Journal of Mathematics Research; Vol. 13, No. 3; June 2021
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Systems Simplicity

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Received: December 27, 2021 Accepted: April 23, 2021 Online Published: May 17, 2021

doi:10.5539/jmr.v13n3p54 URL: https://doi.org/10.5539/jmr.v13n3p54

Abstract

A simple system is a system who has no proper ideals. We prove that every simple system $\mathcal{J}$ have one of the following assertion:

(1) $\mathcal{J}$ is $\mathfrak{h}-$irreducible.

(2) $\mathcal{J} = \mathcal{J}_1 \bigoplus \widetilde{\mathcal{J}_1}$ is the direct summation of two $\mathfrak{h}-$invariant and $\mathfrak{h}-$irreducible subsystems.

Keywords: systems, simplicity, irreducible structure, subsystems

1. Introduction

All the vector spaces considered in this work are of finite dimensions on a field $K = R$ or C. An n-ary algebra E is a vector space with an $n-$linear map from $E \times E \times ... \times E$ on E. A Jordan triple system is defined by the following conditions on the trilinear map:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} \tag{1.1}$$

$$\{u, v, w\} = \{w, v, u\}. \tag{1.2}$$

It is well known that from a triple Jordan system, we can build a system Lie triple on the same underlying vector space. We go to study other connections between all of these structures already mentioned. More precisely, using the construction of Koecher-Kantor -Tits applied to an algebra of Jordan J, we can construct a Lie algebra (J). On the other hand, using similar constructions, also called the KKT construction, one can construct a Lie algebra from a triple Lie or Jordan system. These constructions are due to Kurt Meyberg. The theory of groups and Lie algebras begins at the end of the 19th century with the work of the Norwegian mathematician Sophus Lie. She has known many ramifications (non-Euclidean geometries, homogeneous spaces, harmonic analysis, representation theory, algebraic groups, quantum groups ...) and still remains very active. In addition, these objects also intervene in branches that are a priori more distant from mathematics: in number theory, by means of automorphs and the Langlands program , and in theoretical physics, in particular in particle physics or general relativity.

The triple Lie system represents the linear infinitesimal analog of symmetrical space (Loos, 1969). In other words, the local study of symmetric spaces is equivalent to that of triple Lie systems. The topicality of the system classification problem Lie triples follows from the importance of symmetrical spaces that play a role considerable in several fields of modern science such as physics cosmological (Weinberg, 1982), theoretical mechanics (Doubrovine et al., 1982), differential geometry (Cartan, 1926-27; Helgason, 1962), the theory of continuous loops and that of continuous groups etc.

We are interested in these types of n-ary algebras given their importance on the algebraic level and geometric. In particular, if J is a Euclidean Jordan algebra, then the interior topological of the set $C = \{x^2, x \in J\}$ is a symmetric cone.

The main upshot of this paper is to prove that every simple triple system $\mathcal{J}$ have one of the following assertion:

(1) $\mathcal{J}$ is $\mathfrak{h}-$irreducible.

(2) $\mathcal{J} = \mathcal{J}_1 \bigoplus \widetilde{\mathcal{J}_1}$ is the direct summation of two $\mathfrak{h}-$invariant and $\mathfrak{h}-$irreducible subsystems.

2. Systems Symplicity

Definition 1. A Jordan triple system is a triple system $(\mathcal{J}, \{\ ,\ ,\ \})$ which satisfies
$\forall u, v, w, x, y \in \mathcal{J}$:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} \tag{2.1}$$

$$\{u, v, w\} = \{w, v, u\} \tag{2.2}$$

Example 1. 1. Let V be a linear pace and B be a bilinear form of V. Then, V endowed with the product defined by:

$$\{x, y, z\} = B(y, z)x - B(x, z)y + B(x, y)z,\ \forall x, y, z \in V,$$

is a Jordan triple system.

2. Let $\mathfrak{J}$ be a Jordan algebra. We define the map $\{\ ,\ ,\ \} : \mathfrak{J} \times \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ by:

$$\{x, y, z\} = x(yz) - y(xz) + (xy)z,\ \forall x, y, z \in \mathfrak{J}.$$

Then, $(\mathcal{J}, \{\ ,\ ,\ \})$ is a Jordan triple system.

Remark 1. Let $(\mathcal{J}, \{\ ,\ ,\ \})$ be a Jordan triple system. The equation (1.1) is equivalently written as:

$$[L(x, y), L(u, v)] = L(L(x, y)u, v) - L(u, L(y, x)v),\ \forall x, y, u, v \in \mathcal{J}. \tag{2.3}$$

Thus the subspace $L(\mathcal{J}, \mathcal{J})$ of $End(\mathcal{J})$ spanned by the set $\Big\{L(x, y),\ x, y \in \mathcal{J}\Big\}$ is a Lie subalgebra of $End(\mathcal{J})$.

Let us recall the following result of (Meyberg, 1972):

Theorem 1.

Let $(\mathcal{J}, \{\ ,\ ,\ \})$ be a Jordan triple system and consider the trilinear map $[\ ,\ ,\ ] : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$ defined by:

$$[x, y, z] = \{x, y, z\} - \{y, x, z\},\ \forall x, y, z \in \mathcal{J}.$$

Then, $(\mathcal{J}, [\ ,\ ,\ ])$ is a Lie triple system.

Proof. We have

$$[x, x, z] = \{x, x, z\} - \{x, x, z\} = 0.$$

The other assertion of Lie triple systems is easy to check. □

Corollary 1. *Let $(\mathcal{J}, \{\ ,\ ,\ \})$ be a Jordan triple system and $\overline{\mathfrak{J}}$ be a copy of $\mathfrak{J}$. Then, $\mathcal{L} = \mathfrak{J} \bigoplus \overline{\mathfrak{J}}$ endowed with the trilinear map $[\ ,\ ,\ ] : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$ defined by:*

$$[(x_1, x_2), (y_1, y_2), (z_1, z_2)] = (\{x_1, y_2, z_1\} - \{y_1, x_2, z_1\}, \{x_2, y_2, z_2\} - \{y_2, x_1, z_2\}),$$

$\forall x_1, y_1, z_1 \in \mathcal{J}, \S_\in, \dagger_\in, \ddagger_\in \in \overline{\mathfrak{J}}$ is a Lie triple system.

Now let $\mathcal{J}$ be a Jordan triple system. Put $A = End(\mathcal{J}) \bigoplus End(\mathcal{J})$. The linear space A endowed with the following bracket $[(f, g), (f', g')] = ([f, f'], [g, g'])$ is a Lie algebbra. Denote by $\mathfrak{h}$ the linear space endowed by the set $\Big\{l(x, y) = (L(x, y), -L(y, x)),\ x, y \in \mathfrak{J}\Big\}$. denote also bt $\mathcal{L}$ the Lie triple system $\mathfrak{J} \bigoplus \overline{\mathfrak{J}}$. Then, we can verify that $\mathfrak{h}$ is a Lie subalgebra of $Der(\mathcal{L})$.

Now we can easly obtain,

Theorem 2. *If $\mathcal{J}$ is a Jordan triple system. We consider*

$$\mathfrak{L}(\mathfrak{h}, \mathfrak{J}) = \mathcal{J} \bigoplus \mathfrak{h} \bigoplus \overline{\mathcal{J}}.$$

$\mathfrak{L}(\mathfrak{h}, \mathfrak{J})$ is a Lie algebra called the KKT-algebra of $\mathcal{J}$. The Lie bracket on $\mathfrak{L}(\mathfrak{h}, \mathfrak{J})$ is defined as follow:

$[h, h'] = hh' - h'h, \forall h, h' \in \mathfrak{h}$,

$[x_1 \oplus \overline{x_2}, y_1 \oplus \overline{y_2}] = s(x, y) = l(x_1, y_2) - l(y_1, y_2), \forall x_1 \oplus \overline{x_2}, y_1 \oplus \overline{y_2} \in \mathcal{L}$,

$[(h_1, h_2), x_1 \oplus \overline{x_2}] = h_1 x_1 \oplus \overline{h_2 x_2}, \forall (h_1, h_2) \in \mathfrak{h}, x_1 \oplus \overline{x_2} \in \mathcal{L}$.

Remarks 1.

Let $\mathcal{J}$ be a Jordan triple system and $(\mathfrak{L}(\mathfrak{h}, \mathfrak{J}), [\ ,\ ])$ its standard embedding. Then,

1. $[x, y] = 0,\ \forall x, y \in \mathcal{J}$ et $[\overline{x}, \overline{y}] = 0,\ \forall \overline{x}, \overline{y} \in \overline{\mathcal{J}}$.

2. $[(h_1, h_2), x] = h_1 x,\ \forall (h_1, h_2) \in \mathfrak{h}, x \in \mathcal{J}$ et $[(h_1, h_2), \overline{x}] = \overline{h_2 x},\ \forall (h_1, h_2) \in \mathfrak{h}, \overline{x} \in \overline{\mathcal{J}}$.
3. $[x, \overline{y}] = l(x, y), \forall x \in \mathcal{J}, \overline{y} \in \overline{\mathcal{J}}$.
4. $[x, \overline{y}], z] = \{x, y, z\}, \forall x, z \in \mathcal{J}, \overline{y} \in \overline{\mathcal{J}}$.
5. $\theta : x \oplus (h_1, h_2) \oplus \overline{y} \longmapsto x \oplus (h_2, h_1) \oplus \overline{y}$ is an involutive automorphism of $\mathfrak{L}(\mathfrak{h}, \mathfrak{J})$.
6. $j : x \oplus (h_1, h_2) \oplus \overline{y} \longmapsto -x \oplus (h_2, h_1) \oplus -\overline{y}$ is an involutive of $\mathfrak{L}(\mathfrak{h}, \mathfrak{J})$ and $\mathcal{L}$ is the -1 propre subspace of $\mathfrak{J}$.
7. If $E = (id, -id) \in \mathfrak{h}$, then $(ad_E)^3 = ad_E$ and $\mathfrak{h}, \mathcal{J}, \overline{\mathcal{J}}$ are respectively the propre spaces of ad_E with respective eigenvalues $0, +1, -1$.
8. $(\mathfrak{L}(\mathfrak{h}, \mathfrak{J}), [\ ,\ ])$ is the standard embedding of the Lie triple system $\mathcal{L}$.

Definition 2. A Hom-Jordan triple system is a triple $(\mathcal{J}, [-, -, -], \alpha)$ is defined by a triple linear space $\mathcal{J}$, a trilinear product $\{-, -, -\} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ and a linear map $\alpha : \mathcal{J} \to \mathcal{J}$ which satisfy

$$
\begin{aligned}
&[a, a, b] = 0 \qquad \text{(skewsymmetry)}\\
&[a, b, c] + [b, c, a] + [c, a, b] = 0 \qquad \text{(ternary Jacobi identity)}\\
&[\alpha(s), \alpha(t), [a, b, c]]\\
&= [[s, t, a], \alpha(b), \alpha(c)] + [\alpha(a), [s, t, b], \alpha(c)] + [\alpha(a), \alpha(b), [s, t, c]],
\end{aligned}
$$

for all $a, b, c, s, t \in \mathcal{J}$. If Moreover α satisfies $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$ (resp. $\alpha^2 = id_{\mathcal{J}}$) for all $x, y, z \in \mathcal{J}$, we say that $(\mathcal{J}, [-, -, -], \alpha)$ is a multiplicative (resp. involutive) Hom-Lie triple system.

A Hom-Lie triple system $(\mathcal{J}, [-, -, -], \alpha)$ is called regular if it is endowed with an automorphism α of $\mathcal{J}$.

If we suppose that the application α is equal to the identity map, we find the old notion of Lie triple system [**?**, **?**]. Consequently, We can consider the Lie triple systems as an examples of the cited Hom-Lie triple systems.

Now we mean by $\mathcal{J}$ a Hom-Jordan triple system., $\mathcal{L} = \mathcal{J} \bigoplus \overline{\mathcal{J}}$ and $\mathfrak{g} = \mathfrak{L}(\mathfrak{h}, \mathfrak{J})$.

Definitions 1.

1. For three subsystems $\mathcal{F}, \mathcal{G}, \mathcal{K}$ of $\mathcal{J}$, we denote by $\{\mathcal{F}, \mathcal{G}, \mathcal{K}\}$ the set $\Big\{\{x, y, z\}, x \in \mathcal{F}, y \in \mathcal{G}, z \in \mathcal{K}\Big\}$.
2. Let $\mathcal{V}$ be a linear subspace of $\mathcal{J}$. We define and denote $\widetilde{\mathcal{V}} = \{\mathcal{J}, \mathcal{V}, \mathcal{J}\}$.

Lemma 1.

$\mathcal{U} \subseteq \mathcal{L}$ is an ideal ofr $\mathcal{L}$ if and only if

(i) $\{\mathcal{U}_i, \mathcal{U}, \mathcal{U}\} \subseteq \mathcal{U}_i,\ \forall i \in \{1, 2\}$.

(ii) $\{\mathcal{U}, \mathcal{U}_i, \mathcal{U}\} \subseteq \mathcal{U}_j,\ i \neq j \in \{1, 2\}$.

where $\mathcal{U}_1$ et $\mathcal{U}_2$ are the projections of $\mathcal{U}$ on $\mathcal{J}$ and $\overline{\mathcal{J}}$ respectively.

Corollary 2.

$\mathcal{U} \subseteq \mathcal{J}$ is an ideal of $\mathcal{J}$ if and only if $\mathcal{U} \bigoplus \widetilde{\mathcal{U}}$ is an ideal of $\mathcal{L}$.

Definitions 2.

1. Let $\mathcal{U}$ be a linear subspace of $\mathcal{J}$. Then, $\mathcal{U}$ is said to be $\mathfrak{h}$-invariant if $\{\mathcal{J}, \mathcal{J}, \mathcal{U}\} \subseteq \mathcal{U}$.
2. $\mathcal{J}$ is said $\mathfrak{h}$-irreducible if it contains no $\mathfrak{h}$-invariant non trivial ideal.

Lemma 2.

If $\mathcal{U}$ a $\mathfrak{h}-$invariant linear subspace of $\mathcal{J}$. Then, $\widetilde{\mathcal{U}}$ is a $\mathfrak{h}-$invariant linear subspace of $\mathcal{J}$ and $\widetilde{\widetilde{\mathcal{U}}} \subseteq \mathcal{U}$.

Corollary 3. *If $\mathcal{L}$ is simple, then $\mathcal{J}$ is $\mathfrak{h}-$irreducible.*

Lemma 3. *If $\mathcal{J} \neq \{0\}$ is $\mathfrak{h}-$irreducible and if $E = (id, -id) \in \mathfrak{h}$, then $\mathcal{L}$ is simple.*

Proof 1.

Let $\mathcal{U} = \mathcal{U}_1 \bigoplus \mathcal{U}_2$ be an ideal of $\mathcal{L}$, $u = (u_1, u_2) \in \mathcal{U}, y = (0, y),$ and $z = (z, 0) \in \mathcal{L}$ ($\mathcal{U}_1$ (resp. $\mathcal{U}_2$) be the projection of $\mathcal{U}$ on $\mathcal{J}$ (resp. $\overline{\mathcal{J}}$)).Then, $\{u_1, y, z\} \in \mathcal{U}$ and
$\{u_2, y, z\} \in \mathcal{U}, \forall y, z \in \mathcal{J}$. If $E = \sum_{z,y\in\mathcal{J}} l(z, y)$ then, $id = \sum\limits_{z,y\in\mathcal{J}} L(z, y)$. Thus, $u_1 = idu_1 = \sum\limits_{z,y\in\mathcal{J}} \{z, y, u_1\} \in \mathcal{U}$. By the same argument, $u_2 \in \mathcal{U}$. But, $\mathcal{J}$ is $\mathfrak{h}-$irreducible. Thus, $\mathcal{U}_1 = \{0\}$ or $\mathcal{U}_1 = \mathcal{J}$. If $\mathcal{U}_1 = \{0\}$, then $U_2 = \mathcal{J}$. By Lemma (1), $\{\mathcal{J}, \mathcal{J}, \mathcal{J}\} = \{\mathcal{J}, \mathcal{U}_2, \mathcal{J}\} \subseteq \mathcal{U}_1 = 0$. Let $x \neq 0 \in \mathcal{J}$, then
$x = idx = \sum\limits_{z,y\in\mathcal{J}} L(z, y)x \in \{\mathcal{J}, \mathcal{J}, \mathcal{J}\} = 0.$ which is impossible. Hence, $\mathcal{U}_1 = \mathcal{U}_2 = \{0\}$. By the same way, if $\mathcal{U}_2 = \{0\}$, then $\mathcal{U}_1 = \{0\}$. Consequently,, $\mathcal{U} = \{0\}$ or $\mathcal{U} = \mathcal{L}$. Which entails that $\mathcal{L}$ is simple. □

Theorem 3.

If $\mathcal{J}$ is $\mathfrak{h}-$irreducible and if $E = (id, -id) \in \mathfrak{h}$, then $\mathfrak{g}$ is simple.

Proof 2.

By the previous lemma, $\mathcal{L}$ is simple. Thus,, $(\mathfrak{g}, \eta)$ is a simple involutive Lie algebra, with $\eta(x \oplus h \oplus \overline{y}) = x \oplus -h \oplus \overline{y}$, $\forall x \in \mathcal{J}, h \in \mathfrak{h}, \overline{y} \in \overline{\mathcal{J}}$. Let $\mathfrak{k}$ be a non trivial ideal of $\mathfrak{g}$ and $X = x \oplus h \oplus \overline{y} \in \mathfrak{g}$. Then, $x \oplus \overline{y} = [E, [E, X]] \in \mathfrak{k}$. Thus, $h \in \mathfrak{k}$. Consequently, $\mathfrak{g} = \mathfrak{k}$. Thus, $\mathfrak{g}$ is simple. □

Theorem 4.

If $\mathcal{J}$ is simple, then we have one of the following assertion:

(1) *$\mathcal{J}$ is $\mathfrak{h}-$irreducible.*

(2) *$\mathcal{J} = \mathcal{J}_1 \bigoplus \widetilde{\mathcal{J}_1}$ is the direct summation of two $\mathfrak{h}-$invariant and $\mathfrak{h}-$irreducible subsystems.*

Proof 3.

Since every ideal of $\mathcal{J}$ is $\mathfrak{h}-$invariant, then if $\mathcal{J}$ is $\mathfrak{h}-$irreducible such that
$\{\mathcal{J}, \mathcal{J}, \mathcal{J}\} \neq \{0\}$ then $\mathcal{J}$ is simple. Suppoe that $\mathcal{J}$ is simple and non-$\mathfrak{h}-$irréductible. Let $\mathcal{J}_1$ be a non trivial $\mathfrak{h}-$invariant ideal of $\mathcal{J}$. Then, $\widetilde{\mathcal{J}_1}$ is $\mathfrak{h}-$invariant and $\widetilde{\widetilde{\mathcal{J}_1}} \subseteq \mathcal{J}_1$ (Lemma (2)). Thus, $\mathcal{J}_1 \cap \widetilde{\mathcal{J}_1}$ and $\mathcal{J}_1 + \widetilde{\mathcal{J}_1}$ are two ideals of $\mathcal{J}$. Consequently, $\mathcal{J}_1 \cap \widetilde{\mathcal{J}_1} = 0$ and $\mathcal{J} = \mathcal{J}_1 \bigoplus \widetilde{\mathcal{J}_1}$. □

3. Conclusion

We have proved that every simple system $\mathcal{J}$ have one of the following assertion:

(1) $\mathcal{J}$ is $\mathfrak{h}-$irreducible.

(2) $\mathcal{J} = \mathcal{J}_1 \bigoplus \widetilde{\mathcal{J}_1}$ is the direct summation of two $\mathfrak{h}-$invariant and $\mathfrak{h}-$irreducible subsystems.

In the future we can try to prove similar characterizations of semi-simple systems.

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