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The Infimum Norm of Completely Positive Maps

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Abstract

Let A be a unital C^*-algebra, let $L: A \to B(H)$ be a linear map, and let $\emptyset$: $A \to B(H)$ be a completely positive linear map. We prove the property in the following: $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 \\ L^* & \emptyset & L \\ 0 & L^* & \emptyset \end{pmatrix}$ is completely positive$\} = inf\ \{\|T^*T + TT^*\|^{\frac{1}{2}}: L = V^*T\pi V$ which is a minimal commutant representation with isometry$\}$. Moreover, if $L = L^*$, then $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 \\ L & \emptyset & L \\ 0 & L & \emptyset \end{pmatrix}$ is completely positive$\} = \sqrt{2}\|L\|_{cb}$. In the paper we also extend the result $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L \\ L^* & \emptyset \end{pmatrix}$ is completely positive$\} = inf\{\|T\|: L = V^*T\pi V\}$ [3 , Corollary 3.12].

Keywords: positive operators, completely positive maps, completely bounded maps

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1. Introduction

Let M_n denote the C^*-algebra of complex $n \times n$ matrices and $B(H)$ the algebra of all bounded linear operators on a Hilbert space H.Let A and B be C^*- algebras and let $L: A \to B$ be a bounded map linear map. The map L is called positive if $L(a)$ is positive whenever a is positive. The map L is called completely positive if $L \otimes I_n: A \otimes M_n \to B \otimes M_n$ defined by $L \otimes I_n(a \otimes b) = L(a) \otimes b$ is positive for all n. From [3], $\|L\|_{w_\rho} = sup\{w_\rho(L(a)): w_\rho(a) \leq 1\}$. The map L is w_ρ completely bounded if $sup_n\|L \otimes I_n\|_{w_\rho}$ is finite. Notice that $\|L\|_{cb} = \|L\|_{w_1 cb}$ and $\|L\|_{wcb} = \|L\|_{w_2 cb}$. The map $L = L^*$ if $L(a) = L(a^*)^*$. From [2], we know that every completely bounded map L from A to $B(H)$ has a minimal commutant representation $L = V^*T\pi V(m.c.r.i)$with T in the commutant of $\pi(A)$ and isometry V. In the paper we obtain a lower bound for the set $\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 & 0 \\ L^* & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L^* & \emptyset \end{pmatrix}_{m \times m}$ is completely positive with $m \geq 2\}$ which extends the property [3] $inf\ \{\|\emptyset\|: \begin{pmatrix} \emptyset & L \\ L^* & \emptyset \end{pmatrix}$ is completely positive$\} = inf\{\|T\|: L = V^*T\pi V\ (m.c.r.i)\}$. In particular, we have the value of $inf\ \{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 \\ L^* & \emptyset & L \\ 0 & L^* & \emptyset \end{pmatrix}$ is completely positive$\}$ in the paper.

2. Infimum Norm

Proposition 2.1. Let A be a unital C^*-algebra and $L: A \to B(H)$ be a completely bounded map, then $inf\ \{\|\emptyset\|:$

$\begin{pmatrix} \emptyset & L & 0 & 0 \\ L^* & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L^* & \emptyset \end{pmatrix}_{m\times m}$ is completely positive$\}= inf\ \{k$: $\begin{pmatrix} k & T & 0 & 0 \\ T^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & k \end{pmatrix}_{m\times m}$ is positive, $L = V^*T\pi V\ (m.c.r.i)\}$,

$m \geq 2$. **Proof**. If $\begin{pmatrix} \emptyset & L & 0 & 0 \\ L^* & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L^* & \emptyset \end{pmatrix}_{m\times m}$ is completely positive, from the proof of [4, Theorem 2.6], the matrix $\begin{pmatrix} \|\emptyset\| & T & 0 & 0 \\ T & \|\emptyset\| & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & \|\emptyset\| \end{pmatrix}_{m\times m}$ is positive, where $L = V^*T\pi V$ (m.c.r.i) with an isometry V, a *-representation π, and T in the commutant of $\pi(A)$. Conversely, if $\begin{pmatrix} k & T & 0 & 0 \\ T^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & k \end{pmatrix}_{m\times m}$ is positive and $L = V^*T\pi V$ (m.c.r.i), by [2, Proposition 2.6], we have $\begin{pmatrix} kV^*\pi V & L & 0 & 0 \\ L^* & kV^*\pi V & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L^* & kV^*\pi V \end{pmatrix}_{m\times m}$ is completely positive, where $V^*\pi V$ is a unital completely positive.

Corollary 2.2. [3] $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L \\ L^* & \emptyset \end{pmatrix}$ is completely positive$\} = inf\ \{\|T\|: L = V^*T\pi V$ (m.c.r.i)$\} = \|L\|_{w_2cb}$.

Proof. $Let\ m = 2$ in Proposition 2.1.

Lemma 2.3. $\begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix} \geq 0$ if and only if $k^2I \geq T^*T + TT^*$ where $k > 0$.

Proof. Since $\begin{pmatrix} k & T \\ T^* & k \end{pmatrix} \geq \frac{1}{k}(T,0)^*(T,0)$, we have $\begin{pmatrix} k - \frac{1}{k}T^*T & T \\ T^* & k \end{pmatrix} \geq 0$.

Since $\begin{pmatrix} k & T^* \\ T & k - \frac{1}{k}T^*T \end{pmatrix} \geq 0$, we have $k - \frac{1}{k}T^*T \geq \frac{1}{k}TT^*$.

Lemma 2.4. [5] $min\ \{k: \begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix} \geq 0\} = \|T^*T + TT^*\|^{\frac{1}{2}}$.

Proof. If $\begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix} \geq 0$, applying Lemma 2.3, we have $k \geq \|T^*T + TT^*\|^{\frac{1}{2}}$.

Since $(\|T^*T + TT^*\|^{\frac{1}{2}})^2 I \geq T^*T + TT^*$, applying Lemma 2.3, we have

$$\begin{pmatrix} \|T^*T + TT^*\|^{\frac{1}{2}} & T & 0 \\ T^* & \|T^*T + TT^*\|^{\frac{1}{2}} & T \\ 0 & T^* & \|T^*T + TT^*\|^{\frac{1}{2}} \end{pmatrix} \geq 0.$$

Theorem 2.5. $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 \\ L^* & \emptyset & L \\ 0 & L^* & \emptyset \end{pmatrix}$ is completely positive$\} = inf\{\|T^*T + TT^*\|^{\frac{1}{2}}$: $L = V^*T\pi V$ (m.c.r.i)$\}$.

Proof. Let $m = 3$ in Proposition 2.1, applying Lemma 2.4, we have the Theorem.

Lemma 2.6. Let $T \in B(H)$. Then $2w(S_m)S(L) \le inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 & 0 \\ L^* & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L^* & \emptyset \end{pmatrix}_{m\times m}$ is completely positive $\} \le 2w(S_m)\|L\|_{w_2cb}$ $(m \ge 2)$, where $S_m = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & . & 0 & 0 \end{pmatrix}_{m\times m}$, $S(L) = inf\{w(T): L = V^*T\pi V\ (m.c.r.i)\}$, and [3, Corollary 3.12] $\|L\|_{w_2cb} = inf\{\|T\|: L = V^*T\pi V\ (m.c.r.i)\}$.

Proof. From [6], we know that $2w(S_m)w(T) \le inf\{k: \begin{pmatrix} k & T & 0 & 0 \\ T^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & k \end{pmatrix}_{m\times m}$ is positive$\} \le 2w(S_m)\|T\|$ with $m \ge 2$. Applying Proposition 2.1, we have $2w(S_m)inf\{w(T): L = V^*T\pi V\ (m.c.r.i)\} \le inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 & 0 \\ L^* & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L^* & \emptyset \end{pmatrix}_{m\times m}$ is completely positive$\} \le 2w(S_m)inf\{\ \|T\|: L = V^*T\pi V\ (m.c.r.i)\}$. Applying [4, Theorem 2.6] and [3, Corollary 3.12], we have the Lemma.

Theorem 2.7. If $L = L^*$, then $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 & 0 \\ L & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L & \emptyset \end{pmatrix}_{m\times m}$ is completely positive $\} = inf\{k: \begin{pmatrix} k & T & 0 & 0 \\ T & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T & k \end{pmatrix}_{m\times m}$ is positive and $L = V^*T\pi V\ (m.c.r.i)\} = 2w(S_m)\|L\|_{cb}$.

Proof. Applying Proposition 2.1, Lemma 2.6, and [3, Corollary 3.3], we have the Theorem.

Corollary 2.8. If $L = L^*$, then $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 \\ L & \emptyset & L \\ 0 & L & \emptyset \end{pmatrix}$ is completely positive$\} = \sqrt{2}\|L\|_{cb}$.

Proof. From [1], $w(S_3) = cos\frac{\pi}{4}$.

Corollary 2.9. If $L = L^*$, then $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 & 0 \\ L & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L & \emptyset \end{pmatrix}_{m\times m}$ is completely positive for all $m \ge 2\} = 2\|L\|_{cb}$.

Proof. $lim_{m\to\infty}w(S_m) = lim_{m\to\infty}cos\frac{\pi}{m+1}$.

Example 2.10. Let $L: C \to M_2(C)$ be defined by $L(z) = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$ and $\emptyset: C \to M_2(C)$ be defined by $\emptyset(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$.

Then L is completely bounded and $\emptyset$ is a unital completely positive. Since $L^* = L$ and the map $\begin{pmatrix} \emptyset & L \\ L & \emptyset \end{pmatrix}$ is completely positive, by [3, Corollary 3.3 and Corollary 3.12], we have $\|L\|_{cb} = \|L\|_{wcb} = \|\emptyset\| = 1$. Hence $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 \\ L & \emptyset & L \\ 0 & L & \emptyset \end{pmatrix}$ is completely positive$\} = \sqrt{2}$, $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 & 0 \\ L & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L & \emptyset \end{pmatrix}_{m\times m}$ is completely positive with $m \ge 2\} = 2cos\frac{\pi}{m+1}$, and $inf\{\|\emptyset\|: \begin{pmatrix} \emptyset & L & 0 & 0 \\ L & \emptyset & \ddots & 0 \\ 0 & \ddots & \ddots & L \\ 0 & 0 & L & \emptyset \end{pmatrix}_{m\times m}$ is completely positive for all $m \geq 2\} = 2.$

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