On Notations for Conic Hulls and Related Considerations on Tangent Cones

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Abstract

We propose two different notations for cones generated by a set and for convex cones generated by a set, usually denoted by a same notation. We make some remarks on the Bouligand tangent cone and on the Clarke tangent cone for star-shaped sets and for locally convex sets. We give some applications of these remarks to a differentiable optimization problem with an abstract constraint.

Keywords: conic hulls, convex conic hulls, generated cones, star-shaped sets, Bouligand tangent cone, Clarke tangent cone

1. Introduction

In the first part of this paper, i. e. in Section 2, we recall some basic definitions concerning the various types of combinations of vectors of a set $S \subset \mathbb{R}^n$; we point out that in the current literature it is often used the same notation to denote the *conic hull* of S and the *convex conic hull* of S, i. e. *cone*(S), which may generate some confusions and misunderstandings. We propose two different notations and make some remarks concerning some particular cases. In the second part of this paper, i. e. in Section 3, we make some remarks on the *Bouligand tangent cone (or contingent cone)* to S at $x^0 \in S$ and on the *Clarke tangent cone* to S at $x^0 \in S$, when the related set S is *convex* or *locally convex at $x^0 \in S$* or *star-shaped at $x^0 \in S$*. In Section 4 we give some applications of the previous remarks to a differentiable optimization problem with an abstract constraint.

2. Conic Hulls and Convex Conic Hulls

We recall that a set $K \subset \mathbb{R}^n$ is a *cone* with vertex at the origin if

$$x \in K \implies \lambda x \in K, \ \ \forall \lambda > 0.$$

Many authors, however, do include the vertex in the cone by letting $\lambda \geqq 0$ in the definition. It may be useful to accept both definitions, in order to have more flexibility in the various cases and problems encountered in treating this topic. It is well-known that a cone $K \subset \mathbb{R}^n$ is *convex* if and only if

$$x, y \in K \implies x + y \in K.$$

Let $x^1, x^2, ..., x^k$ be vectors of $\mathbb{R}^n$. We recall that a *linear combination* of the vectors $x^1, x^2, ..., x^k$ is any expression of the form

$$\sum_{i=1}^{k} \alpha_i x^i \tag{1}$$

where each $\alpha_i \in \mathbb{R}$.

When $\sum_{i=1}^{k} \alpha_i = 1$, the linear combination (1) is called *affine combination* of the vectors $x^1, x^2, ..., x^k$.

When it holds that each $\alpha_i \geqq 0$, the linear combination (1) is called *convex conic combination or nonnegative combination* of the vectors $x^1, x^2, ..., x^k$.

When in (1) it holds that each $\alpha_i \geqq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$, then the linear combination (1) is called a *convex combination* of the vectors $x^1, x^2, ..., x^k$.

- The collection of all linear combinations of vectors of S is said to be the *linear space generated by S* or *linear span*

of S or also *linear hull of* S and it is denoted by

$$lin(S) \text{ or } span(S):$$

$$lin(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i : \text{ each } x^i \in S, \text{ each } \alpha_i \in \mathbb{R}, k \in \mathbb{N} \right\}.$$

It can be proved that $lin(S)$ is the smallest linear space of $\mathbb{R}^n$ containing S.

- The collection of all affine combinations of the vectors of S is called *affine hull of* S and it is denoted by

$$aff(S):$$

$$aff(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i : \text{ each } x^i \in S, \text{ each } \alpha_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1, k \in \mathbb{N} \right\}.$$

It can be proved that $aff(S)$ is the smallest affine subspace of $\mathbb{R}^n$ containing S.

- The collection of all convex conic combinations of the vectors of S is called *convex conic hull of* S or *convex cone generated by* S or *convex cone spanned by* S and it is denoted, in the present paper, by

$$cone(S):$$

$$cone(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i, \text{ each } x^i \in S, \text{ each } \alpha_i \geq 0, k \in \mathbb{N} \right\}.$$

Some authors (e. g. Bazaraa and Shetty (1976)) require that the coefficients α_i, $i = 1, ..., k$, are not all zero, i. e. $\sum_{i=1}^{k} \alpha_i > 0$. It can be proved that $cone(S)$ is the smallest convex cone of $\mathbb{R}^n$ containing S (i. e. the intersection of all convex cones in $\mathbb{R}^n$ containing S).

- The collection of all convex combinations of the vectors of S is called the *convex hull of* S and it is denoted by

$$conv(S) \text{ or } co(S):$$

$$conv(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i, \text{ each } x^i \in S, \text{ each } \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1, k \in \mathbb{N} \right\}.$$

It can be proved that $conv(S)$ is the smallest convex set of $\mathbb{R}^n$ containing S (i. e. the intersection of all convex sets in $\mathbb{R}^n$ containing S).

We note that if S is any nonempty subset of $\mathbb{R}^n$, we have

$$
\begin{aligned}
S &\subset conv(S) \subset aff(S) \subset lin(S); \\
S &\subset conv(S) \subset cone(S) \subset lin(S).
\end{aligned}
$$

Moreover:

a) S is a linear subspace of $\mathbb{R}^n$ if and only if $S = lin(S)$;

b) S is an affine set of $\mathbb{R}^n$ if and only if $S = aff(S)$;

c) S is a convex cone of $\mathbb{R}^n$ with vertex at $0 \in S$ if and only if $S = cone(S)$;

d) S is a convex set of $\mathbb{R}^n$ if and only if $S = conv(S)$.

The notation "$cone(S)$" is widely used, however some authors use other notations: for example Jeter (1986) uses the notation $coni(S)$, whereas Dhara and Dutta (2012) use the notation $cone\, co(S)$, to denote the convex cone generated by S. Some authors call the convex cone generated (or spanned) by S, the *positive hull of* S.

We remark that $cone(S)$ *need not be a closed set,* even if S is compact. However, it can be proved that $cone(S)$ is closed (and convex) in special cases, such as when S is *finite.* We speak in this case of *finite cone* or *finitely generated cone,* i. e. the closed and convex cone generated by a finite set $C = \{a^1, a^2, ..., a^m\}$ of vectors in $\mathbb{R}^n$:

$$cone(C) = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \alpha_i a^i, \ \alpha_i \geq 0, \ i = 1, ..., m \right\}.$$

The vectors $a^1, a^2, ..., a^m$ of $\mathbb{R}^n$ are said to be the *generators* of *cone(C)*. If $a^i \equiv A_i$, i-th row of a matrix A of order (m, n), then

$$cone(a^1, a^2, ..., a^m) = \{x \in \mathbb{R}^n : \mu A = x, \ \mu \geqq 0\}.$$

The proof that a finitely generated cone is closed is basic in proving the well-known Farkas' lemma, by means of a separation theorem between convex sets, even if several authors give this property for granted and skip the related proof. Some authors use, for the case of a finite number of generators, instead of the notation

$$cone(a^1, a^2, ..., a^m),$$

the notation

$$pos(a^1, a^2, ..., a^m)$$

and, if $a^1, a^2, ..., a^m$ are the m rows of a matrix A of order (m, n), also the notation

$$(\mathbb{R}^m_+)A.$$

It is well-known that a cone described by

$$K = \{x \in \mathbb{R}^n : Bx \leqq 0\},$$

where B is a matrix of order (m, n), is called a *polyhedral (convex) cone*. The *Minkowski-Weyl Theorem* states that a cone is polyhedral if and only if it is finitely generated. See, e. g., Bertsekas (1999), Florenzano and Le Van (2001), Hiriart-Urruty and Lemarechal (1993), Rockafellar (1970).

In the literature it is often introduced the *conic hull* of $S \subset \mathbb{R}^n$ or *radial hull* of S as the smallest cone of $\mathbb{R}^n$ (not necessarily convex!) which contains S. Unfortunately this cone is almost always denoted by *cone(S)*, which can generate some confusions and misunderstandings. As this cone is given by the union of all rays starting from the origin and passing through the element $x \in S$, we propose the notation *ray(S)* :

$$ray(S) = \{y : y = \lambda x, \ x \in S, \ \lambda \geqq 0\}.$$

Some authors require $\lambda > 0$ and some authors use the notation $\mathbb{R}_+(S)$.

Furthermore, some authors speak, for the present case, of *cone generated (or spanned) by S*, however this can be misleading, as in the definition of *cone(S)* the term "convex" is often omitted. Note that the nonempty set $S \subset \mathbb{R}^n$ is a cone if and only if $ray(S) = S$.

The basic relation between *cone(S)* and *ray(S)* is given by the following result.

Theorem 1. If $S \subset \mathbb{R}^n$ is a nonempty convex set, then

$$cone(S) = ray(S).$$

See, e. g., Bazaraa and Shetty (1976).

Moreover, it holds, with $S \subset \mathbb{R}^n$:

$$cone(S) = ray(conv(S)) = conv(ray(S)).$$

It is sometimes also useful to consider cones and convex cones generated by a set $S \subset \mathbb{R}^n$, with reference to a point $x^0 \in S$ (or also with reference to a point $x^0 \in cl(S)$ or also $x^0 \in \mathbb{R}^n$). Let $x^0 \in S$, with S nonempty set of $\mathbb{R}^n$. Then $ray(S, x^0)$ is the smallest cone of $\mathbb{R}^n$ which contains $S - x^0$ and $cone(S, x^0)$ is the smallest convex cone of $\mathbb{R}^n$ which contains $S - x^0$, i. e., with $x^0 \in S$,

$$ray(S, x^0) = ray(S - x^0) = \left\{y : \lambda(x - x^0), \ x \in S, \ \lambda \geqq 0\right\};$$

$$cone(S, x^0) = cone(S - x^0).$$

These cones are also called, respectively, the *cone generated by S at x^0 (or from x^0)* and the *convex cone generated by S at x^0 (or from x^0)*. The cone generated by S at x^0 is called by Palata (1989) the *projection cone of S at x^0*. This cone may be viewed as a (very) rough approximation of the set S in a neighborhood of $x^0 \in S$. In the next section we shall consider two more "refined" local cone approximations of the set S at $x^0 \in S$.

3. Tangent Cones to Convex Sets, Locally Convex Sets and Star-Shaped Sets

There are several local cone approximations of a set $S \subset \mathbb{R}^n$ at $x^0 \in S$, which better than $ray(S, x^0)$ "represent" the structure of S around the point $x^0 \in S$. For a survey, see, e. g., Aubin and Frankowska (1990), Bazaraa and Shetty (1976), Giorgi and Guerraggio (1992, 2002), Palata (1989).

Definition 1. A sequence $\{x^k\} \subset \mathbb{R}^n \setminus \{x^0\}$, with $x^k \longrightarrow x^0$ is called *tangentially convergent* in the direction $y \in \mathbb{R}^n$ to the point x^0 if

$$\lim_{k \longrightarrow +\infty} \frac{x^k - x^0}{\|x^k - x^0\|} = y$$

and we write $x^k \xrightarrow{y} x^0$.

Obviously any convergent sequence $x^k \longrightarrow x^0$ (with $x^k \neq x^0$ for all k) contains at least a tangentially convergent subsequence. The set of all directions y for which there exists a feasible sequence $\{x^k\} \subset S$, with $S \subset \mathbb{R}^n$, tangentially convergent to $x^0 \in S$, form a cone which is a local cone approximation at x^0 of the set S.

Definition 2.   Let $S \subset \mathbb{R}^n$ and $x^0 \in S$; the cone

$$T(S, x^0) = \left\{ \lambda y \in \mathbb{R}^n : \exists \{x^k\} \subset S, \ x^k \xrightarrow{y} x^0, \ \lambda \geqq 0 \right\}$$

is called *Bouligand tangent cone* or *contingent cone* to the set S at the point x^0. If x^0 is an isolated point of S, we set $T(S, x^0) = \{0\}$.

Other equivalent characterizations of $T(S, x^0)$ are the following ones:

$$T(S, x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : \exists \{x^k\} \subset S, \ \lim_{k \longrightarrow +\infty} x^k = x^0, \ \exists \{\lambda_k\} \subset \mathbb{R}_+ \\ \text{such that} \quad y = \lim_{k \longrightarrow +\infty} \lambda_k (x^k - x^0) \end{array} \right\};$$

$$T(S, x^0) = \left\{ y \in \mathbb{R}^n : \exists \{y^k\} \longrightarrow y, \ \exists \{t_k\} \longrightarrow 0, \ \text{such that} \ x^0 + t_k y^k \in S \right\};$$

$$T(S, x^0) = \left\{ y \in \mathbb{R}^n : \forall N(y), \ \forall \lambda > 0, \ \exists t \in (0, \lambda), \ \exists \bar{y} \in N(y) \ \text{such that} \ x^0 + t\bar{y} \in S \right\}.$$

Note that $T(S, x^0)$ is indeed a cone, closed, but not necessarily convex, with $0 \in T(S, x^0)$. We note also that:

i) $T(S, x^0)$ depends only from the structure of S in a neighborhood of x^0, that is

$$T(S, x^0) = T(S \cap U(x^0), x^0)$$

where $U(x^0)$ is any neighborhood of x^0 (the notion of "Bouligand tangent cone" is therefore an "infinitesimal notion").

ii)   If $x^0 \in int(S)$, then $T(S, x^0) = \mathbb{R}^n$.

iii) $T(S, x^0) = T(\bar{S}, x^0)$, where $\bar{S} = cl(S)$.

iv) $T(S_1, x^0) \subset T(S_2, x^0)$, if $x^0 \in S_1 \subset S_2$.

Moreover, we note that

$$T(S, x^0) \subset cl(ray(S - x^0)),$$

however if S is a nonempty *convex set,* we have the following basic result.

Theorem 2.   Let $S \subset \mathbb{R}^n$ be a nonempty convex set and let be $x^0 \in S$, then

$$T(S, x^0) = cl(cone(S - x^0)) = cl(ray(S - x^0)).$$

Therefore, if S is a convex set, $T(S, x^0)$ is a closed convex cone.

Definition 3.   Let $S \subset \mathbb{R}^n$ be a nonempty set and let $x^0 \in S$. Then S is said to be *star-shaped* at x^0 (or *star-convex* at x^0 or *radially convex* at x^0) if

$$\lambda x^0 + (1 - \lambda)x \in S, \quad \forall x \in S, \ \forall \lambda \in [0, 1].$$

Some authors require $x^0 \in cl(S)$ and $\lambda x^0 + (1 - \lambda)x \in S$, $\forall x \in S$, $\forall \lambda \in (0, 1)$. It is evident that a convex set is star-shaped at every point $x^0 \in S$ and conversely. We note, moreover, that the closure of a star-shaped set at $x^0 \in S$ is a star-shaped set at x^0, but the interior of a star-shaped set at x^0 is not necessarily a star-shaped set. It can be proved the following result (see, e. g., Bazaraa and Shetty (1976)).

Theorem 3.   Let Let $S \subset \mathbb{R}^n$ be a nonempty set and let be $x^0 \in S$. Suppose that S is *star-shaped* at x^0. Then

$$T(S, x^0) = cl(ray(S, x^0)).$$

If we introduce the notion of (negative) *polar cone* S^* of the nonempty set $S \subset \mathbb{R}^n$:

$$S^* = \{y \in \mathbb{R}^n : yx \leqq 0, \ \ \forall x \in S\},$$

then it is not difficult to prove that if S is convex at $x^0 \in S$ (in particular if S is a convex set), then $y \in (T(S, x^0))^*$ if and only if $y(x - x^0) \leqq 0$, $\forall x \in S$. This fact will be used further to develop some considerations on optimality criteria for set-constrained minimization problems.

We have already remarked that in general $T(S, x^0)$ is not a convex cone. Guignard (1969) has introduced the following notion, in order to overcome this "defect", without assuming convexity of S. The closed convex hull of $T(S, x^0)$ is called the *pseudotangent cone* to S at x^0 and is denoted $P(S, x^0)$:

$$P(S, x^0) = cl(conv(T(S, x^0))).$$

In studying the sufficiency of first-order optimality conditions Guignard (1969) has introduced the concept of pseudoconvexity of a set $S \subset \mathbb{R}^n$ at a point $x^0 \in S$.

Definition 4.   A nonempty set $S \subset \mathbb{R}^n$ is said to be *pseudoconvex at* $x^0 \in S$ if

$$S - x^0 \subset P(S, x^0).$$

It is easy to see that when S is star-shaped at x^0, then S is pseudoconvex at x^0 and that if S is convex, then S is pseudoconvex at x^0 for all $x^0 \in S$. Guignard (1969) remarks that the union of several convex sets, the intersection of which is not empty, is pseudonvex at any point of this intersection. This is another example of a pseudoconvex set which is not necessarily convex.

Another definition that sometimes is encountered in the literature is the following one.

Definition 5.   Let be given the nonempty set $S \subset \mathbb{R}^n$ and let be $x^0 \in S$. Then S is said to be *locally convex at* x^0 if there exists a neighborhood $V(x^0)$ such that the set $S \cap V(x^0)$ is convex.

We recall that the first-order local cone approximations depend only on the structure of the set S near x^0. Hence, if $S \cap V(x^0)$ is a convex set, i. e. if S is locally convex at x^0, then $T(S, x^0)$ is convex too and it will hold

$$T(S, x^0) = cl(cone(S \cap V(x^0)), x^0).$$

It must be remarked that the cone in the right-hand side of this last equality is in general smaller than $cl(cone(S - x^0))$. Simple examples can be created.

We have seen that one of the drawbacks of the Bouligand tangent cone is that it is not convex in general. To overcome this fact, Clarke (see, e. g., Clarke (1975, 1983)) has introduced a cone which is always closed and convex. This cone can be represented in various ways, e. g., by means of sequences (see, e. g., Giorgi and Guerraggio (1992, 2002), Hiriart-Urruty (1979)).

Definition 6.   Let be given the nonempty set $S \subset \mathbb{R}^n$ and let be $x^0 \in S$. The cone

$$T_C(S, x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : \forall t_k \longrightarrow 0^+, \ \forall x^k \longrightarrow x^0, \ \text{with} \ \{x^k\} \subset S, \\ \exists y^k \longrightarrow y, \ \ \text{such that} \ \ x^k + t_k y^k \in S \end{array} \right\}$$

is called *Clarke tangent cone* to S at x^0.

Theorem 4. For any nonempty set $S \subset \mathbb{R}^n$ the Clarke tangent cone is *closed and convex.* Moreover,

$$T_C(S, x^0) \subset T(S, x^0).$$

If S is a convex set, we have

$$T_C(S, x^0) = T(S, x^0) = cl(cone(S, x^0)) = cl(ray(S, x^0)).$$

Remark 1.   The two cones $T_C(S, x^0)$ and $T(S, x^0)$ are equal (and convex!) also in the case S is a *locally convex set* at $x^0 \in S$. We have in this case

$$T_C(S, x^0) = T(S, x^0) = cl(cone(S \cap V(x^0), x^0)).$$

In general, when $T_C(S, x^0) = T(S, x^0)$, we say that S is *Clarke regular* at x^0 : see, e. g., Rockafellar and Wets (2009).

Remark 2.   If $S \subset \mathbb{R}^n$ is star-shaped at $x^0 \in S$, it is no longer true that $T_C(S, x^0) = T(S, x^0)$, as asserted in Palata (1989) and in Giorgi and Guerraggio (1992). When S is star-shaped at x^0, we can assert, for example (see Bazaraa and Shetty (1976)), that the Bouligand tangent cone is equal to the *Kuhn-Tucker tangent cone* (called also *cone of attainable directions*), but not, as in Giorgi and Guerraggio (1994), that the Bouligand tangent cone is convex. If S is *locally convex at* x^0, then the convexity of $T(S, x^0)$ follows, as previously explained. We take the opportunity to point out an oversight appearing in Hiriart-Urruty (1979) and in a footnote of R. Cambini (2002). These authors assume $S \subset \mathbb{R}^n$ to be *locally convex at* $x^0 \in S$, but write the equality

$$T_C(S, x^0) = T(S, x^0) = cl(cone(S, x^0))$$

instead of

$$T_C(S, x^0) = T(S, x^0) = cl(cone(S \cap V(x^0), x^0)).$$

Summing up:

- If $S \subset \mathbb{R}^n$ is star-shaped at $x^0 \in S$, then S is pseudoconvex at x^0.

- If $S \subset \mathbb{R}^n$ is convex, then S is locally convex at any $x^0 \in S$.

- If $S \subset \mathbb{R}^n$ is locally convex at $x^0 \in S$, then $T(S, x^0) = T_C(S, x^0)$ (and therefore $T(S, x^0)$ is a convex set).

4. Applications to Optimization Problems

Let us consider the following minimization problem with an *abstract constraint:*

$$(P):\quad \min f(x), \quad x \in S,$$

where $S \subset \mathbb{R}^n$ is a nonempty set and $f(x)$ is differentiable at a point $x^0 \in S$.

With regard to problem (P) the following result, due to Gould and Tolle (1971), Guignard (1969), Varaiya(1967) is of basic importance.

Theorem 5.   If $x^0 \in S$ is a local solution of (P), then

$$-\nabla f(x^0) \in (T(S, x^0))^* \tag{2}$$

i. e.

$$\nabla f(x^0)y \geqq 0, \quad \forall y \in T(S, x^0).$$

The result obtained by Guignard (1969):

$$-\nabla f(x^0) \in (P(S, x^0))^*$$

is not sharper than the thesis of Theorem 5, because for any cone $C \subset \mathbb{R}^n$, it holds $C^* = (cl(conv(C)))^*$.

If S is a nonempty *convex set,* relation (2) of Theorem 5 can be written as

$$\nabla f(x^0)(x - x^0) \geqq 0, \quad \forall x \in S. \tag{3}$$

On the grounds of what previously remarked on star-shaped sets, if S is *star-shaped at* $x^0 \in S$ and x^0 is a local solution of (P), then (3) holds.

If (3) holds and $f(x)$ is *pseudoconvex at* $x^0 \in S$ with respect to S, i. e. (see Mangasarian (1969)

$$\nabla f(x^0)(x - x^0) \geqq 0 \Longrightarrow f(x) \geqq f(x^0), \quad \forall x \in S,$$

then (3) becomes also a sufficient condition for the global optimality of x^0 for (P).

It is easy to prove that the class of differentiable convex functions at $x^0 \in S$ with respect to S, is strictly contained in the class of pseudoconvex functions at $x^0 \in S$ with respect to S. Unfortunately, there are no inclusion relationships between quasi-convexity, semistrict quasiconvexity and pseudoconvexity at a point $x^0 \in S$ with respect to S (see Cambini and Martein (2009)). However, if $f(x)$ is *continuous* on the open set $S \subset \mathbb{R}^n$, and $f(x)$ is differentiable and quasiconvex at $x^0 \in S$ with respect to S, with $\nabla f(x^0) \neq 0$, then $f(x)$ is pseudoconvex at x^0 with respect to S.

We remark that Guignard (1969) has proved that relation (2) is sufficient for the global optimality of $x^0 \in S$ for (P) if $f(x)$ is pseudoconvex at x^0 with respect to S and S is *pseudoconvex at* x^0. We note also that in absence of generalized convexity assumptions on $f(x)$, a stronger form of (2) is a sufficient optimality condition for $x^0 \in S$ to be a local strict minimizer for (P). The following result is due to Hestenes (1966, 1975).

Theorem 6. If for $x^0 \in S$ it holds
$$\nabla f(x^0)y > 0, \quad \forall y \in T(S, x^0) \setminus \{0\}, \tag{4}$$
then x^0 is a strict local minimizer for (P). More precisely, there exist a positive number α and a neighborhood $N(x^0)$ of x^0 such that
$$f(x) - f(x^0) \geqq \alpha \left\| x - x^0 \right\|, \quad \forall x \in N(x^0) \cap S. \tag{5}$$

It must be noted that condition (4) requires for its application that $T(S, x^0)$ is a *pointed cone,* i. e. a cone which does not contain any line through the origin. Relation (5) in other words says that x^0 is a *strict local minimizer of order 1* or *strong local minimizer* or *sharp local minimizer* (see, e. g., Still and Streng (1996), Polyak (1987)).

Another relationship between (P) and the Bouligand tangent cone to S at $x^0 \in S$ is given in the following result (see, e. g., Giorgi (1985), Martein (1985)).

Theorem 7. If $x^0 \in S$ is a local solution of (P), then x^0 is also solution of the problem

$$\min_{x \in T(S, x^0)} \left\{ f(x^0) + \nabla f(x^0)(x - x^0) \right\}.$$

The optimality conditions expressed by the previous results involve vectors belonging to the Bouligand tangent cone $T(S, x^0)$. In general, however, there is no relationship between the optimality of $x^0 \in S$ for (P) and for the problem

$$(P_1): \qquad \min f(x), \quad x \in T(S, x^0) \cap S.$$

For example, consider the problem

$$\begin{cases} \min_{x \in S} f(x, y) = y \\ S = \left\{ (x, y) \in \mathbb{R}^2 : x^3 + y \geqq 0 \right\}. \end{cases}$$

Here (P) has no solution, whereas $x^0 = (0, 0)$ is a local minimizar for (P_1).

If we consider the problem

$$\begin{cases} \min_{x \in S} f(x, y) = y - x^2 \\ S = \left\{ (x, y) \in \mathbb{R}^2 : x^2 - y \leqq 0 \right\}, \end{cases}$$

the point $x^0 = (0, 0)$ is a local minimizer for (P), whereas the problem (P_1) has no solution.

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