^{\circ}} \frac{L^{2}}{4 \sqrt{5-2 \sqrt{5}}}=4.2340313299 \times \frac{9,365.959212^{2}}{4 \sqrt{5-2 \sqrt{5}}}=127,801,973.3 \mathrm{~km}^{2} \tag{42}
\end{equation*}
$$

And for hexagon ABCDEF , the side $L=F A$ (in Figure 12b) would measure $7,364.837069 \mathrm{~km}$ giving the area

$$
\begin{equation*}
A_{\odot}(N=6)=\pi_{120^{\circ}} \frac{\sqrt{3}}{4} L^{2}=5.4413998027 \times \frac{\sqrt{3}}{4} \times(7,364.837069)^{2}=127,801,973.3 \mathrm{~km}^{2} \tag{43}
\end{equation*}
$$


(b)


Figure 12. Measuring the projected area of Earth using revolving (a) pentagons and (b) hexagons [not to scale]
As another example, imagine a constellation of six equidistant satellites transiting around the Earth in the same circular orbit (Figure 13). This exercise aims to provide insight into the simple application of the method onto a realistic scenario - like for example, a properly geometrically-spaced GPS satellite constellation orbiting the Earth (Moorefield, Jr. 2020). It is worth noting that in reality such a constellation has typically 24 satellites disposed in a 3D configuration, however, for the purpose of this simplistic exercise the chosen 2D case of six satellites is deemed sufficient. It is also assumed that they are in a Low Earth Orbit at an altitude of 400km. Imagine now that two adjacent satellites A and B measure the distance to each other (by means of laser, for example) to be $A B=6,778.137 \mathrm{~km}$, which is the side of the hexagon ABCDEF inscribed inside the orbit defined by the six satellites (i.e., the distance of a corner of a hexagon to its centre is the same as the radius R , which incidentally is the same as the side of the hexagon).


Figure 13. Constellation of angularly equidistant satellites in a circular orbit around Earth
However, the distance AC between two alternating satellites A and C (forming altogether the triangular configuration ACE ) is unknown. The computation of distance AC (by means of the present method) is achieved by equating the area of the circular orbit using the triangular approach [with Eq.(33)] to that using the hexagonal approach [with Eq.(39)], resulting in

$$
\begin{equation*}
A_{\odot}=\pi_{120^{\circ}} \frac{\sqrt{3}}{4}\left(\frac{6,778.137}{\cos 30^{\circ}}\right)^{2}=\pi_{60^{\circ}} \frac{\sqrt{3}}{12}\left(\frac{A C}{\cos 60^{\circ}}\right)^{2}=144,334,634.8 \mathrm{~km}^{2} \tag{44}
\end{equation*}
$$

The length of the side of the regular hexagon circumscribing the orbit is $6,778.137 / \cos 30^{\circ}=7826.718 \mathrm{~km}$ (that is, one fractal level outwards), and the length of the side of the regular triangle also circumscribing the orbit is $A C / \cos 60^{\circ}$ (also located one fractal level outwards). As explained before - as part of Eq.(21) for Figure 2a, and Eq.(27) for Figure 6 a - the sides of two polygons of subsequent fractal levels are related via the multiplication/division coefficient $\cos \left(180^{\circ} / N\right)$, where $N$ is the number of sides of the polygon. As an outcome, the distance between the satellites in a triangular formation is $A C=11,740.077 \mathrm{~km}$. While there may be other trigonometrical ways to compute this, the present method offers a solution that uses a circular medium - i.e., the area of a circle - to interlink two linear lengths AB and AC pertaining to inscribed polygons, and this is done without having to know the altitude or radius R of the orbit. Similar relations can be achieved between other polygons.

### 3.5 Relativity of $\pi$

The general expression for the area of a circle $A_{\odot}(N)$ circumscribed by a $N$-sided regular polygon of area $A_{\text {polygon }}(N)$, is extrapolated from Table 3 as being

$$
\begin{equation*}
A_{\odot}(N)=\pi_{\theta} \frac{A_{\text {polygon }}(N)}{N} \tag{45}
\end{equation*}
$$

where the function $\pi_{\theta}$ has a specific value depending on the selected polygon, here identified by its internal angle $\theta=180^{\circ}-360^{\circ} / \mathrm{N}$. Equation (45) means that for any given circle, if the area of the circumscribing N -sided regular polygon (of side $L$ ) used as reference $A_{\text {polygon }}(N)$ changes, then the specific value of the function $\pi_{\theta}$ for that polygon also needs to change accordingly in order to obtain the same circle area being considered. In so doing, the area of a circle is now connected to any polygon (not just squares, as done traditionally).

$$
\begin{equation*}
A_{\text {polygon }}(N)=\frac{N}{4 \tan \left(\frac{180}{N}\right)} L^{2} \tag{46}
\end{equation*}
$$

Figure 14a shows the specific values of $\pi_{\theta}$ (defined from CAD ) for the first seven polygons, plot as a function of the internal angle $\theta$. When curve fitting, the relation takes on the form of the scaled tangent

$$
\begin{equation*}
\pi_{\theta}=\pi_{90^{\circ}} \tan \left(\frac{\theta}{2}\right) \tag{47}
\end{equation*}
$$

This relation highlights the function $\pi_{\theta}$ to be dependent on the chosen reference system of axes (i.e., the internal angle $\theta$ of $90^{\circ}$ for orthogonal, $60^{\circ}$ for triangular, etc). The reason why $\pi_{90^{\circ}}$ appears as the reference value in Eq.(47) against which others are defined - is because trigonometric functions sine and cosine are inherently defined using an orthogonal $x-y$ axes system. If another system of axes would be used (e.g., triangular), that would be the new reference against which all other values of function $\pi_{\theta}$ (including $\pi_{90^{\circ}}$ ) would be defined against. Figure 15 b plots the same values of function $\pi_{\theta}$ for the first seven polygons, but this time with respect to the number of sides $N$ of the polygon. The purpose is to highlight the underlying relation in Eq.(45) that suggest that as the area of the polygon $A_{\text {polygon }}(N)$ tends to that of the circle $A_{\odot}(N)$ - and the number of sides N tends to infinity - the value of the function $\pi_{\theta}$ tends to the number of sides $N$ of the infinite polygon. Seen from the other way around - still from the perspective of Figure 14 b - the discrepancy between the value $3.1415 \ldots$ and 4 reflects the discrepancy in properties (i.e., perimeter and area) from an infinitely-sided polygon $N=\infty$ (i.e., the circle) and the corresponding circumscribing finite-sided polygon with four sides $N=4$ (i.e., the square circumscribing the circle). Rewriting Eq.(45) below allows another interpretation

$$
\begin{equation*}
\frac{A_{\odot}(N)}{\pi_{\theta}}=\frac{A_{\text {polygon }}(N)}{N} \tag{48}
\end{equation*}
$$

In this equality of ratios, the function $\pi_{\theta}$ has the purpose of defining the amount of area of the circle that is necessary to match the area of a sector of the circumscribing polygon. In other words, if the area of a sector of a circumscribing polygon $A_{\text {polygon }}(N) / N$ would be defined in a circular manner, it would become the portion of the area of the circle $A_{\odot}(N) / \pi_{\theta}$.
(a)

(b)


Figure 14. Convergence of function $\pi_{\theta}$ for evolving polygon's (a) internal angle $\theta$ and (b) number of sides $N$
Since the properties of a circle are all interconnected, it is plausible to assume that there is an equation similar to Eq.(45) relating perimeters instead of areas - i.e., the perimeter of a circle $P_{\odot}(N)$ to that of its circumscribed polygon $P_{\text {polygon }}(N)=N \times L$. The results is a general expression of the perimeter of circle circumscribed by an N -sided polygon

$$
\begin{equation*}
P_{\odot}(N)=\pi_{\theta} \frac{P_{\text {polygon }}(N)}{N} \tag{49}
\end{equation*}
$$

It is worth noting that a former observation for the area is again present for the perimeter - by rewriting Eq.(49) into the ratio format in Eq. (50) - in that the function $\pi_{\theta}$ has the purpose of segmenting the circle's perimeter necessary to match a sector of the perimeter of the circumscribing polygon. In other words, if the perimeter of a sector of a
circumscribing polygon $P_{\text {polygon }}(N) / N$ would be defined in a circular manner, it would become the portion of the perimeter of the circle $P_{\odot}(N) / \pi_{\theta}$.

$$
\begin{equation*}
\frac{P_{\odot}(N)}{\pi_{\theta}}=\frac{P_{\text {polygon }}(N)}{N} \tag{50}
\end{equation*}
$$

As a first verification that Eq.(50) is valid, consider the polygon to be a square with $N=4$ and $\theta=90^{\circ}$. Here, Eq.(50) simplifies to the classical $P_{\odot}(N=4)=\pi_{90^{\circ}} D$, where $D$ is both the side of the square and the diameter of its inscribed circle. As it happened before in Eq.(48) [for the area of the circle], Eq.(50) establishes an equality between the ratio of the circle's perimeter over the polygon's perimeter, and the ratio of the corresponding value of the function $\pi_{\theta}$ and the number of sides $N$ of the polygon. The variants of Eq.(50) for triangles, pentagons and hexagons are listed in Table 4. Computer-Aided Design (CAD) software - such as Geogebra (Feng 2013) or FreeCAD (van Havre et al. 2021) - can be used to validate the equations in Table 4. Start by plotting a unit-side regular triangle, square, pentagon and hexagon (i.e., $L=D=1$ ), all sharing a common side (Figure 15). Inscribing in each a circle, and measuring the perimeter gives the values for the corresponding $\pi_{\theta}$, as computed earlier using the revolving polygon approach - thus confirming that Eq.(50) is valid for up to at least $N=6$. Concluding, from a one-dimensional perspective, the function $\pi_{\theta}$ is the ratio of the perimeter of a circle $P_{\odot}(N)$ to the fraction of the perimeter of the circumscribed N -sided unit regular polygon $P_{\text {polygon }}(N) / N$. From a two-dimensional perspective (and as seen before), the function $\pi_{\theta}$ is the ratio of the area of a circle $A_{\odot}(N)$ to the fraction of the area of the circumscribed N -sided unit regular polygon $A_{\text {polygon }}(N) / N$.

Table 4. Ratio of the perimeter of a circle over the perimeter of the corresponding circumscribed polygon

| Triangle | Square | Pentagon | Hexagon |
| :---: | :---: | :---: | :---: |
| $\frac{\pi_{60^{\circ}} L}{3 L}=\frac{\pi_{60^{\circ}}}{3}$ | $\frac{\pi_{90^{\circ}} D}{4 D}=\frac{\pi_{90^{\circ}}}{4}$ | $\frac{\pi_{108^{\circ}} L}{5 L}=\frac{\pi_{108^{\circ}}}{5}$ | $\frac{\pi_{120^{\circ} L}}{6 L}=\frac{\pi_{120^{\circ}}}{6}$ |



Figure 15. Relativity of $\pi_{\theta}$ as a function of the $N$-sided unit regular polygon circumscribing a circle
Hence, both definitions result in joining Eq.(48) and Eq.(50) as

$$
\begin{equation*}
\frac{A_{\odot}(N)}{A_{\text {polygon }}(N)}=\frac{\pi_{\theta}}{N}=\frac{P_{\odot}(N)}{P_{\text {polygon }}(N)} \tag{51}
\end{equation*}
$$

The relativity of the function $\pi_{\theta}$ also offers an expansion of the definition of the angular measure radian - which is a unit of angular measure quantifying the arc length covered when it is equal to the linear radius $R$ (or half the diameter $D / 2$ ). Here, the usage of radius implies an orthogonal relation (between circular and linear) which indicates that the
definition of radian is governed by a square approach. Using other polygons, the relation implied by the radian changes as the linear reference radius R (as previously highlighted in Figure 11a) alters to half the characteristic length of the polygon L/2 (as highlighted for a triangle in Figure 11b, and for pentagon and hexagon in Figure 12a and 12b). Since the etymology of the word radian is derivative from radius, a new term to express the new relation is required, being one possibility an equivalent derivation from the key word polygon, as polyan. Here the radian is comprised as a special case of a polyan, where the side of the polygon circumscribing the circle (i.e., a square) is $2 R=D$. As a conclusion, the relativity of the properties of a circle (i.e., perimeter, area, etc) emerges from the acknowledgment that there is more than one linear reference - that being, the side of a family of regular polygons circumscribing the circle - against which said properties can be readily quantified, and thus to which they are relative.

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```
Annex
clc, clear
% SETUP
format long % Increases decimal place output
N=4; % Sides of polygon
p=100000; % Number of polygons % Line 5
if p>10 probe=0; else probe=1; end % Enable extra details
% Identification of which polygon approach
if N==3 display('TRIANGLE Approach') else
    if N==4 display('SQUARE Approach') else
        if N==5 display('PENTAGON Approach') else % Line 10
            if N==6 display('HEXAGON Approach') else
                if N==7 display('HEPTAGON Approach')
                        else display('HIGHER POLYGON Approach')
end end end end end
display('--- Sides in a polygon and number revolving'), display(N), display(p) % Line 15
display('--- Angle per sector (degrees)'), delta=360/(N*p)
display('--- Internal angle of polygon (degrees)'), theta=180-360/N
display('--- Area scaling factor due to fractality'), f=\operatorname{cosd}(180/N)^2
display('--- Area of the N-sided unit regular polygon'), Apoly=[N/(4*tand(180/N))]
% SIDES X(N) and Z(N) % Line 20
Alpha1=(180-delta)/2-theta/2;
display('--- Sides of most outward triangle in the strip')
for i=1:(p-1)
    Q(i)=sqrt(1-cosd(theta+(i-1)*delta))/sqrt(1-cosd(theta+(i)*delta));
    if i~=(p-1) % Line 25
        K(i)=sqrt(1-\operatorname{cosd}(theta+(i-1)*delta))/sqrt(1-\operatorname{cosd}(theta+(i+1)*delta));
end end
sQ=prod(Q); sK(1)=1;
for m=1:(p-2) % Line 30
    sK(m+1)=prod(K(1:m));
end
x=1/(2*(sum(sK))+sQ*sqrt(2*(1-cosd(theta+delta)))); x1=x(1)
z=x1*sqrt(2*(1-cosd(theta+delta))); z1=z(1)
for v=1:(p-1) % Line 35
    z(v+1)=sqrt(1-cosd(((theta))+(v-1)*delta))/sqrt(1-cosd(theta+(v)*delta))*z(v);
    if v~=(p-1)
        x(v+1)=sqrt(1-cosd(((theta))+(v-1)*delta))/sqrt(1-cosd(theta+(v+1)*delta))*x(v);
    else }\textrm{x}=\textrm{x}(1:v)
end end % Line 40
if probe==1
    display('--- Sides of subsequent triangles'), display(z); display(x);
```


## end

```
% AREAS : TRIANGLE, STRIP & FIRST RING
s=0; l=0; % Line 45
for i=1:(p-1)
    for j=1:2
        s=s+1; At(s)=z(i+j-1)/2*sqrt(x(i)^2-1/4*z(i+j-1)^2);
end end
if probe==1 disp('--- Area of each triangle in outward strip'); disp(At) end % Line 50
display('--- Area of most outward strip'), As=sum(At)
display('--- Area of most outward ring'), Ar=As*(N*p)
% FRACTAL SCALING SUM OF RINGS
SumScale=0; % Starting variable for loop
n=1000; % Number of inward rings considered (ideally infinite) % Line 55
for k=1:n
SumScale=SumScale+f}\mp@subsup{\textrm{f}}{}{\wedge}\textrm{k};%\mathrm{ Formation of the series
end
display('--- Scaling series due to inward fractality'), Fseries=SumScale
if probe==1 disp('Scaling series'); disp(Fseries) end % Line }6
% APPROX. AREA OF CIRCLE
display('--- Area of circle'), Ac=Ar*Fseries
% RATIO OF AREAS
display('--- Ratio of areas circle/polygon'), Ratio=Ac/Apoly
% FUNCTION PI FOR SELECTED POLYGON % Line }6
display('--- Value of Pi (for the chosen polygon)'), PiN=Ratio*N
```


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