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The Rank of $\mathcal{U}_V$-Generated Modules

Fitriani[1], Indah Emilia Wijayanti[2], Budi Surodjo[2] & Ahmad Faisol[1]

[1]Department of Mathematics, Universitas Lampung, Lampung, Indonesia

[2]Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia

Correspondence: Fitriani, Department of Mathematics, Universitas Lampung, Lampung, Indonesia.
E-mail: fitriani.1984@fmipa.unila.ac.id

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Abstract

Let $\mathcal{U}$ be a nonempty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$ for all $\lambda \in \Lambda$. A $\mathcal{U}_V$ generated module is a generalization of $\mathcal{U}$-generated module by using the concept of V-coexact sequence. We say that an R-module N is generated by $\mathcal{U}_V$ if there is an epimorphism from V to N. In this paper, we introduce the definition of rank of $\mathcal{U}_V$-generated modules. Furthermore, we investigate some properties of rank of $\mathcal{U}_V$-generated modules.

Keywords: $\mathcal{U}$-generated module, $\mathcal{U}_V$-generated module, rank

1. Introduction

Let R be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules, i.e. $Im\ f = Ker\ g(= g^{-1}(0))$. This exact sequence can be generalized to a quasi-exact sequence by replacing the submodule 0 with any submodule $U \subseteq C$ (Davvaz and Parnian-Garamaleky, 1999). In this case, the sequence is called U-exact (in B). As a dual of a U-exact sequence, V-coexact sequence (V a submodule of A) is defined as follows. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is V-coexact if $f(V) = Ker\ g$ (Davvaz and Parnian-Garamaleky, 1999). The quasi-exact sequences can be used to generalize the Schanuel Lemma (Anvariyeh dan Davvaz, 2005). Furthermore, this sequence is used to generalize some notions in homological algebra (Davvaz and Shabani-Solt, 2002). In 2002, the U-split sequence is introduced, and the connection between this sequence and projective modules (Anvariyeh and Davvaz, 2002).

Motivated by the generalization of the exact sequence to U-exact sequence and V-coexact sequence, a sub exact sequence is introduced (Fitriani et al., 2016). Furthermore, a sub-exact sequence is used to establish the X-sub-linearly independent module as a generalization of linearly independent module relative to a family of R-modules (Fitriani et al., 2017).

Let $\mathcal{U}$ be a family of R-modules. Then, as a dual of an X-sub-linearly independent module, a $\mathcal{U}_V$-generated module is introduced as the generalization of a $\mathcal{U}$-generated module (Fitriani et al., 2018a). Furthermore, a basis and free module relative to a family of R-module is established by using the concept of X-sub-linearly independent module, and a $\mathcal{U}_V$-generated module (Fitriani et al., 2018b). The motivation of the definition of $\mathcal{U}_V$-generated module is from a generator class of modules (Anderson and Fuller, 1992).

The rank of a finitely generated module M is defined as the number of minimal generators of M (Adkins and Weintraub, 1992). In this paper, we introduce the definition of the rank of $\mathcal{U}_V$-generated modules, and we investigate some properties of the rank of $\mathcal{U}_V$-generated modules.

2. Results

Let $\mathcal{U}$ be a non-empty set of R-modules and R-module M be a finitely $\mathcal{U}_V$-generated module. Hence, there is an epimorphism from V to M, where V is a submodule of $\{U_\lambda\}_\Lambda$. The set $\{U_\lambda\}_\Lambda$ is *$\mathcal{U}_V$-generator* for N. Furthermore, the set $\{U_\lambda\}_\Lambda$ is *minimal $\mathcal{U}_V$-generator* for M if $\Lambda = min\{\Lambda_V | M$ is $\mathcal{U}_V$-generated, $V \subseteq \oplus_{\Lambda_V} U_\lambda\}$. We define rank of $\mathcal{U}_V$-generated modules as follows:

Definition 1 *Let $\mathcal{U}$ be a non-empty set of R-modules and M be a finitely $\mathcal{U}_V$-generated R-module, for some submodule V of $\oplus_\Lambda U_\lambda$ with modules $U_\lambda \in \mathcal{U}$, for all $\lambda \in \Lambda$. The rank of M relative to $\mathcal{U}$, we denote it by $rank(M)_{\mathcal{U}}$, is the minimal cardinality of Λ, where Λ is the index of minimal $\mathcal{U}_V$-generators of M.*

Let $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a non-empty set of R-modules and R-module M is finitely $\mathcal{U}_V$-generated. Then there exists a finite index set $E \subseteq \Lambda$ such that M is $\mathcal{U}_V$-generated and $V \subseteq \oplus_E U_e$, for all $e \in E$. We have $\{U_e\}_{e \in E}$ is $\mathcal{U}_V$-generator for M. If E is a minimal $\mathcal{U}_V$-generator for M which has the minimal cardinality, i.e. $E = min\{\Lambda_V | M$ is $\mathcal{U}_V$generated, $V \subseteq \oplus_{\Lambda_V} U_\lambda\}$, then

$$rank(M)_{\mathcal{U}} = |\{U_e\}_E|.$$

Then, we give some examples of the rank of module generated by $\mathcal{U}_V$, where $\mathcal{U}$ is a family of R-modules.

Example 1 Let $\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime}\}$ be a family of $\mathbb{Z}$-modules and M be an abelian group of order q^2, where q prime. We assume that M is a $\mathcal{U}_V$ generated module. If q prime and M is group of order q^2, then $M \cong \mathbb{Z}_{q^2}$ or $M \cong \mathbb{Z}_q \times \mathbb{Z}_q$.
If $M \cong \mathbb{Z}_{q^2}$, then M is not $\mathcal{U}_V$ generated module. So, we have $M \cong \mathbb{Z}_q \times \mathbb{Z}_q$ and hence the number of minimal $\mathcal{U}_V$-generators ($V = \mathbb{Z}_q \times \mathbb{Z}_q$) of M is 2. Therefore, $rank(M)_{\mathcal{U}} = 2$.

Example 2 Let $\mathcal{U} = \{\mathbb{Z}_{p^n} | p \text{ prime}, n \in \mathbb{N}\}$ be a family of $\mathbb{Z}$-modules and M be an abelian group of order q^2, where q prime. If $M \cong \mathbb{Z}_{q^2}$, then $rank(M)_{\mathcal{U}} = 1$ (where $V = \mathbb{Z}_{q^2}$). If $M \cong \mathbb{Z}_q \times \mathbb{Z}_q$, then $rank(M)_{\mathcal{U}} = 2$ (where $V = \mathbb{Z}_q \times \mathbb{Z}_q$).

Example 3 Let $\mathcal{U} = \{\mathbb{Z}_{p^n} | p \text{ prime}, n \in \mathbb{N}\}$ be a family of $\mathbb{Z}$-modules and M be an abelian group of order 8. If M is an abelian group of order 8, then M is isomorphic to exactly one of the following groups: $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We have the following conditions:

1. If $M \cong \mathbb{Z}_8$, then $rank(M)_{\mathcal{U}} = 1$ (where $V = \mathbb{Z}_{2^3}$).
2. If $M \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, then $rank(M)_{\mathcal{U}} = 2$ (where $V = \mathbb{Z}_{2^2} \times \mathbb{Z}_2$).
3. If $M \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $rank(M)_{\mathcal{U}} = 3$ (where $V = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$).

We recall that $\mu(\{0\}) = 0$ and if R is PID, then any R-submodule M of R is an ideal, so $\mu(M) = 1$ (Adkins and Weintraub, 1992). For $\mathcal{U}_V$-generated modules, we have the following properties:

Remark 1 *Let $\mathcal{U}$ be a family of R-modules.*

1. $rank(0)_{\mathcal{U}} = 1$*;*
2. $rank(W)_{\mathcal{U}} = 1$*, for any direct summand W of $U \in \mathcal{U}$.*
3. *If $\mathcal{U}$ a family of all complemented R-modules, then* $rank(W)_{\mathcal{U}} = 1$*, for any submodule W of V, $V \in \mathcal{U}$.*
4. *If $\mathcal{U}$ a family of all free R-modules, then* $rank(P)_{\mathcal{U}} = 1$*, for any projective R-module P.*

If R-module N is $\mathcal{U}_V$-generated, then N' is $\mathcal{U}_V$-generated, for every homomorphic image N' of N (Fitriani et al., 2018a). Therefore, we have the following proposition.

Proposition 1 *Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$ with modules $U_\lambda \in \mathcal{U}$, for all $\lambda \in \Lambda$ and R-module M is a finitely $\mathcal{U}_V$-generated module. Then, $rank(N)_{\mathcal{U}} \leq rank(M)_{\mathcal{U}}$, for every homomorphic image N of M.*

Proof. Let M be a finitely $\mathcal{U}_V$-generated module, and N be a homomorphic image of M. Hence, N is an $\mathcal{U}_V$-generated module (Fitriani et al., 2018a). In other words, every U_V-generator of M is U_V-generator of N and hence $rank(N)_{\mathcal{U}} \leq rank(M)_{\mathcal{U}}$.

In general, a submodule of an $\mathcal{U}_V$-generated module need not be an $\mathcal{U}_V$-generated. For example, if we take $\mathcal{U} = \{\mathbb{Q}\}$, then $\mathbb{Z}$-module $\mathbb{Q}$ is an $\mathcal{U}_{\mathbb{Q}}$-generated module. However, since we can not define an epimorphism from $\mathbb{Q}$ to $\mathbb{Z}$, $\mathbb{Z}$-module $\mathbb{Z}$ is not an $\mathcal{U}_{\mathbb{Q}}$-generated module. Nevertheless, in case M is semisimple, we have the following corollary is a consequence of Proposition 1.

Corollary 1 *Let $\mathcal{U}$ be a non-empty set of R-modules and R-module M be a finitely $\mathcal{U}_V$-generated module. If M is semisimple, then $rank(N)_{\mathcal{U}} \leq rank(M)_{\mathcal{U}}$, for every submodule N of M.*

Proof. Since every submodule of semisimple module is a direct summand, submodule N of M is a homomorphic image of M. By Proposition , we have $rank(N)_{\mathcal{U}} \leq rank(M)_{\mathcal{U}}$.

Proposition 2 *Let $\mathcal{U}$ be a non-empty set of R-modules, V_1, V_2 be submodules of $\oplus_\Lambda U_\lambda$ with modules $U_\lambda \in \mathcal{U}$, for all $\lambda \in \Lambda$. If R-module M_1 and M_2 are finitely $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_1}$-generated, respectively. Then,*

$$rank(M_1 \oplus M_2)_{\mathcal{U}} \leq rank(M_1)_{\mathcal{U}} + rank(M_2)_{\mathcal{U}}.$$

Proof. Let $\{U_a\}_A$ and $\{U_b\}_B$ be minimal $\mathcal{U}_V$-generators for N_1 and N_2, respectively. If M_1 is $\mathcal{U}_{V_1}$-generated and M_2 is $\mathcal{U}_{V_1}$-generated, then $M_1 \oplus M_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-generated. Therefore, we have $\{U_a\}_A \cup \{U_b\}_B$ is $\mathcal{U}_{V_1 \oplus V_2}$-generator of $M_1 \oplus M_2$. Hence, $rank(M_1 \oplus M_2)_{\mathcal{U}} \leq rank(M_1)_{\mathcal{U}} + rank(M_2)_{\mathcal{U}}$.

Now, we give the properties of pullback and pushout of $\mathcal{U}_V$-generated modules.

Proposition 3 *Let $\mathcal{U}$ be a non-empty set of R-modules, V_1, V_2 be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. If R-modules N_1 and N_2 are $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_2}$-generated, respectively, $g_1 : X \to N_1$ and $g_2 : X \to N_2$ be morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-generated modules, and Q be a pushout of a pair of morphisms (g_1, g_2). Then*

$$rank(Q)_{\mathcal{U}} \leq rank(N_1)_{\mathcal{U}} + rank(N_2)_{\mathcal{U}}.$$

Proof. Pushout Q of a pair of morphisms (g_1, g_2) is a factor module of $N_1 \oplus N_2$ (Wisbauer, 1991). Therefore, Q is an $\mathcal{U}_{V_1 \oplus V_2}$-generated module. By Proposition 1 and Proposition 2, we have $rank(Q)_{\mathcal{U}} \leq rank(N_1)_{\mathcal{U}} + rank(N_2)_{\mathcal{U}}$.

Proposition 4. *Let $\mathcal{U}$ be a non-empty set of R-modules, V_1, V_2 be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. If R-modules N_1 and N_2 are $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_2}$-generated, respectively, $N_1 \oplus N_2$ be a semisimple module, and P be a pullback of a pair of morphisms (f_1, f_2), where $f_1 : N_1 \to N$ and $f_2 : N_2 \to N$ are morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-generated modules. Then*

$$rank(P)_{\mathcal{U}} \leq rank(N_1)_{\mathcal{U}} + rank(N_2)_{\mathcal{U}}.$$

Proof. Pullback P of a pair of morphisms (g_1, g_2) is a submodule of $N_1 \oplus N_2$ (Wisbauer, 1991). Since $N_1 \oplus N_2$ is a semisimple module, P is a direct summand of $N_1 \oplus N_2$ and hence P is a homomorphic image of $N_1 \oplus N_2$. By Proposition 1 and Proposition 2, we have $rank(P)_{\mathcal{U}} \leq rank(N_1)_{\mathcal{U}} + rank(N_2)_{\mathcal{U}}$.

It is possible that an R-module M is a $\mathcal{U}_{V_1}$-generated and a $\mathcal{U}_{V_2}$-generated module. In the following proposition, we will show the connection between V_1 and V_2 by using Five Lemma (Wisbauer, 1991).

Proposition 5 *Let $\mathcal{U}$ be a non-empty set of R-modules, V_1, V_2 be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. If R-modules M is $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_2}$-generated, i.e there are epimorphisms $p_1 : V_1 \to M$ and $p_2 : V_2 \to M$. Let V_1 be a V_2-projective module, i.e there is morphism $p : V_1 \to V_2$ such that $p_2 \circ p = p_1$. If we define $\alpha = p|_{Ker\ p_1}$ and we assume that $\alpha(Ker\ p_1) \subseteq Ker\ p_2$, then we have:*

1. *If α is monomorphism, then V_1 is isomorphic to a submodule of V_2;*
2. *If α is epimorphism, then V_2 is a $\mathcal{U}_{V_1}$-generated module.*

Proof. If R-modules M is $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_2}$-generated, then there are epimorphisms $p_1 : V_1 \to M$ and $p_2 : V_2 \to M$. Since V_1 is V_2-projective, there is morphism $p : V_1 \to V_2$ such that $p_2 \circ p = p_1$.
We define $\alpha = p|_{Ker\ p_1}$ and we assume that $\alpha(Ker\ p_1) \subseteq Ker\ p_2$. Hence, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ker\ p_1 & \xrightarrow{i_1} & V_1 & \xrightarrow{p_1} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle p} & & \| & & \\
0 & \longrightarrow & Ker\ p_2 & \xrightarrow{i_2} & V_2 & \xrightarrow{p_2} & M & \longrightarrow & 0
\end{array}$$

By Five Lemma, if α is a monomorphism, then p is a monomorphism. So, V_1 is isomorphic to a submodule of V_2. If α is an epimorphism, then p is an epimorphism, and hence V_2 is a $\mathcal{U}_{V_1}$-generated module.

3. Conclusions

Let $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a non-empty set of R-modules and R-module M is finitely $\mathcal{U}_V$-generated. Then there exists a finite index set $E \subseteq \Lambda$ such that M is $\mathcal{U}_V$-generated and $V \subseteq \oplus_E U_e$, for all $e \in E$. We have $\{U_e\}_{e \in E}$ is $\mathcal{U}_V$-generator for M. If E is a minimal $\mathcal{U}_V$-generator for M which has the minimal cardinality, then $rank(M)_{\mathcal{U}} = |\{U_e\}_E|$. Furthermore, $rank(N)_{\mathcal{U}} \leq rank(M)_{\mathcal{U}}$, for every homomorphic image N of M.

If R-module M_1 and M_2 are finitely $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_1}$-generated, respectively. Then, $rank(M_1 \oplus M_2)_{\mathcal{U}} \leq rank(M_1)_{\mathcal{U}} + rank(M_2)_{\mathcal{U}}$. This implies if R-modules N_1 and N_2 are $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_2}$-generated, respectively, $g_1 : X \to N_1$ and $g_2 : X \to N_2$ be morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-generated modules, and Q be a pushout of a pair of morphisms (g_1, g_2). Then $rank(Q)_{\mathcal{U}} \leq rank(N_1)_{\mathcal{U}} + rank(N_2)_{\mathcal{U}}$. Besides that, if R-modules N_1 and N_2 are $\mathcal{U}_{V_1}$-generated and $\mathcal{U}_{V_2}$-generated, respectively, $N_1 \oplus N_2$ be a semisimple module, and P be a pullback of a pair of morphisms (f_1, f_2), where $f_1 : N_1 \to N$ and $f_2 : N_2 \to N$ are morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-generated modules. Then $rank(P)_{\mathcal{U}} \leq rank(N_1)_{\mathcal{U}} + rank(N_2)_{\mathcal{U}}$.

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