

Superlinear Convergence of a Modified Newton's Method for Convex Optimization Problems With Constraints

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Abstract

We consider the constrained optimization problem defined by:

$$f(x^*) = \min_{x \in X} f(x) \quad (1)$$

where the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex on a closed bounded convex set X .

To solve problem (1), most methods transform this problem into a problem without constraints, either by introducing Lagrange multipliers or a projection method. The purpose of this paper is to give a new method to solve some constrained optimization problems, based on the definition of a descent direction and a step while remaining in the X convex domain. A convergence theorem is proven. The paper ends with some numerical examples.

Keywords: nonlinear optimization, modified Newton's method

1. Introduction

In applied mathematics such as in many scientific fields, we are often led to solve nonlinear optimization problems with constraints. Several authors have studied the solution of nonlinear optimization problems with constraints, such as (Dennis & Schnabel, 1983; Ortega & Rheinboldt, 1970; Laurent, 1972; Culioli, 1994; Rhanizar, 2002; Rhanizar, 2020). Among the methods used to solve the problem (1) by transforming it to an unconstrained problem, we can cite the projection methods defined by :

$$x_{k+1} = P_X(x_k - \alpha_k \nabla f(x_k))$$

with

$$\|x - P_X(x)\| = \min_{y \in X} \|x - y\|$$

This method is only applicable if one can easily compute the projection P_X , for example if $X = \prod_{i=1}^m [a_i, b_i]$ is a block of \mathbb{R}^n . But if X is defined by constrained inequalities, it is not easy in general to use this method.

We also find the external penalization method which introduces a function:

$\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ having the following properties:

$$\begin{cases} \psi \text{ is continuous and convex} \\ \psi(x) \geq 0 \quad \forall x \in \mathbb{R}^n \\ \psi(x) = 0 \iff x \in X \end{cases}$$

The method considers $\forall \epsilon > 0$ a function $f_\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by:

$$f_\epsilon(x) = f(x) + \frac{1}{\epsilon} \psi(x)$$

The method consists in minimizing $f_\epsilon(x)$ on \mathbb{R}^n with ϵ tending to 0.

This method is applicable if it is easy to build a function ψ with its properties.

We also find the method SQP (Sequential Quadratic Programming). By introducing the Lagrangian, this method consists in solving series of quadratic problems given by:

$$\min_{d \in X_k} \frac{1}{2} d^T \nabla_{xx}^2 L(x_k, \lambda_k) d + \nabla f(x_k)^T \cdot d$$

By adding the Lagrange multipliers, the number of variables increases.

In the present paper we present an optimization method with constraints without reducing the problem to the case without constraints. In (Rhanizar, 2020), we have developed an optimization method with constraints defined by:

$$\begin{cases} x_{k+1} = x_k + \alpha_k d_k \\ d_k = y_k - x_k \\ (\nabla f(x_k), y_k) = \min_{y \in X} (\nabla f(x_k), y) \\ \alpha_k \in]0, 1[\end{cases}$$

This method has a geometric convergence and the sequence (x_k) checks:

$$\|x_k - x^*\| \leq Cq^k, \quad C > 0 \text{ and } q \in]0, 1[$$

In general, gradient methods have slow convergence. This is due to the fact that the admissible directions used are obtained from first-order approximations of the function to be optimized. Hence the idea of obtaining admissible directions from second order approximations. This idea has been widely developed for the unconstrained problems. In this paper, we propose to develop a second order method applied to constrained optimization problems. Let us now give some notations to be used throughout this article:

(\cdot, \cdot) denotes the usual scalar product $(x, y) = \sum_{i=1}^n x_i y_i$ for x and y two vectors of \mathbb{R}^n .

We shall also use the Euclidean vector norm $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.

The induced matrix norm $\|A\| = \max_{\|x\|=1} \|Ax\|$.

$\nabla f(x) = \left(\frac{\partial f}{\partial x_i}(x)\right)_{1 \leq i \leq n}$: the gradient of the function f .

$\nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial^2 x_i x_j}(x)\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$: the Hessian matrix.

The remainder of the paper is organized as follows: The introduction is presented in section 1. In section 2, we describe how to choose the direction of descent and a new algorithm that solves problem (1). Section 3 is devoted to results of convergence of the new method. We then study the speed of convergence in section 4. In section 5, some numerical examples are elaborated. The conclusions are given in section 6.

2. Searching for a Direction of Descent

Instead of using a first order approximation of the function to be optimized, we will determine the admissible directions from second order approximations. Hence the idea of defining the directions d_k as follows:

For each approximation x_k we define:

$d_k = y_k - x_k$ with y_k the solution of the following problem :

$$\min_{y \in X} \frac{1}{2} \left(\nabla^2 f(x_k)(y - x_k), y - x_k \right) + \left(\nabla f(x_k), y - x_k \right) \tag{4}$$

and we define $x_{k+1} = x_k + \alpha_k d_k$ with $0 < \alpha_k \leq 1$

For each k , we consider the function g_k defined by:

$$g_k(y) = \frac{1}{2} \left(\nabla^2 f(x_k)(y - x_k), y - x_k \right) + \left(\nabla f(x_k), y - x_k \right) \quad \forall y \in X \tag{5}$$

Using the fact that y is a minimum, we get $g_k(y_k) \leq g_k(x_k) = 0$, so:

$$\left(\nabla f(x_k), d_k \right) \leq -\frac{1}{2} \left(\nabla^2 f(x_k) d_k, d_k \right) \leq 0 \tag{6}$$

For the direction $d_k = y_k - x_k$, we have two cases to consider:

First case: suppose $d_k = 0$, so $y_k = x_k$ and $g_k(y_k) = 0$, thus from (4), we have:

$$0 \leq \frac{1}{2} \left(\nabla^2 f(x_k)(y - x_k), y - x_k \right) + \left(\nabla f(x_k), y - x_k \right), \forall y \in X$$

On the other hand:

$$f(y) - f(x_k) \approx \left(\nabla f(x_k), y - x_k \right) + \frac{1}{2} \left(\nabla^2 f(x_k)(y - x_k), y - x_k \right) \geq 0$$

and thereafter:

$$f(y) \geq f(x_k) \forall y \in X$$

therefore x_k is the solution to the problem (1).

Second case: If $d_k \neq 0$

Using relation (6) and the hypothesis $\nabla^2 f(x_k)$ is positive definite, we have $(\nabla f(x_k), d_k) < 0$

which gives d_k is a direction of descent, which implies that $f(x_{k+1}) - f(x_k) < 0$

But this condition is not sufficient for convergence (Rondepierre, 2017). This is why we are going to impose :

$$f(x_{k+1}) - f(x_k) \leq \frac{1}{2} \alpha_k g_k(y_k) \tag{7}$$

With $\alpha_k \in]0, 1]$.

So we have the following algorithm:

Algorithm

1. Choose $x_0 \in X, \epsilon \in]0, 1[, \alpha_0 \in]0, 1[$ and $k := 0$
2. At step k
 - 2.1. Compute $\nabla f(x_k)$
 - 2.2. Compute $y_k \in X$ by (4)
 - 2.3. Set $d_k = y_k - x_k$
3. If $|(\nabla f(x_k), d_k)| \leq \epsilon$, then:
 - 3.2. Set $x^* = x_k$,
 - 3.3. Stop
- end If
4. Compute $x_k + \alpha_k d_k$
5. If $f(x_k + \alpha_k d_k) - f(x_k) \leq \frac{1}{2} \alpha_k g_k(y_k)$
 - 5.1. $x_{k+1} = x_k + \alpha_k d_k$
 - 5.2. Set $k := k + 1$ and go to 2
- end If
6. $\alpha_k = \frac{1}{2} \alpha_k$ and go to 4.

3. Convergence Study

The following theorem shows the possible choice of α_k verifying (7) and the convergence of the sequence $(g_k(y_k))_k$ to the solution 0.

Theorem 1 Let f be of class C^2 on X convex bounded, and suppose that there exists $m > 0$ and $M > 0$, such that:

$$m\|y\|^2 \leq \left(\nabla^2 f(x)y, y \right) \leq M\|y\|^2 \quad \forall x, y \in X \tag{8}$$

Then:

- I. $x_k \in X \quad \forall k \geq 0$

2. Condition $f(x_k + \alpha_k d_k) - f(x_k) \leq \frac{1}{2} \alpha_k (\nabla f(x_k), d_k)$ is satisfied for $0 < \alpha_k \leq \frac{-g_k(y_k)}{(M-m)\|y_k - x_k\|^2}$

3. $f(x_k)$ is a convergent sequence

4. $g_k(y_k) \rightarrow 0$ when $k \rightarrow +\infty$

Proof.

1) $x_0 \in X$, we suppose $x_k \in X$. We have $y_k \in X$ and $\alpha_k \in]0,1]$, then $x_{k+1} = x_k + \alpha_k(y_k - x_k) \in X$

2) By applying Taylor formula to f , we have:

$$f(x_{k+1}) - f(x_k) = (\nabla f(x_k), x_{k+1} - x_k) + \frac{1}{2} (\nabla^2 f(t_k)(x_{k+1} - x_k), x_{k+1} - x_k)$$

with $t_k = x_k + s(x_{k+1} - x_k)$ and $0 < s < 1$.

Using the relation (7), the convexity of g_k , and $g_k(x_k) = 0$, we have:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= g_k(x_{k+1}) - \frac{1}{2} (\nabla^2 f(x_k)(x_{k+1} - x_k), x_{k+1} - x_k) + \frac{1}{2} (\nabla^2 f(t_k)(x_{k+1} - x_k), x_{k+1} - x_k) \\ &= g_k((1 - \alpha_k)x_k + \alpha_k y_k) - \frac{1}{2} (\nabla^2 f(x_k)(x_{k+1} - x_k), x_{k+1} - x_k) + \frac{1}{2} (\nabla^2 f(t_k)(x_{k+1} - x_k), x_{k+1} - x_k) \\ &\leq \alpha_k g_k(y_k) - \frac{1}{2} (\nabla^2 f(x_k)(x_{k+1} - x_k), x_{k+1} - x_k) + \frac{1}{2} (\nabla^2 f(t_k)(x_{k+1} - x_k), x_{k+1} - x_k) \end{aligned} \tag{9}$$

By the relation (9) and the assumption (8) we get:

$$f(x_{k+1}) - f(x_k) \leq \alpha_k g_k(y_k) + \frac{1}{2} \alpha_k^2 (M - m) \|y_k - x_k\|^2$$

For the (7) condition to be verified, it is sufficient that:

$$\alpha_k g_k(y_k) + \frac{1}{2} \alpha_k^2 (M - m) \|y_k - x_k\|^2 \leq \frac{1}{2} \alpha_k g_k(y_k)$$

so α_k verifies:

$$\alpha_k \leq \frac{-g_k(y_k)}{(M - m) \|y_k - x_k\|^2}$$

We can choose $\alpha_k = \alpha_0 2^{-i}$ with i as the first clue that verifies:

$$2^{-i} \alpha_0 \leq \frac{-g_k(y_k)}{(M - m) \|y_k - x_k\|^2}$$

It's always possible indeed:

$$\frac{-g_k(y_k)}{(M - m) \|y_k - x_k\|^2} > 0$$

and $2^{-n} \alpha_0 \rightarrow 0$ when $n \rightarrow +\infty$.

3) We have:

$$f(x_{k+1}) - f(x_k) \leq \frac{1}{2} \alpha_k g_k(y_k) \leq 0$$

Then: $(f(x_k))$ is a declining sequence, so it converges.

4) The condition on α_k , gives:

$$2^{-i+1} \alpha_0 > \frac{-g_k(y_k)}{(M - m) \|y_k - x_k\|^2}$$

Then:

$$2\alpha_k > \frac{-g_k(y_k)}{(M - m)\|y_k - x_k\|^2} \tag{10}$$

So:

$$\frac{1}{2}\alpha_k g_k(y_k) < \frac{-(g_k(y_k))^2}{4(M - m)\|y_k - x_k\|^2}$$

But:

$$f(x_{k+1}) - f(x_k) \leq \frac{1}{2}\alpha_k g_k(y_k),$$

Then:

$$f(x_{k+1}) - f(x_k) \leq \frac{-(g_k(y_k))^2}{4(M - m)\|y_k - x_k\|^2}$$

As a result:

$$\begin{aligned} 0 \leq (g_k(y_k))^2 &\leq 4(M - m)\|y_k - x_k\|^2(f(x_k) - f(x_{k+1})) \\ &\leq 4(M - m)D^2(f(x_k) - f(x_{k+1})) \end{aligned}$$

with $D = \max_{(x,y) \in X^2} \|x - y\|$, and using 3) of the theorem 1 we have:

$$g_k(y_k) \xrightarrow[k \rightarrow +\infty]{} 0.$$

4. Convergence Speed Assessment

The demonstration of the theorem that gives the speed of convergence of the sequence $(x_k)_k$ requires the following lemma which is a result on Banach’s fixed point theorem for contractions.

Lemma 1 *Let $(x_k)_k$ be a sequence verifying the following hypothesis:*

$$\forall k \geq 0 \exists \beta_k \text{ such as } \|x_{k+1} - x_k\| \leq \beta_k \|x_k - x_{k-1}\|$$

with $\beta_k \xrightarrow[k \rightarrow +\infty]{} 0$

Then :

1. x_k is a convergent sequence
2. $\exists K > 0, \exists C > 0, \forall k \geq K : \|x_k - x^*\| \leq C\beta_k\beta_{k+1}\dots\beta_K$
with $x^* = \lim_{k \rightarrow +\infty} x_k$

Lemma 2 *Under the same assumptions as in Theorem 1, we have:*

$$\|y_k - x_k\| \xrightarrow[n \rightarrow +\infty]{} 0$$

Proof.

$g_k(y_k) = \min_{y \in X} g_k(y)$ implies $(\nabla g_k(y_k), y_k - x_k) \leq 0$

Which gives:

$$\left(\nabla f(x_k), y_k - x_k\right) + \left(\nabla^2 f(x_k)(y_k - x_k), y_k - x_k\right) \leq 0$$

So:

$$\left(\nabla f(x_k), y_k - x_k\right) \leq -\left(\nabla^2 f(x_k)(y_k - x_k), y_k - x_k\right) \leq 0$$

As a result:

$$\left(\nabla f(x_k), y_k - x_k\right) + \frac{1}{2}\left(\nabla^2 f(x_k)(y_k - x_k), y_k - x_k\right) \leq -\frac{1}{2}\left(\nabla^2 f(x_k)(y_k - x_k), y_k - x_k\right)$$

Using the relation (8) we have:

$$g_k(y_k) \leq -\frac{m}{2}\|y_k - x_k\|^2 \tag{12}$$

Thereafter:

$$\|y_k - x_k\|^2 \leq -\frac{2}{m}g_k(y_k)$$

Using 4) of Theorem 1, we obtain:

$$\|y_k - x_k\| \xrightarrow{n \rightarrow +\infty} 0$$

Theorem 2 Under the same assumptions as in Theorem 1, we have:

1. $\exists K > 0, \forall k \geq K$, we have $x_{k+1} = y_k$
2. x_k converges to a x^* limit a super-linear way.
3. x^* is the solution to problem (1).

Proof.

1. By the relation (9) and equality $x_{k+1} - x_k = \alpha_k(y_k - x_k)$, we have:

$$f(x_{k+1}) - f(x_k) \leq \alpha_k g_k(y_k) + \frac{1}{2}\alpha_k^2 \|\nabla^2 f(x_k) - \nabla^2 f(t_k)\| \|y_k - x_k\|^2$$

To have the inequality (7) it is enough that the following condition is verified

$$\alpha_k g_k(y_k) + \frac{1}{2}\alpha_k^2 \|\nabla^2 f(x_k) - \nabla^2 f(t_k)\| \|y_k - x_k\|^2 \leq \frac{1}{2}\alpha_k g_k(y_k)$$

so α_k verifies:

$$\alpha_k \leq \frac{-g_k(y_k)}{\|\nabla^2 f(x_k) - \nabla^2 f(t_k)\| \|y_k - x_k\|^2} \tag{13}$$

On the other hand by the relation (12) we have:

$$\frac{m}{2\|\nabla^2 f(x_k) - \nabla^2 f(t_k)\|} \leq \frac{-g_k(y_k)}{\|\nabla^2 f(x_k) - \nabla^2 f(t_k)\| \|y_k - x_k\|^2} \tag{14}$$

By the Lemma 2, we also have:

$$\|t_k - x_k\| = \|s(x_{k+1} - x_k)\| \leq \|y_k - x_k\| \xrightarrow{k \rightarrow +\infty} 0.$$

Which implies:

$$\|\nabla^2 f(x_k) - \nabla^2 f(t_k)\| \xrightarrow{k \rightarrow +\infty} 0. \tag{15}$$

And:

$$\frac{m}{2\|\nabla^2 f(x_k) - \nabla^2 f(t_k)\|} \xrightarrow{k \rightarrow +\infty} +\infty$$

And using (14) we also have:

$$\frac{-g_k(y_k)}{\|\nabla^2 f(x_k) - \nabla^2 f(t_k)\| \|y_k - x_k\|^2} \xrightarrow{k \rightarrow +\infty} +\infty.$$

So $\exists K > 0$ such as $\forall k \geq K$ we have :

$$1 \leq \frac{-g_k(y_k)}{\|\nabla^2 f(x_k) - \nabla^2 f(t_k)\| \|y_k - x_k\|^2}$$

Using the relation 13 and condition $0 < \alpha_k \leq 1$ we get: For $k \geq K$ the inequality (7) is checked for $\alpha_k = 1$ which gives:

$$x_{k+1} = y_k \quad \forall k \geq K \tag{16}$$

2. By the convexity of g_k , the generalized Lagrange formula (Kolmogorov & Fomine, 1979), and the relation $g_k(x_k) = 0$, we have:

$$\begin{aligned} g_k(x_{k+1}) &\geq \left(\nabla g_k(x_k), x_{k+1} - x_k \right) \\ &= \left(\nabla f(x_k), x_{k+1} - x_k \right) + \left(\nabla^2 f(x_k)(x_{k+1} - x_k), x_{k+1} - x_k \right) \\ &\geq \left(\nabla f(x_k), x_{k+1} - x_k \right) \\ &= \left(\nabla f(x_k) - \nabla f(x_{k-1}), x_{k+1} - x_k \right) + \left(\nabla f(x_{k-1}), x_{k+1} - x_k \right) \\ &= \left(\nabla^2 f(t_k)(x_k - x_{k-1}), x_{k+1} - x_k \right) + \left(\nabla f(x_{k-1}), x_{k+1} - x_k \right) \\ &= \left(\nabla f(x_{k-1}) + \nabla^2 f(x_{k-1})(x_k - x_{k-1}), x_{k+1} - x_k \right) + \left((\nabla^2 f(t_k) - \nabla^2 f(x_{k-1}))(x_k - x_{k-1}), x_{k+1} - x_k \right) \end{aligned}$$

So:

$$g_k(x_{k+1}) \geq \left(\nabla g_{k-1}(x_k), x_{k+1} - x_k \right) + \left((\nabla^2 f(t_k) - \nabla^2 f(x_{k-1}))(x_k - x_{k-1}), x_{k+1} - x_k \right) \tag{17}$$

with $t_k = x_{k-1} + s(x_k - x_{k-1})$ and $s \in [0, 1]$, which gives Using the relation (16) and the fact that

$$\min_{x \in X} g_{k-1}(x) = g_{k-1}(y_{k-1}) = g_{k-1}(x_k)$$

we obtain:

$$\left(\nabla g_{k-1}(x_k), x_{k+1} - x_k \right) \geq 0$$

And from the relation (17), it follows that:

$$g(x_{k+1}) \geq \left((\nabla^2 f(t_k) - \nabla^2 f(x_{k-1}))(x_k - x_{k-1}), x_{k+1} - x_k \right)$$

Which gives:

$$-g(x_{k+1}) \leq \|\nabla^2 f(t_k) - \nabla^2 f(x_{k-1})\| \|x_k - x_{k-1}\| \|x_{k+1} - x_k\|$$

Moreover, by (12) and (16), we have:

$$\|x_{k+1} - x_k\| \leq \frac{2\|\nabla^2 f(t_k) - \nabla^2 f(x_k)\|}{m} \|x_k - x_{k-1}\|$$

By having:

$$\beta_k = \frac{2\|\nabla^2 f(t_k) - \nabla^2 f(x_k)\|}{m}$$

and using (15), we have:

$$\|x_{k+1} - x_k\| \leq \beta_k \|x_k - x_{k-1}\|$$

With $\beta_k \xrightarrow[k \rightarrow +\infty]{} 0$

Using Lemma 1, we finally obtain that: x_k is a convergent sequence

And:

$$\exists K > 0, \exists C > 0, \forall k \geq K \quad \|x_k - x^*\| \leq C\beta_k\beta_{k+1}\dots\beta_k$$

With: $x^* = \lim_{k \rightarrow +\infty} x_k$

3. Let us show that x^* is the solution of problem (1).

Using relation (4) we have:

$$\left(\nabla g_k(y_k), y_k - y\right) \leq 0 \quad \forall y \in X$$

Which gives:

$$\left(\nabla f(x_k), y_k - y\right) + \left(\nabla^2 f(x_k)(y_k - x_k), y_k - y\right) \leq y \quad \forall y \in X$$

Using the Lemma (2) and the fact that: $x^* = \lim_{k \rightarrow +\infty} x_k$

We have thus:

$$\left(\nabla f(x^*), x^* - y\right) \leq 0 \quad \forall y \in X$$

Which proves that x^* is the solution to problem (1).

5. Numerical Examples

In this section, we present some numerical experiments. We compare the new method **N.M.** with a quadratic programming method **Q.M.**. This comparison is summarized in the tables which give the number of iterations, the associated residual norms for each method and the convergence time.

Example 1:

We consider the following problem:
$$\begin{cases} \text{Minimize } 0.5x_1^2 + 0.5x_2^2 - 2x_1x_2 - x_1 - 2x_2 \\ \text{subject to : } \begin{cases} x_1 + x_2 \leq 1 \\ x_1 \geq 0, \text{ and } x_2 \geq 0. \end{cases} \end{cases}$$

Table 1. Numerical results of example 1, $x_0 = (0.6, 0.9)$

iterations	N.M.	Q.M.
1	0.354338193757822	0.354 e+00
2	2.342287234782921e-05	7.165e-01
3	4.950612686125601e-10	1.679e-03
4		3.021e-08
5		
CPU	0.499203199999954 s	0.951606100000006 s

$$x^* = (0.333333333463281, 0.666666666355120)$$

Example 2:

In this example, we have chosen the Resonbrock function defined by:

$$f(x) = \sum_{i=1}^{\frac{n}{2}} [(x_{2i-1} - 1)^2 + 100(x_{2i-1}^2 - x_{2i})^2]$$

We consider the following problem:

$$\left\{ \begin{array}{l} \text{Minimize } f(x) = \sum_{i=1}^{\frac{n}{2}} \left((x_{2i-1} - 1)^2 + 100(x_{2i-1}^2 - x_{2i})^2 \right) \\ \text{subject to : } \left\{ \begin{array}{l} AX \leq b \\ x_i \geq 0, \quad i = 1, \dots, n \end{array} \right. \end{array} \right.$$

Where: $A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & \dots & 1 & 1 \\ 0 & 0 & . & \dots & . & . \\ 0 & 0 & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} n \\ n-1 \\ . \\ . \\ . \\ 2 \\ 1 \end{pmatrix}$ For $n = 2$

Table 2. Numerical results of example 2. $X_0 = (0.7, 0.8)$

iterations	N.M.	Q.M.
1	0.028478407097865	0.028478e+00
2	0.012305422981477	3.370370e-02
3	0.0028790327150207	4.682192e-05
4	3.753682455549126e-04	2.582192e-07
5	9.362730949709947e-06	
6	4.992750200674966e-08	
7	4.795137971350407e-09	
CPU	0.6708430000000 s	0.967206199999993 s

$$x^* = (0.999492283237436, 0.998982719563765)$$

For $n = 4$

Table 3. Numerical results of example 2. $X_0 = (0.8, 0.9, 2, 3)$.

iterations	N.M.	Q.M.
1	60.208398256230012	60.208e+00
2	10.951976263434688	1.646e+01
3	0.085938289646478	5.140e-02
4	9.721017738673661e-06	9.191e-05
5	4.159793037111924e-08	5.159e-07
CPU	0.577203700000041 s	1.060806799999966 s

$$x^* = (0.999754865648997, 0.999508812425961, 0.999325348316186, 0.998648593607337)$$

For $n = 6$

Table 4. Numerical results of example 2. $X_0 = (1, 1, 2, 3, -1, 1)$

iterations	N.M.	Q.M.
1	9.482806440169231	9.482e+00
2	1.203531412605563	1.389e-01
3	0.952791678359950	6.529e-05
4	0.622953650681599	7.737e-07
5	0.239340264855938	
6	0.418995706912790	
7	0.145547118767705	
8	0.064522231068026	
9	0.031789110436983	
10	0.009208499425393	
11	0.002450547757329	
12	2.002389648644385e-04	
13	3.512172376494795e-06	
14	2.613283061750371e-08	
CPU	0.951606099999992 s	1.310408399999972 s

$x^* =$

Columns 1 through 4

0.999832428389446 , 0.999664221762951 , 0.999606232212638 , 0.999211176526036

Columns 5 through 6

0.999232165748957 , 0.998462090227239

6. Conclusions

1. The method described in this paper minimizes a sequence of quadratic problems under constraints without using Lagrange multipliers, which does not increase the number of variables.
2. With this method, we can solve an important class of problems encountered in numerical analysis which are formulated as follows:

$$\begin{cases} \text{Min}f(x) \\ Ax \leq b \\ x \geq 0 \end{cases}$$

where A is the matrix that defines the constraints, x is the vector of variables and b is the vector of bounds of the variables . At each iteration x_k , we determine the direction $d_k = y_k - x_k$ by solving the following problem

$$\begin{cases} \text{Min}\frac{1}{2}(H_k y, y) + (c_k, y) \\ Ay \leq b \\ y \geq 0 \end{cases}$$

where $H_k = \nabla^2 f(x_k)$ and $c_k = \nabla f(x_k)$.

3. What is also important is that in the sequence of quadratic problems solved by the method, what changes is the objective function. This gives a gain in memory space and execution time.

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