The Rank of $\mathcal{U}_V$-Generated Modules

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Abstract

Let $\mathcal{U}$ be a nonempty set of $R$-modules and $V$ be a submodule of $\oplus_{\lambda} U_\lambda$, $U_\lambda \in \mathcal{U}$ for all $\lambda \in \Lambda$. A $\mathcal{U}_V$ generated module is a generalization of $\mathcal{U}$-generated module by using the concept of $V$-coexact sequence. We say that an $R$-module $N$ is generated by $\mathcal{U}_V$ if there is an epimorphism from $V$ to $N$. In this paper, we introduce the definition of rank of $\mathcal{U}_V$-generated modules. Furthermore, we investigate some properties of rank of $\mathcal{U}_V$-generated modules.

Keywords: $\mathcal{U}$-generated module, $\mathcal{U}_V$-generated module, rank

1. Introduction

Let $R$ be a ring and let $f: A \rightarrow B \xrightarrow{g} C$ be an exact sequence of $R$-modules, i.e. $\text{Im } f = \ker g = \ker g(0)$. This exact sequence can be generalized to a quasi-exact sequence by replacing the submodule $0$ with any submodule $U \subseteq C$ (Davvaz and Parnian-Garamaleky, 1999). In this case, the sequence is called $U$-exact (in $B$). As a dual of a $U$-exact sequence, $V$-coexact sequence ($V$ a submodule of $A$) is defined as follows. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is $V$-coexact if $f(V) = \ker g$ (Davvaz and Parnian-Garamaleky, 1999). The quasi-exact sequences can be used to generalize the Schanuel Lemma (Anvariyeh dan Davvaz, 2005). Furthermore, this sequence is used to generalize some notions in homological algebra (Davvaz and Shabani-Solt, 2002). In 2002, the $U$-split sequence is introduced, and the connection between this sequence and projective modules (Anvariyeh and Davvaz, 2002).

Motivated by the generalization of the exact sequence to $U$-exact sequence and $V$-coexact sequence, a sub exact sequence is introduced (Fitriani et al., 2016). Furthermore, a sub-exact sequence is used to establish the $X$-sub-linearly independent module as a generalization of linearly independent module relative to a family of $R$-modules (Fitriani et al., 2017).

Let $\mathcal{U}$ be a family of $R$-modules. Then, as a dual of an $X$-sub-linearly independent module, a $\mathcal{U}_V$-generated module is introduced as the generalization of a $\mathcal{U}$-generated module (Fitriani et al., 2018a). Furthermore, a basis and free module relative to a family of $R$-module is established by using the concept of $X$-sub-linearly independent module, and a $\mathcal{U}_V$-generated module (Fitriani et al., 2018b). The motivation of the definition of $\mathcal{U}_V$-generated module is from a generator class of modules (Anderson and Fuller, 1992).

The rank of a finitely generated module $M$ is defined as the number of minimal generators of $M$ (Adkins and Weintraub, 1992). In this paper, we introduce the definition of the rank of $\mathcal{U}_V$-generated modules, and we investigate some properties of the rank of $\mathcal{U}_V$-generated modules.

2. Results

Let $\mathcal{U}$ be a non-empty set of $R$-modules and $R$-module $M$ be a finitely $\mathcal{U}_V$-generated module. Hence, there is an epimorphism from $V$ to $M$, where $V$ is a submodule of $\{U_\lambda\}_\Lambda$. The set $\{U_\lambda\}_\Lambda$ is $\mathcal{U}_V$-generator for $N$. Furthermore, the set $\{U_\lambda\}_\Lambda$ is minimal $\mathcal{U}_V$-generator for $M$ if $\Lambda = \min \{\Lambda | V \in \mathcal{U}_V\}$, $V \subseteq \oplus_{\Lambda} U_\lambda$. We define rank of $\mathcal{U}_V$-generated modules as follows:

**Definition 1** Let $\mathcal{U}$ be a non-empty set of $R$-modules and $M$ be a finitely $\mathcal{U}_V$-generated $R$-module, for some submodule $V$ of $\oplus_{\Lambda} U_\lambda$ with modules $U_\lambda \in \mathcal{U}_V$ for all $\lambda \in \Lambda$. The rank of $M$ relative to $\mathcal{U}_V$ is denoted by $\text{rank}(M)_{\mathcal{U}_V}$, is the minimal cardinality of $\Lambda$, where $\Lambda$ is the index of minimal $\mathcal{U}_V$-generators of $M$.

Let $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a non-empty set of $R$-modules and $R$-module $M$ is finitely $\mathcal{U}_V$-generated. Then there exists a finite index set $E \subseteq \Lambda$ such that $M$ is $\mathcal{U}_V$-generated and $V \subseteq \oplus_{E} U_\lambda$, for all $e \in E$. We have $\{U_\lambda\}_{\lambda \in E}$ is $\mathcal{U}_V$-generator for $M$. If $E$ is a minimal $\mathcal{U}_V$-generator for $M$ which has the minimal cardinality, i.e. $E = \min \{\Lambda | V \in \mathcal{U}_V\}$, $V \subseteq \oplus_{\Lambda} U_\lambda$,
then

\[ \text{rank}(M)_{\mathcal{U}} = |[U_i]_E|. \]

Then, we give some examples of the rank of module generated by \( \mathcal{U}_V \), where \( \mathcal{U} \) is a family of \( R \)-modules.

**Example 1** Let \( \mathcal{U} = \{ Z_p | p \text{ prime} \} \) be a family of \( Z \)-modules and \( M \) be an abelian group of order \( q^2 \), where \( q \) prime. We assume that \( M \) is an \( \mathcal{U}_V \) generated module. If \( q \) prime and \( M \) is group of order \( q^2 \), then \( M \cong \mathbb{Z}_{q^2} \) or \( M \cong \mathbb{Z}_q \times \mathbb{Z}_q \).

If \( M \cong \mathbb{Z}_{q^2} \), then \( M \) is not \( \mathcal{U}_V \) generated module. So, we have \( M \cong \mathbb{Z}_q \times \mathbb{Z}_q \) and hence the number of minimal \( \mathcal{U}_V \)-generators (\( V = \mathbb{Z}_q \times \mathbb{Z}_q \)) of \( M \) is 2. Therefore, \( \text{rank}(M)_{\mathcal{U}} = 2 \).

**Example 2** Let \( \mathcal{U} = \{ Z_{p^n} | p \text{ prime}, n \in \mathbb{N} \} \) be a family of \( Z \)-modules and \( M \) be an abelian group of order \( q^2 \), where \( q \) prime. If \( M \cong Z_{q^2} \), then \( \text{rank}(M)_{\mathcal{U}} = 1 \) (where \( V = Z_{q^2} \)). If \( M \cong Z_q \times Z_q \), then \( \text{rank}(M)_{\mathcal{U}} = 2 \) (where \( V = Z_q \times Z_q \)).

**Example 3** Let \( \mathcal{U} = \{ Z_{p^n} | p \text{ prime}, n \in \mathbb{N} \} \) be a family of \( Z \)-modules and \( M \) be an abelian group of order 8. If \( M \) is an abelian group of order 8, then \( M \) is isomorphic to exactly one of the following groups: \( Z_8, Z_4 \times Z_2, \) or \( Z_2 \times Z_2 \times Z_2 \). We have the following conditions:

1. If \( M \cong Z_8 \), then \( \text{rank}(M)_{\mathcal{U}} = 1 \) (where \( V = Z_8 \)).
2. If \( M \cong Z_4 \times Z_2 \), then \( \text{rank}(M)_{\mathcal{U}} = 2 \) (where \( V = Z_8 \times Z_2 \)).
3. If \( M \cong Z_2 \times Z_2 \times Z_2 \), then \( \text{rank}(M)_{\mathcal{U}} = 3 \) (where \( V = Z_2 \times Z_2 \times Z_2 \)).

We recall that \( \mu((0)) = 0 \) and if \( R \) is PID, then any \( R \)-submodule \( M \) of \( R \) is an ideal, so \( \mu(M) = 1 \) (Adkins and Weintraub, 1992). For \( \mathcal{U}_V \)-generated modules, we have the following properties:

**Remark 1** Let \( \mathcal{U} \) be a family of \( R \)-modules.

1. \( \text{rank}(0)_{\mathcal{U}} = 1 \);
2. \( \text{rank}(W)_{\mathcal{U}} = 1 \), for any direct summand \( W \) of \( U \) in \( \mathcal{U} \);
3. If \( \mathcal{U} \) is a family of complemented \( R \)-modules, then \( \text{rank}(W)_{\mathcal{U}} = 1 \), for any submodule \( W \) of \( V \) in \( \mathcal{U} \);
4. If \( \mathcal{U} \) is a family of all free \( R \)-modules, then \( \text{rank}(P)_{\mathcal{U}} = 1 \), for any projective \( R \)-module \( P \).

If \( R \)-module \( N \) is \( \mathcal{U}_V \)-generated, then \( N' \) is \( \mathcal{U}_V \)-generated, for every homomorphic image \( N' \) of \( N \) (Fritiani et al., 2018a).

Therefore, we have the following proposition.

**Proposition 1** Let \( \mathcal{U} \) be a non-empty set of \( R \)-modules, \( V \) be a submodule of \( \oplus_{\lambda} U_{\lambda} \) with modules \( U_{\lambda} \in \mathcal{U} \), for all \( \lambda \in \Lambda \) and \( R \)-module \( M \) is a finitely \( \mathcal{U}_V \)-generated module. Then, \( \text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}} \), for every homomorphic image \( N \) of \( M \).

**Proof.** Let \( M \) be a finitely \( \mathcal{U}_V \)-generated module, and \( N \) be a homomorphic image of \( M \). Hence, \( N \) is an \( \mathcal{U}_V \)-generated module (Fritiani et al., 2018a). In other words, every \( U_V \)-generator of \( M \) is \( U_V \)-generator of \( N \) and hence \( \text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}} \).

In general, a submodule of an \( \mathcal{U}_V \)-generated module need not be an \( \mathcal{U}_V \)-generated. For example, if we take \( \mathcal{U} = \{ \mathbb{Q} \} \), then \( \mathbb{Z} \)-module \( \mathbb{Q} \) is an \( \mathcal{U}_V \)-generated module. However, since we can not define an epimorphism from \( \mathbb{Q} \) to \( \mathbb{Z} \), \( \mathbb{Z} \)-module \( \mathbb{Z} \) is not an \( \mathcal{U}_V \)-generated module. Nevertheless, in case \( M \) is semisimple, we have the following corollary is a consequence of Proposition 1.

**Corollary 1** Let \( \mathcal{U} \) be a non-empty set of \( R \)-modules and \( R \)-module \( M \) be a finitely \( \mathcal{U}_V \)-generated module. If \( M \) is semisimple, then \( \text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}} \), for every submodule \( N \) of \( M \).

**Proof.** Since every submodule of semisimple module is a direct summand, submodule \( N \) of \( M \) is a homomorphic image of \( M \). By Proposition 1, we have \( \text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}} \).

**Proposition 2** Let \( \mathcal{U} \) be a non-empty set of \( R \)-modules, \( V_1, V_2 \) be submodules of \( \oplus_{\lambda} U_{\lambda} \) with modules \( U_{\lambda} \in \mathcal{U} \), for all \( \lambda \in \Lambda \). If \( R \)-module \( M_1 \) and \( M_2 \) are finitely \( \mathcal{U}_{V_1} \)-generated and \( \mathcal{U}_{V_2} \)-generated, respectively. Then,

\[ \text{rank}(M_1 \oplus M_2)_{\mathcal{U}} \leq \text{rank}(M_1)_{\mathcal{U}} + \text{rank}(M_2)_{\mathcal{U}}. \]
Proof. Let \{U_{a}\}_{a} and \{U_{b}\}_{b} be minimal \(U_{V}\)-generators for \(N_{1}\) and \(N_{2}\), respectively. If \(M_{1}\) is \(U_{V_{1}}\)-generated and \(M_{2}\) is \(U_{V_{2}}\)-generated, then \(M_{1} \oplus M_{2}\) is \(U_{V_{1} \oplus V_{2}}\)-generated. Therefore, we have \(\{U_{a}\}_{a} \cup \{U_{b}\}_{b}\) is \(U_{V_{1} \oplus V_{2}}\)-generator of \(M_{1} \oplus M_{2}\). Hence, \(\text{rank}(M_{1} \oplus M_{2})_{U} \leq \text{rank}(M_{1})_{U} + \text{rank}(M_{2})_{U}\).

Now, we give the properties of pullback and pushout of \(U_{V}\)-generated modules.

**Proposition 3** Let \(\mathcal{U}\) be a non-empty set of \(R\)-modules, \(V_{1}, V_{2}\) be submodules of \(\oplus_{A} U_{A}, U_{A} \in \mathcal{U}\), for every \(\lambda \in \Lambda\). If \(R\)-modules \(N_{1}\) and \(N_{2}\) are \(U_{V_{1}}\)-generated and \(U_{V_{2}}\)-generated, respectively, \(g_{1} : X \to N_{1}\) and \(g_{2} : X \to N_{2}\) be morphisms of \(U_{V_{1} \oplus V_{2}}\)-generated modules, and \(Q\) be a pushout of a pair of morphisms \((g_{1}, g_{2})\). Then

\[
\text{rank}(Q)_{U} \leq \text{rank}(N_{1})_{U} + \text{rank}(N_{2})_{U}.
\]

**Proof.** Pushout \(Q\) of a pair of morphisms \((g_{1}, g_{2})\) is a factor module of \(N_{1} \oplus N_{2}\) (Wisbauer, 1991). Therefore, \(Q\) is an \(U_{V_{1} \oplus V_{2}}\)-generated module. By Proposition 1 and Proposition 2, we have \(\text{rank}(Q)_{U} \leq \text{rank}(N_{1})_{U} + \text{rank}(N_{2})_{U}\).

**Proposition 4.** Let \(\mathcal{U}\) be a non-empty set of \(R\)-modules, \(V_{1}, V_{2}\) be submodules of \(\oplus_{A} U_{A}, U_{A} \in \mathcal{U}\), for every \(\lambda \in \Lambda\). If \(R\)-modules \(N_{1}\) and \(N_{2}\) are \(U_{V_{1}}\)-generated and \(U_{V_{2}}\)-generated, respectively, \(N_{1} \oplus N_{2}\) be a semisimple module, and \(P\) be a pullback of a pair of morphisms \((f_{1}, f_{2})\), where \(f_{1} : N_{1} \to N\) and \(f_{2} : N_{2} \to N\) are morphisms of \(U_{V_{1} \oplus V_{2}}\)-generated modules. Then

\[
\text{rank}(P)_{U} \leq \text{rank}(N_{1})_{U} + \text{rank}(N_{2})_{U}.
\]

**Proof.** Pullback \(P\) of a pair of morphisms \((g_{1}, g_{2})\) is a submodule of \(N_{1} \oplus N_{2}\) (Wisbauer, 1991). Since \(N_{1} \oplus N_{2}\) is a semisimple module, \(P\) is a direct summand of \(N_{1} \oplus N_{2}\) and hence \(P\) is a homomorphic image of \(N_{1} \oplus N_{2}\). By Proposition 1 and Proposition 2, we have \(\text{rank}(P)_{U} \leq \text{rank}(N_{1})_{U} + \text{rank}(N_{2})_{U}\).

It is possible that an \(R\)-module \(M\) is a \(U_{V_{1}}\)-generated and a \(U_{V_{2}}\)-generated module. In the following proposition, we will show the connection between \(V_{1}\) and \(V_{2}\) by using Five Lemma (Wisbauer, 1991).

**Proposition 5.** Let \(\mathcal{U}\) be a non-empty set of \(R\)-modules, \(V_{1}, V_{2}\) be submodules of \(\oplus_{A} U_{A}, U_{A} \in \mathcal{U}\), for every \(\lambda \in \Lambda\). If \(R\)-modules \(M\) is \(U_{V_{1}}\)-generated and \(U_{V_{2}}\)-generated, i.e. there are epimorphisms \(p_{1} : V_{1} \to M\) and \(p_{2} : V_{2} \to M\). Let \(V_{1}\) be a \(V_{2}\)-projective module, i.e. there is morphism \(p : V_{1} \to V_{2}\) such that \(p_{2} \circ p = p_{1}\). If we define \(\alpha = p|_{\text{Ker } p_{1}}\) and we assume that \(\alpha(\text{Ker } p_{1}) \subseteq \text{Ker } p_{2}\), then we have:

1. If \(\alpha\) is monomorphism, then \(V_{1}\) is isomorphic to a submodule of \(V_{2}\);

2. If \(\alpha\) is epimorphism, then \(V_{1}\) is a \(U_{V_{1}}\)-generated module.

**Proof.** If \(R\)-modules \(M\) is \(U_{V_{1}}\)-generated and \(U_{V_{2}}\)-generated, then there are epimorphisms \(p_{1} : V_{1} \to M\) and \(p_{2} : V_{2} \to M\). Since \(V_{1}\) is \(V_{2}\)-projective, there is morphism \(p : V_{1} \to V_{2}\) such that \(p_{2} \circ p = p_{1}\). We define \(\alpha = p|_{\text{Ker } p_{1}}\), and we assume that \(\alpha(\text{Ker } p_{1}) \subseteq \text{Ker } p_{2}\). Hence, we have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ker } p_{1} & \xrightarrow{i} & V_{1} & \xrightarrow{p_{1}} & M & \longrightarrow & 0 \\
& & \alpha & \downarrow p & \downarrow & & & \\
0 & \longrightarrow & \text{Ker } p_{2} & \xrightarrow{i} & V_{2} & \xrightarrow{p_{2}} & M & \longrightarrow & 0
\end{array}
\]

By Five Lemma, if \(\alpha\) is a monomorphism, then \(p\) is a monomorphism. So, \(V_{1}\) is isomorphic to a submodule of \(V_{2}\). If \(\alpha\) is an epimorphism, then \(p\) is an epimorphism, and hence \(V_{2}\) is a \(U_{V_{1}}\)-generated module.

### 3. Conclusions

Let \(\mathcal{U} = \{U_{\lambda}\}_{\lambda}\) be a non-empty set of \(R\)-modules and \(R\)-module \(M\) is finitely \(U_{V}\)-generated. Then there exists a finite index set \(E \subseteq \Lambda\) such that \(M\) is \(U_{V}\)-generated and \(V \subseteq \oplus_{e} U_{e}\), for all \(e \in E\). We have \(\{U_{\lambda}\}_{\lambda \in E}\) is \(U_{V}\)-generator for \(M\). If \(E\) is a minimal \(U_{V}\)-generator for \(M\) which has the minimal cardinality, then \(\text{rank}(M)_{U} = \|U_{\lambda}\|_{E}\). Furthermore, \(\text{rank}(N)_{U} \leq \text{rank}(M)_{U}\), for every homomorphic image \(N\) of \(M\).

If \(R\)-module \(M_{1}\) and \(M_{2}\) are finitely \(U_{V_{1}}\)-generated and \(U_{V_{2}}\)-generated, respectively. Then, \(\text{rank}(M_{1} \oplus M_{2})_{U} \leq \text{rank}(M_{1})_{U} + \text{rank}(M_{2})_{U}\). This implies if \(R\)-modules \(N_{1}\) and \(N_{2}\) are \(U_{V_{1}}\)-generated and \(U_{V_{2}}\)-generated, respectively, \(g_{1} : X \to N_{1}\) and \(g_{2} : X \to N_{2}\) be morphisms of \(U_{V_{1} \oplus V_{2}}\)-generated modules, and \(Q\) be a pushout of a pair of morphisms \((g_{1}, g_{2})\). Then \(\text{rank}(Q)_{U} \leq \text{rank}(N_{1})_{U} + \text{rank}(N_{2})_{U}\). Besides, that if \(R\)-modules \(N_{1}\) and \(N_{2}\) are \(U_{V_{1}}\)-generated and \(U_{V_{2}}\)-generated, respectively, \(N_{1} \oplus N_{2}\) be a semisimple module, and \(P\) be a pullback of a pair of morphisms \((f_{1}, f_{2})\), where \(f_{1} : N_{1} \to N\) and \(f_{2} : N_{2} \to N\) are morphisms of \(U_{V_{1} \oplus V_{2}}\)-generated modules. Then \(\text{rank}(P)_{U} \leq \text{rank}(N_{1})_{U} + \text{rank}(N_{2})_{U}\).
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