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$N(2,2,0)$ Algebras and Related Topic

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Abstract

In this paper, we investigate the elementary properties of the $N(2,2,0)$-algebra. Especially, some properties of nilpotent $N(2,2,0)$-algebras are presented. Also some relationships between nilpotent $N(2,2,0)$-algebra and other algebras with the type of $(2,0)$ are obtained.

Keywords: $N(2,2,0)$ algebra, nilpotent$N(2,2,0)$-algebras, semigroup, relationship

1. Introduction

In the past years, fuzzy algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. The implication connective plays a crucial role in fuzzy logic and reasoning(Andrzej, M and Jayaram, B, 2008), (Mas, M, and all, 2007). Recently, some authors studied fuzzy implications from different perspectives (Massanet, S, and all, 2013), (Pei, D, 2014). Naturally, it is meaningful investigating the common properties of some important fuzzy implications used in fuzzy logic. Consequentially, Professor Wu (Wu, D, 1999) introduced a class of fuzzy implication algebras abbreviating FI-algebras, in 1990.

In the past two decades, some authors focused on FI-algebras. Various interesting properties of FI-algebras(Li, Z and Li, G, 2008), (Li, Z and Li, G, 2000), regular FI-algebras(Chen, W, 2001), (Li, Z and all, 2002), (Wu, W, 1990), commutative FI-algebras [34], W_d- FI -algebras(Deng, F, 1996), (Deng, F, 2017)and other kinds of FI-algebras were reported.

Influenced by Wu's idea(Wu, W, 1990), Deng and Xu(Deng, F and Xu, Y, 1996)defines an new notion, called a N(2, 2, 0) algebras which is related to several classical algebras such as BCK/BCI-algebras, B-algebras, CI-algebras, BM-algebras and so on(Andrzej and Walendziak, 2008), (Chang, B, 2012), (Deng, F, and all, 2017), (Howie, J, 1976), (Kim, C and Kim, H, 2006),(Neggers, J and Kim, H, 2001), (Neggers, J, and all, 2001), (Ravi, K and Rafi, N, 2012), (Sharp and Jack, C, 1977). Several basic properties of $N(2,2,0)$ algebra $(S,*,\triangle,0)$ are obtained (Deng, F, and all, 2018), (Li, X and Ma, S, 2006), (Li, X and Song, X, 2012), (Li, X, 2005). We proved that if the operations $*$ is idempotent, then $(S,*,\triangle,0)$ is a rewriting systems. Analogously, if the operations $*$ is nilpotent, then $(S,*,0)$ is a associated BCI-algebra(Deng, F and Xu, Y, 1996), (Deng, F, and all, 2016).

In this paper, our aim is to discuss further relations between $N(2,2,0)$-algebras and other classes of algebraic structure such as BM-algebra, Q-algebra and so on. And give some properties with respective to the $N(2,2,0)$-algebra.

The present paper is organized as follows:

In section 2, we review some basic concepts and facts about $N(2,2,0)$-algebra which will be used later. In section 3, we characterize $N(2,2,0)$-algebra and give some characterizations with respective to $N(2,2,0)$-algebras. In section 4, we investigate some relationships between the $N(2,2,0)$-algebras and the related fuzzy logic algebras.

2. Notation and Terminology

Unless specified otherwise, throughout S denotes an arbitrary semigroup $(S,*)$. The dual of S is the semigroup T defined on the same set as S with reversed multiplication, that is $a*b=b\diamond a$ for all $a,b\in S$. We denote by $E(S)$ the set of all idempotents of S.

An element z of S is a left (respectively right) zero of S if $z*s=z$ (respectively, $s*z=z$) for all $s\in S$; z is a zero of S if it is both a left and a right zero of S, the usual symbol for zero is 0. If S has a 0, then $S^*=S\setminus\{0\}$ denotes the set S without the zero with the product defined only when $ab\neq 0$ in S.

In this part, we firstly review some relevant concepts and definitions as follows.

Definition 2.1 (Wu, W, 1990) Let X be a universe set and $0 \in X$, $\rightarrow$ be a binary operation on X. A (2,0)-type algebra $(X, \rightarrow, 0)$ is called a fuzzy implication algebra, shortly, FI-algebra, if the following five conditions hold for all $x, y, z \in X$:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(I2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;

(I3) $x \rightarrow x = 1$;

(I4) If $x \rightarrow y = y \rightarrow x = 1$, then $x = y$;

(I5) $0 \rightarrow x = 1$, where $1 = 0 \rightarrow 0$.

Then $N(2, 2, 0)$ algebra is an algebra of type (2,2,0). This notion was first formulated in 1996 by Deng and some properties were obtained(Deng, F and Xu, Y, 1996). This notion was originated from the motivation based on fuzzy implication algebra introduced by Wu(Wu, W, 1990). He proved that in a fuzzy implication algebra$(X, \rightarrow, 0)$, the order relation $\leq$ satisfying $x \leq y$ iff $x \rightarrow y = 1$ is a partial order. In(Deng,F and Xu,Y,1996), Deng introduced a binary operation $*$ defined on fuzzy implication algebra $(X, \rightarrow, 0)$, such that for all $a, b, u \in X$,

$$u \leq a \rightarrow b \Leftrightarrow a * u \leq b. \tag{1}$$

Where $(*, \rightarrow)$ is an adjoint pair on X.

In the corresponding fuzzy logic, the operation $*$ is recognized as logic connective "conjunction" and $\rightarrow$ is considered as "implication". If the above expression holds for a product $*$, then $\rightarrow$ is the residunm of $*$. For a product $*$ the corresponding residunm $\rightarrow$ is uniquely defined by

$$a \rightarrow b = \vee\{x | a * x \leq b\}.$$

Let us note that $a \rightarrow b$ is the greatest element of the set $\{u | a * u \leq b\}$.

We proved that $\forall a, b, u \in X$, if the following two equations hold :

$$u \rightarrow (a * b) = b \rightarrow (u \rightarrow a), \tag{2}$$

$$(a * u) \rightarrow b = u \rightarrow (a \rightarrow b). \tag{3}$$

Then $(X, *)$ is a semigroup.

In fact, the multiplication defined as above is associative.

It was shown in(Wu,W,1990)that for a fuzzy implication algebra $(X, \rightarrow, 0)$, considering any $a \in X$, there is $1 \rightarrow a = a$,

$$a * (b * c) = 1 \rightarrow (a * (b * c)) = (b * c) \rightarrow (1 \rightarrow a) = (b * c) \rightarrow a = c \rightarrow (b \rightarrow a),$$

$$(a * b) * c) = 1 \rightarrow ((a * b) * c) = c \rightarrow (1 \rightarrow (a * b)) = c \rightarrow (a * b) = b \rightarrow (c \rightarrow a).$$

Since $b \rightarrow (c \rightarrow a) = c \rightarrow (b \rightarrow a)$. Then we have $(a * b) * c = a * (b * c)$. So, it can be concluded that $(X, *)$ is a semigroup.

By generalizing the expressions (2) and (3), we obtain the basic equations of $N(2, 2, 0)$ algebra.

Definition 2.2 (Deng, F and Xu, Y, 1996) An algebra $(S, *, \triangle, 0)$ of type (2,2,0) is called $N(2, 2, 0)$ algebra if it satisfies the following axioms :

(N_1) $a * (b \triangle c) = c * (a * b)$,

(N_2) $(a \triangle b) * c = b * (a * c)$,

(N_3) $0 * a = a$, for all $a, b, c \in S$.

By substituting $*$ and $\triangle$ in expressions (N_1) and (N_2) with $\rightarrow$ and $*$, respectively, we arrive at the expressions (2) and (3).

Example 2.1 Let $S = \{0, a, b, c, d, e\}$ be equipped with the operation $*$ defined by the following Caylay's table.

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	a	b	c	d	e
b	b	b	b	b	b	b
c	c	c	b	c	b	e
d	b	b	b	b	b	b
e	e	e	b	e	b	e

Δ	0	a	b	c	d	e
0	0	0	b	c	b	e
a	a	a	b	c	b	e
b	b	b	b	b	b	b
c	c	c	b	c	b	e
d	d	d	b	b	b	b
e	e	e	b	e	b	e

Then $(S, *, \Delta, 0)$ is a $N(2, 2, 0)$ algebra.

Example 2.2 Let $S = \{0, a, b, c, d\}$ be equipped with the operation $*$ defined by the following Caylay's table.

$*$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	0	a	b	c	d
c	0	a	b	c	d
d	0	a	b	c	d

Δ	0	a	b	c	d
0	0	a	0	0	0
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	c	c	c
d	d	a	d	d	d

Clearly, $(S, *, \Delta, 0)$ is a $N(2, 2, 0)$ algebra and $(S, *, 0)$ is an inverse semigroup.

3. Properties of $N(2, 2, 0)$ Algebras

Based on the motivations of (Deng, F, and all, 2016), (Howie, J, 1976), (Heinz, M, 1983), (Ravi, K, 2012), in this section, we site the important definitions that will be used in the latter.

Theorem 3.1(Deng, F and Xu, Y, 1996) Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$ algebra. Then for all $a, b, c \in S$, the following equations hold:

(1) $a * b = b\triangle a$;

(2) $(a * b) * c = a * (b * c)$,$(a\triangle b)\triangle c = a\triangle(b\triangle c)$;

(3) $a * (b * c) = b * (a * c)$, $(a\triangle b)\triangle c = (a\triangle c)\triangle b$.

Corollary 3.2 (Deng, F and Xu, Y, 1996) If $(S, *, \triangle, 0)$ is a $N(2, 2, 0)$ algebra, then both $(S, *, 0)$ and $(S, \triangle, 0)$ are semigroups.

Therefore, the $N(2, 2, 0)$ algebra is an algebra system with a pair of dual semigroups. Several interesting properties of $N(2, 2, 0)$ algebra have been discussed earlier in (Deng,F and Xu,Y,1996).

Theorem 3.3 Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$ algebra. If $x * x = 0$ for all $x \in S$, then we have

(1) $x * 0 = x$;

(2) $x * y = y * x$, i.e., $(S, *, 0)$ is a commutative semigroups;

(3) $(S, *, 0)$ is an abelian group.

Proof. (1) Since $(S, *, \triangle, 0)$ is a $N(2, 2, 0)$ algebra, $0*x = x$ for any $x \in S$. Hence $x*0 = x*(x*x) = (x*x)*x = 0*x = x$. Then we have $x * 0 = x$.

(2) Since $x, y \in S$, which shows that $x * y = x * (y * 0) = y * (x * 0) = y * x$. This implies that $x * y = y * x$. This yields $(S, *, 0)$ is a commutative semigroups.

(3) By (1) and (2), we have 0 is the identity element for $(S, *, 0)$. Since $x * x = 0$, the inverse of x is itself, i.e., $x^{-1} = x$. By Theorem 3.1, the associative law holds, hence $(S, *, 0)$ is a group.

So, in a $N(2, 2, 0)$ algebra $(S, *, \triangle, 0)$, for any x in S, if $x * x = 0$ holds, then $(S, *, 0)$ and $(S, \triangle, 0)$ is the same, and is a abelian group as well.

Definition 3.1 (Somayeh, M and Lida, T, 2017) A residuated poset is a structure $(A; \leq, \rightarrow, ., 0, 1)$ such that

(R_1) $(A; \leq, 0, 1)$ is a bounded poset,

(R_2) $(A; ., 1)$ is a commutative monoid,

(R_3) it satisfies the adjointness property, i.e.,

$$x \cdot y \leq z \Longleftrightarrow x \leq y \to z.$$

Applying the Definition and Theorem 3.3 to $N(2, 2, 0)$ algebra, we immediately obtain the following remarks:

Remark 1 Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$ algebra. If $(S, *, 0)$ with the induced order $\leq$, i.e. $x \leq y \Leftrightarrow x \to y = 1$, for all $x, y \in S$, where $1 = 0 \to 0$ and $x * 0 = x$, for all $x \in S$. Then semigroup $(S, *, 0)$ is a residuated poset.

Remark 2 If a fuzzy implication algebra $(X, \to, 0)$ with a partial order " $\leq$ ", such that any $a, b, u \in X, a \leq b \Leftrightarrow a \to b = 1$, $u \leq a \to b \Leftrightarrow a * u \leq b$ and for all $a, b, u \in X$, the following conditions hold:

$$u \to (a * b) = b \to (u \to a),$$

$$(a * u) \to b = u \to (a \to b).$$

Then $(X, \to, *, 0)$ is a $N(2, 2, 0)$ algebra.

Remark 3 Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$ algebra, then semigroups $(S, *, 0)$ and $(S, \triangle, 0)$ are a pair of dual semigroups. A pair of dual operations $(*, \triangle)$ form an adjoint pair $(\to, *)$, ie. $u \leq a \to b \Leftrightarrow a * u \leq b$, for every $a, b, u \in S$.

Theorem 3.4 Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$ algebra and x, y, z in S. Then the following hold:

$$x \triangle (y * z) = y * (x \triangle z), x * (y \triangle z) = y \triangle (x * z).$$

Proof. According to the Theorem 3.1, we have $x \triangle y = y * x$ in S. Hence, $x \triangle (y * z) = (y * z) * x = y * (z * x) = y * (x \triangle z)$. Similarly, we obtain $x * (y \triangle z) = y \triangle (x * z)$.

Theorem 3.5 Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$ algebra. If $x * 0 = 0$ for all $x \in S$, then $x^2 = x$.

Proof. From $x * 0 = 0$, by Definition 2.2 and Theorem 3.1, we conclude that $x^2 = x$.

Definition 3.3 Any groupoid (algebra, binary system) $(X, *, 0)$ of type (2,0) is said to be 0-commutative if $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$.

Corollary 3.6 Let $(S, *, 0)$ be a semi-group of $N(2, 2, 0)$ algebra $(S, *, \triangle, 0)$. Then the following statements hold.

(1) If $x * 0 = 0$ for any $x \in S$, then $(S, *, 0)$ is a right zero-semigroup and it is also a inverse semigroup.

(2) If $x * x = 0$ for any $x \in S$, then $(S, *, 0)$ is a 0-commutative semigroup.

Definition 3.2 (Ravi, K, 2012) A semigroup S is called anti-regular if for each element a in S, there is an element b in S such that $aba = b$ and $bab = a$. The elements a and b are then called anti-inverses.

Example 3.1 Let $S = \{0, a, b, c\}$ be equipped with the operation $*$ defined by the following Caylay's table.

$*$	0	a	b	c
0	0	a	b	c
a	a	a	b	a
b	b	b	a	b
c	c	a	b	0

Δ	0	a	b	c
0	0	a	b	c
a	a	a	b	a
b	b	b	a	b
c	c	a	b	0

It is to check that $(S, *, \Delta, 0)$ is a $N(2, 2, 0)$ algebra , $(S, *, 0)$ is an regular semigroup and anti-regular semigrouop.

Example 3.2 Let $S = \{0, a, b, c\}$ be equipped with the operation $*$ defined by the following Caylay's table.

$*$	0	a	b	c
0	0	a	b	c
a	a	b	a	a
b	b	a	b	b
c	b	a	b	b

Δ	0	a	b	c
0	0	a	b	b
a	a	b	a	a
b	b	a	b	b
c	c	a	b	b

Clearly $(S, *, \Delta, 0)$ is a $N(2, 2, 0)$ algebra , but $(S, *, 0)$ is not a anti-regular semigroup. Since

$$a * c * a = a * a = b (\neq c),$$

$$b * c * b = b * b = b(\neq c),$$

$$c * c * c = b * c = b(\neq c).$$

So, c has no anti-inverse in S.

Theorem 3.7 If semigroup $(S, *, 0)$ of $N(2, 2, 0)$ algebra$(S, *, \triangle, 0)$ is an anti-regular semigroup, i.e., any $x \in S$, there is an element b in S such that $a * b * a = b$ and $b * a * b = a$. Then for any $a, b \in S$,

$$a * b = b * a, a^2 = b^2, a^5 = a, b^5 = b.$$

Proof. (1) Since a and b are anti-inverses, we have $a * b * a = b$ and $b * a * b = a$. Thus

$$a * b = (b * a * b) * b = b * (a * b * b) = b * (b * a * b) = b * a.$$

(2) Since $a * b * a = b$ and $b * a * b = a$, thus

$$a^2 = a * a = a * (b * a * b) = (a * b * a) * b = b * b = b^2.$$

(3) $a^5 = a^2 * a * a^2 = b^2 * a * b^2 = b * (b * a * b) * b = b * a * b = a.$

Similarly, we get $b^5 = b$.

4. N(2,2,0)-Algebra with $x * x = 0$

Many type of algebras connected with nonclassical logics have one binary operation $*$ and one constant denoted by 0 or by 1. This operation with the corresponding constant is connected by the axiom $x * x = 0$ or by $x * x = 1$ and satisfies the identity $(x * y) * z = (x * z) * y$ (resp., $x * (y * z) = y * (x * z)$). This means that it plays a very important role to study such algebras with the constant 0 satisfying the identity $x * x = 0$.

In this section, we discuss some relations between $N(2, 2, 0)$-algebras and $BH/BRk/BCH/$
BCI/BG/B/BM-algebras.

Definition 4.1 (Neggers, J and Kim, H, 2002) A B-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfies the following axioms:

(B_1) $x * x = 0$,

(B_2) $x * 0 = x$,

(B_3) $(x * y) * z = x * (z * (0 * y))$, for all $x, y, z \in X$.

Theorem 4.1 If $(S, *, \triangle, 0)$ is a $N(2, 2, 0)$-algebra with $x * x = 0$, then $(S, *, 0)$ is a B-algebra, but the converse need not be true.

Proof. By theorem 3.1 and theorem 3.3, we obtain $(B_1), (B_2), (B_3)$ hold.

Example 4.1 Let $S = \{0, a, b, c, d, e\}$ be equipped with the operation $*$ defined by the following Caylay's table:

$*$	0	a	b	c	d	e
0	0	b	a	c	d	e
a	a	0	b	d	e	c
b	b	a	0	d	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

It is easy to see that $(X, *, 0)$ is a B-algebra and X is not a $N(2, 2, 0)$-algebra, where $0 * x = x$ is not satisfied.

Definition 4.2 (Andrzej and Walendziak,2008) A BM-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B_1) $x * 0 = x$,

(BM) $(z * x) * (z * y) = y * x$, for any $x, y, z \in X$.

Theorem 4.2 Every BM-algebra is a B-algebra, but the converse need not be true.

From the Theorem 2.6(Kim,C and Kim,H,2006), it follows that every BM-algebra is a B-algebra.

Example 4.2 Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

It is to check that $(X, *, 0)$ is a B-algebra. Since $(2 * 0) * (2 * 1) = 2 * 2 = 0 \neq 1 * 0 = 1$, axiom BM is not satisfied. Hence, $(X, *, 0)$ is not BM-algebra.

Theorem 4.3 If $(S, *, \triangle, 0)$ is a $N(2, 2, 0)$-algebra with $x * x = 0$, then $(S, *, 0)$ is a BM-algebra.

Proof. Suppose $(S, *, \triangle, 0)$ is a nilpotent $N(2, 2, 0)$-algebra. Then for any $x, y \in S$, by $x * x = 0$ and by Theorem 3.3, we have $x * 0 = x, x * y = y * x$. Therefore

$$(z * x) * (z * y) = z * ((z * x) * y = (z * z) * (x * y) = 0 * (x * y) = x * y = y * x.$$

Hence, every nilpotent $N(2, 2, 0)$-algebra is a BM-algebra.

Theorem 4.4 If $(A, *, 0)$ is a BM-algebra, then the following conditions hold:

(1) $x * x = 0$;

(2) If $*$ satisfies associative law, then we have $0 * x = x$ and $y * z = z * y$.

Proof. (1) If we let $y = z = 0$ in axiom (BM), then

$$x * x = (x * 0) * (x * 0) = 0 * 0 = 0 \Longrightarrow x * x = 0.$$

(2) By $*$ satisfying associative law, then $0 * x = (x * x) * x = x * (x * x) = x * 0 = x$.

If we let $x = 0$ in axiom (BM), then $z * y = (x * y) * (x * z) = (0 * y) * (0 * z) = y * z$. Hence $y * z = z * y$.

Corollary 4.5 If a BM-algebra $(A, *, 0)$ satisfies associative law, then $(A, *, 0)$ is a nilpotent $N(2, 2, 0)$-algebra.

Definition 4.3 (Kim, C and Kim, H, 2006) A BH-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B_1) $x * x = 0$,

(B_2) $x * 0 = x$,

(BH) $x * y = y * x = 0 \Rightarrow x = y$, for any $x, y, z \in X$.

Definition 4.4 (Li, Z, and all, 2002) A BH-algebra with the condition $(BCH)(x*y)*z = (x*z)*y$ is called a BCH-algebra.

A BH-algebra with the identity $(BCI)((x * y) * (x * z)) * (z * y) = 0$ is called a BCI-algebra.

A BCK-algebra is a BCI-algebra satisfying the following additional axiom:

(BCK) $0 * x = 0$.

Remark 4.6 (Andrzej and Walendziak, 2014) Every BCK-algebra is a BCI-algebra and every BCI-algebra is a BCH-algebra and every BCH-algebra is a BH-algebra.

From Theorem 3.3 and Definition 4.3, 4.4, we have

Remark 4.7 Every nilpotent $N(2, 2, 0)$-algebra is a BH-algebra, also it is a BCH-algebra.

Definition 4.5 (Andrzej and Walendziak, 2014) Let $(X, *, 0))$ be an algebra of type (2,0) satisfying axioms (B_1) and (B_2). We say that X is a BF-algebra (resp. BG) if X obeys axiom $(BF)/(BG)$, where

(BF) $0 * (x * y) = y * x$;

(BG) $x = (x * y) * (0 * y)$.

Remark 4.8 From the Remark 2.2 (Andrzej and Walendziak, 2014), it follows that every BM-algebra is a B-algebra. Every B-algebra is a BG-algebra and every BG-algebra is a BH-algebra. Consequently, BM-algebras are BH-algebras. It is easy to see that (BM) implies (BCI). Therefore, the class of BM- algebras is a subclass of the class of BCI-algebras.

Definition 4.6 (Chang, B and Hee, S, 2012) A BO-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B_1) $x * x = 0$,

(B_2) $x * 0 = x$,

(BO) $x * (y * z) = (x * y) * (0 * y)$, for any $x, y, z \in X$.

From (Chang, B, 2012), every BO-algebra is both a BG-algebra and a BH-algebra.

Theorem 4.9 Let $(S, *, 0)$ be a semi-group of $N(2, 2, 0)$ algebra $(S, *, \triangle, 0)$. If $x * x = 0$, for any $x \in S$, then the following three statements hold:

(1) $(S, *, 0)$ is a BO-algebra, but the converse need not be true;

(2) $(S, *, 0)$ is a BF-algebra, but the converse need not be true;

(3) $(S, *, 0)$ is a BG-algebra.

Proof. (1) By Theorem 3.3, we have $(B_1), (B_2)$ hold and $0 * x = x$. Hence $(x * y) * (0 * z) = (x * y) * z = x * (y * z)$. Therefore (BO) hold.

Example 4.3 Let $S = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

It is to check that $(S, *, 0)$ is a BO-algebra. And it is also a nilpotent $N(2, 2, 0)$-algebra. In fact, $(S, *, 0)$ is the Klein's four group.

Example 4.4 Let $S = \{0, 1, 2, 3, 4\}$ be a set with the following table:

$*$	0	1	2	3	4
0	0	2	1	4	3
1	1	0	3	2	4
2	2	4	0	3	1
3	3	1	4	0	2
4	4	3	2	1	0

It is to check that $(S, *, 0)$ is a BO-algebra. Clearly, $(S, *, 0)$ is not a nilpotent $N(2, 2, 0)$-algebra, since for all $x \in S$, $0 * x = x$ is not hold.

Proof. (2) By Theorem 3.3, we have$(B_1), (B_2)$ hold and $0 * x = x$. Hence $0 * (x * y) = x * y = y * x$. Therefore (BF) hold.

Example 4.5 Let $S = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

It is to check that $(S, *, 0)$ is a BF-algebra. Clearly, $(S, *, 0)$ is not a nilpotent $N(2, 2, 0)$-algebra. Since $1 * 2 = 2 * 1 = 0 \Rightarrow 1 \neq 2$. Therefore $x, y \in S$, $x * y = y * x = 0 \Rightarrow x = y$ is not hold.

(3) By theorem 3.3 we have B_1, B_2 hold and $0 * x = x$. Hence $(x * y) * (0 * y) = (x * y) * y = x * (y * y) = x * 0 = x$. Therefore (BG) hold.

Example 4.6 Let $S = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

It is to check that $(S, *, 0)$ is a BG-algebra. Clearly, $(S, *, 0)$ is not a nilpotent $N(2, 2, 0)$-algebra, since for all $x, y \in S$, $x * y = y * x$ is not hold.

Definition 4.7 (Neggers, J, and all, 2001) A Q-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B_1) $x * x = 0$,

(B_2) $x * 0 = x$,

(Q) $(x * y) * z = (x * z) * y$, for any $x, y, z \in X$.

A Q-algebra X is said to be a QS-algebra if it satisfies the additional relation:

(QS) $(x * y) * (x * z) = z * y$ for any $x, y, z \in X$.

Remark 4.10 From Theorem 3.1 and Theorem 3.3, every nilpotent semigroup $(S, \triangle, 0)$ of $N(2, 2, 0)$-algebra $(S, *, \triangle, 0)$ is a Q-algebra. But the converse needs not be true.

Example 4.7 Let $S = \{0, 1, 2\}$, in which $\triangle$ is defined by

$\triangle$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(S, \triangle, 0)$ is a Q-algebra, which is not a nilpotent semigroup $(S, \triangle, 0)$ of $N(2, 2, 0)$-algebra $(S, *, \triangle, 0)$.

Definition 4.8 (Ravi, K and Rafi, B, 2012) A G-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B_1) $x * x = 0$,

(G) $x * (x * y) = y$, for all $x, y, z \in X$.

From (Neggers, J and Kim, H, 2002), Every commutative B-algebra is a G-algebra, but the converse need not be true. Every QS-algebra is a G-algebra, but the converse need not be true.

Remark 4.11 From (Ravi, K and Rafi, B, 2012), it follow that every BCH-algebra is a Q-algebra, but not conversely. Every B-algebra is a BF-algebra, but not conversely.

The following theorem can be proved easily.

Theorem 4.12 (Ravi, K and Rafi, B, 2012) (1) Every G-algebra satisfying QS is a BCI-algebra.

(2) Every G-algebra satisfying QS is a BCH-algebra.

(3) Every G-algebra satisfying QS is a Q-algebra.

(4) Every G-algebra satisfying Q is a BF-algebra.

Theorem 4.13 (Ravi, K and Rafi, B, 2012) Let $(X, *, 0)$ be a G-algebra. Then the following are equivalent:

(5) X is a Q-algebra.

(6) X is a QS-algebra.

(7) X is a BCH-algebra.

Definition 4.9 (Kyung, H, 2011) A CI-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(CI_1) $x * x = 0$,

(CI_2) $0 * x = x$,

(CI_3) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

Note that every BE-algebra is a CI-algebra, but the converse is not true.

Definition 4.10 (Ravi, K, 2012) A BRK-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B_1) $x * x = 0$,

(BRK) $(x * y) * x = 0 * y$ for any $x, y \in X$.

From (Ravi,K,2012), we can observe that every Q-algebra is a BRK-algebra, but the converse need not be true. Every BM-algebra is a BRK-algebra, but converse needs not be true. By Theorem 3.1 and Theorem 3.3, the following theorem can be proved easily.

Theorem 4.14 Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$-algebra, if semigroup $(S, *, 0)$ is a nilpotent, i.e., for any $x \in S, x * x = 0$. Then $(S, *, 0)$ are G-algebra, B-algebra£Q-algebra, CI-algebra, BRK-algebra.

Definition 4.11 (Pairote,Y and Piyada, W, 2018) A $PSRU$-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(a) $x * 0 = 0$,

(b) $x * (y * z) = y * (x * z)$ for any $x, y, z \in X$.

From [28], we know that every BE-algebra is $PSRU$-algebra, but the converse need not be true. Hence $PSRU$-algebra is a generalization of BE-algebra.

Theorem 4.15 Let $(S, *, \triangle, 0)$ be a $N(2, 2, 0)$-algebra, if semigroup $(S, *, 0)$ is a right zero semigroup, i.e., for any $x, y \in S, x * y = y$, then semigroup $(S, *, 0)$ is a $PSRU$-algebra.

Proof. Applying Theorem 3.3, we obtain that (B) hold. From any $x, y \in S, x * y = y$, we conclude that when $y = 0$, then $x * 0 = 0$. Therefore, S satisfies (a) and $(S, *, 0)$ is a $PSRU$-algebras.

In fact, it is easy to show that if $x * 0 = 0$ for any $x \in S$, then $x^2 = x$. Hence semigroup $(S, *, 0)$ is a idempotent semigroup.

Finally, we give some interrelationships between some algebras mentioned above which are depicted in Figure 1 and Figure 2.

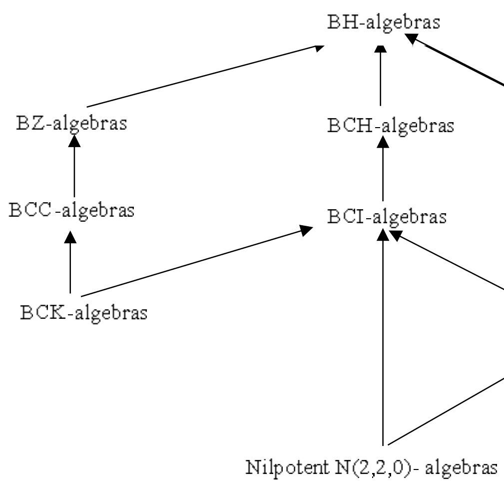

Figure 1. The interrelationship of algebras I

5. Conclusion and Future Research

N(2,2,0) algebra is an new algebra system of type (2,2,0). In this paper, we investigate some properties of $N(2, 2, 0)$-algebra. Also we obtain the interrelationship between $N(2, 2, 0)$-algebra and other algebras of type (2,0). It is our hope that this work would provide foundations for further study of the theory of $N(2, 2, 0)$- algebras. In future, we will study the following topics:

(1) To get more results in $N(2, 2, 0)$-algebras and application;

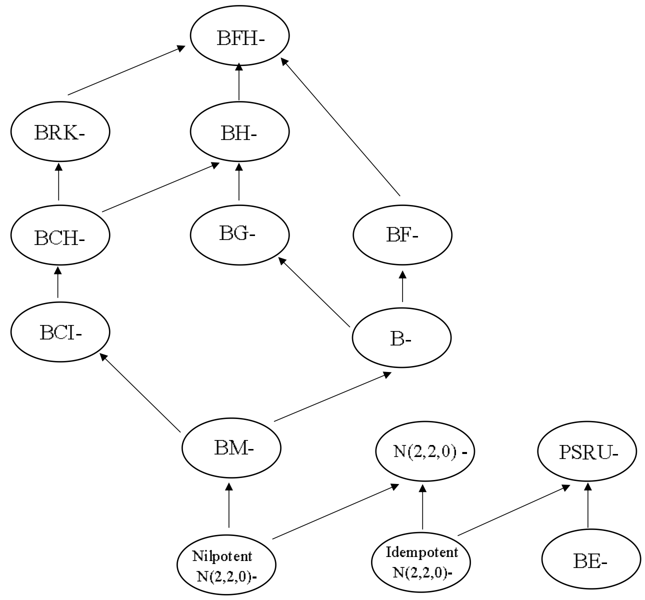

Figure 2. The interrelationship of algebras *II*

(2) To get more for idempotent semigroup of $N(2,2,0)$-algebra;

(3) To get more connection between $N(2,2,0)$-algebra and other $N(2,2,0)$-algebra algebras of type (2,0).

Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

References

Andrzej, W. (2014). BM-algebras and related topics. *Math. Slovaca, 64*, 1075-1082. https://doi.org/10.2478/s12175-014-0259-x

Baczynski, M., & Jayaram, B. (2008). *Fuzzy implications.* New York.

Chen, W. (2001). Some chararacters of regular fuzzy implication algebras. *Fuzzy system and Mathematics, 15*(4), 24-27 (in Chinese).

Chang, B. K., & Hee, S. K. (2012). On BO-algebras. *Math. Slovaca, 62*(5),855-864. https://doi.org/10.2478/s12175-012-0050-9

Deng, F., & Li, J. L. (1996). W_d- fuzzy implication algebras. *J.Ha'erbin Norm. Univ.(Nat. Sci. Ed.), 12*(2), 18-21 (in Chinese). https://doi.org/10.1109/ICISCE.2017.113

Deng, F., Ren, S., & Chen, L. (2017). A note on W_d- fuzzy implication algebras. *Fuzzy system and Mathematics, 31*(3),22-26 (in Chinese).

Deng F., & Xu, Y. (1996). On N(2,2,0)algebra. *The Journal of Southwest Jiaotong University, l31*, 457-463.

Deng, F., Chen, L., Tuo, S., & Ren, S. (2016). Characterizations of N(2,2,0)-algebras. *Algebras,* (1), 1-7. https://doi.org/10.1155/2016/2752681

Deng, F., Chen, L., & Zheng H.(2018). N(2,2,0)-algebras Theory and Application, Beijing, China Science Press.

Howie, J. M. (1976). An Introduction to Semigroup Theory, Academic Press, San Diego.

Heinz, M. (1983). Semigroups and their lattice of congruences. *Semigroup Forum, 26*, 1-63. https://doi.org/10.1007/BF02572819

Kim, C. B., & Kim, H. S. (2006). On BM-algebras. *Sci. Math. Japo., 1,* 215-221.

Kim, C. B., & Kim, H. S. (2006). On BM-algebras. *Sci. Math. Jpn., 63*, 421-427.

Kyung, H. K. (2011). A Not on CI-algebras. *Int. Math. Forum, 6*(1), 1-5.

Li, Z. W., & Li, G. H. (2008). Structure properties of fuzzy implication algebras. *J. Math, 28*(6), 701-705 (in Chinese).

Li, Z. W., & Li, G. H. (2000). Some properties of fuzzy implication algebras. *Fuzzy system and Mathematics, 14*, 19-21 (in Chinese).

Li, X. D., & Ma, S. X. (2006). A congruence decomposition on N(2,2,0) algebras, *42*(3), 15-17.

Li, X. D., & Song, X. M. (2012). An ideal of N(2,2,0) algebras. *J. of Hua Zhong Normal University, 46*(1), 1-5.

Li, X. D. (2005). Two classes of congruence decompositions on N(2, 2, 0) algebras. *Journal of Lanzhou University, 5*, 137-14.

Li, X. D. (2005). Image and Converse Image of Translation Transform of N(2,2,0) Algebra. *Journal of Mathematical Research with Applications, 2*, 148-153.

Li, Z. W. , Sun, L. M., & Zheng, C. Y. (2002). Regular fuzzy implication algebras. *Fuzzy system and Mathematics, 16*(2), 22-26 (in Chinese).

Mas, M., Monserrat, M., Torrens, J., & Trillas, E. (2007). A Survey of fuzzy implication functions. *IEEE Trans, Fuzzy Syst, 15*(6), 1107-1121. https://doi.org/10.1109/TFUZZ.2007.896304

Massanet, S., Mayor, G., Mesiar, R., & Torrens, J. (2013). On fuzzy implications: an axiomatic approach. *International Journal of Approximate Reasoning, 54,* 1471-1482. https://doi.org/10.1016/j.ijar.2013.06.001

Meng, B. L. (2009), CI-algebras. *Sci. Math. Jpn., 1,* 695-701.

Neggers, J., & Kim, H. S. (2002). On B-algebras. *Mat. Vesnik, 54*, 21-29.

Neggers, J., Ahn, S. S., & Kim, H. S. (2001). On Q-algebras. *IIJMMS, 27*, 749-757. https://doi.org/10.1155/S0161171201006627

Pei, D. W. (2014). A Survey of fuzzy implication algebras and their axiomatization. *International Journal of Approximate Reasoning, 55*, 1643-1658. https://doi.org/10.1016/j.ijar.2014.05.008

Pairote, Y., & Piyada W. (2018). A new generalization of BE-algebras, 4. https://doi.org/10.1016/j.heliyon.2018.e00863

Ravi, K. B., & Rafi, N. (2012). On G-algebras. *Scientia Magna., 8*(3), 1-7.

Ravi, K. B. (2012). On BRK-algebras. *International Journal of Mathematics and Mathematical Science, 1*, Artical ID 9526541, 1-12. https://doi.org/10.1155/2012/952654

Sharp, & Jack, C. (1977). Anti-regular semigroups. *Notices Amer. Math.Soc., 2*, 266.

Somayeh, M., & Lida, T. (2017). A new class of *BL*-algebras. *Soft Comput, 21*, 687-698.

Wu, W. M. (1990). Fuzzy implication algebra. *Fuzzy systems and Mathematics, 1,* 56-64.

Wu, D. (1999). Commutative fuzzy implication algebras. *Fuzzy system and Mathematics, 13*(1), 27-30. (In Chinese).