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Hom Leibniz Superalgebras With Supersymmetric Invariant Nondegenerate Bilinear Forms

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Abstract

In this paper we study Hom Leibniz superalgebras endowed with a supersymmetric, nondegenerate and invariant bilinear form. Such Hom Leibniz superalgebras are called quadratic Hom Leibniz superalgebras. We show that every quadratic Hom Leibniz superalgebra is symmetric. After introducing representation of Hom Leibniz superalgebras, we prove that Hom Leibniz superalgebras are quadratic if and only if the adjoint representation and its dual representation are equivalent. We also extend the notion of T^*-extension to Hom Leibniz superalgebras. Finally, by using double extension, we describe inductively quadratic regular Hom Leibniz superalgebras.

Keywords: Hom Leibniz superalgebra, T^*-extension, representation, quadratic structures, double extension

1. Introduction

A left (resp. right) Hom Leibniz superalgebra is a triplet $(\mathcal{L}, [\ ,\ ], \eta)$ where $\mathcal{L} := \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ is a $\mathbb{Z}_2-$graded vector space, $[\ ,\ ] : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{L}$ an even bilinear map and $\eta : \mathcal{L} \rightarrow \mathcal{L}$ is an even linear map such that $[\eta r, [s, t]] = [[r, s], \eta t] + (-1)^{|r||s|}[\eta s, [r, t]]$ (resp. $[\eta r, [s, t] = [[r, s], \eta t] - (-1)^{|s||t|}[[r, t], \eta s])$ for all $r, s, t \in \mathcal{L}$. The Hom Leibniz superalgebra is said to be symmetric if it is simultanouesly a left and right Hom Leibniz superalgebra.

The notion of Hom algebra appears first in the study of algebraic structures characterizing some $q-$deformations of Witt and Virasoro algebras where authors in (Hartwig, J., Larsson, D. & Silvestrov, S.(2006)) introduced the notion of Hom Lie algebra. In this last decade, we notice an important evolution in the area of Hom algebraic structures. In particular, many results about Lie algebras have been generalized to Hom Lie (super)-algebras and Hom Leibniz algebras see (Benayadi, S. & Makhlouf, A.(2014)),(Gaparayi, D., Attan, S. & Issa, A. N.),(Yau, D.), (Guan, B., Chen, L. & Sun, B.(2015)) and (Larsson, D. & Silvestrov, S.(2005)) for more details.

By following this trend, we aim to investigate Hom Leibniz superalgebras endowed with a supersymmetric, invariant and nondegenerate bilinear form. Such Hom Leibniz superalgebras are called quadratic Hom Leibniz superalgebras. Quadratic structures have been studied for many non associative (super)-algebras due to its interaction with other problems in physics and other areas in mathematics. In order to study quadratic Hom Leibniz superalgebras, we introduce the notion of double extension for Hom Leibniz superalgebras, which plays an important role in the study of quadratic structures. This notion has been introduced by author in (Kac, V.(1985)) on his study of solvable Lie algebras. A.Medina and P.Revoy used this notion to characterize all quadraric Lie algebras see (Medina, A. & Revoy, P.). The classification of solvable Lie algebras has been done by using the notion of double extension by authors in (Pinczon, G., Duong, M.T. & Ushirobira, R.). This notion appears almost in every study of quadratic structures see (Benayadi, S. & Bouarroudj, S.(2018)), (Sanchez, O. A., Valenzuela, R., Delgado, G. & Salgado, G.), (Benayadi, S. & Hidri, S.(2014)) and (Vishnyakova, E.(2016)). We will also generalize the notion of T^*-extension to Hom Leibniz superalgebras. This extension plays an important role also in the study of quadratic structure. For example in (Martin, B.(1997)), Bordmann used this method to construct finite dimensional nilpotent quadratic algebras. Another purpose of this paper is to give a necessary and sufficient conditions for a quadratic Hom Leibniz superalgebra to be isomorph to a T^*-extension of another Hom Leibniz superalgebra superalgebra.

This paper is organized in 4 sections. The frist section is devoted to basic defintions and elementary results on Hom Leibniz superalgebras. We show that all quadratic Hom Leibniz superalgebras are symmetric, that is simultaneously left and right Hom Leibniz superalgebras. In section 2, we define the notion of representation of Hom Leibniz superalgebra and study the adjoint representation and its dual representation. We also show that a Hom Leibniz superalgebra is quadratic if and only if the adjoint representation and its dual representation are equivalent. In section 3, we generalize the notion of T^*-extension to Hom Leibniz superalgebra and study the equivalence and isometry of two different T^*-extensions of a given Hom Leibniz superalgebra. We also give necessary and sufficient conditions for a quadratic Hom Leibniz superalgebra to be isomorph to a T^*-extension. In section 4, we etablish the notion of double extension for Hom Leibniz superalgebras and use it to give an inductive description of odd quadratic regular Hom Leibniz superalgerba. In other words, we show that every odd quadratic regular Hom Leibniz superalgebra can be obtained from a finite number of one dimensional abelian Hom Lie superalgebras and a Hom Lie superalgebra.

For the notation, we shall keep the notation in (Bajo, I. , Benayadi, S. & Bordemann, M.(2007)).

2. Preliminaries

In this section we present the basic definitions and elementary results.

Definition 2.1. *Let $\mathcal{L}, \mathcal{L}'$ be two $\mathbb{Z}_2-$graded vector spaces and $f : \mathcal{L} \to \mathcal{L}'$ a linear map. The map f is said to be homogeneous of degree $\mid f \mid \in \mathbb{Z}_2$ if $f(\mathcal{L}_\alpha) \subseteq \mathcal{L}'_{\alpha+|f|}$ for all $\alpha \in \mathbb{Z}_2$.*

Definition 2.2. *A left Hom Leibniz superalgebras is a triplet $(\mathcal{L}, [.,.], \eta)$ where $\mathcal{L}$ is a $\mathbb{Z}_2-$graded vector space, $[.,.]$ an even bilinear map over $\mathcal{L}$ and $\eta \in (End(\mathcal{L}))_{\bar{0}}$ such that $[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$ and*

$$[\eta r, [s, t]] = [[r, s], \eta t] + (-1)^{|r||s|} [\eta s, [r, t]] \tag{1}$$

for all $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$ and $t \in \mathcal{L}$.

The triplet $(\mathcal{L}, [.,.], \eta)$ is called right Hom Leibniz superalgebra if we have

$$[\eta r, [s, t]] = [[r, s], \eta t] - (-1)^{|s||t|} [[r, t], \eta s] \tag{2}$$

for all $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$ and $t \in \mathcal{L}$. A Hom Leibniz superalgebra is called symmetric if it is simultaneously a left and right Hom Leibniz superalgebra.

Lemma 2.3. *A left or right Hom Leibniz superalgebra is symmetric if and only if*

$$[\eta r, [s, t]] = -(-1)^{|r|(|s|+|t|)} [[s, t], \eta r] \tag{3}$$

for all $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$ and $t \in \mathcal{L}_{|t|}$

Proof. Let us prove the lemma for a left Hom Leibniz superalgebra $(\mathcal{L}, [.,.], \eta)$. Let us assume that the relation (3) is satisfied, then

$$\begin{aligned} [\eta r, [s, t]] &= [[r, s], \eta t] + (-1)^{|r||s|} [\eta s, [r, t]] \\ &= [[r, s], \eta t] + (-1)^{|r||s|} \left(-(-1)^{|s|(|r|+|t|)} [[r, t], \eta s]\right) \\ &= [[r, s], \eta t] - (-1)^{|s||t|} [[r, t], \eta s], \end{aligned}$$

hence $(\mathcal{L}, [.,.], \eta)$ is also a right Hom Leibniz superalgebra, so it is symmetric. Conversely, if $(\mathcal{L}, [.,.], \eta)$ is symmetric then the relations (1) and (2) are satisfied. So by applying the later relations to the triplet (s, r, t), we obtain

$$[[s, r], \eta t] + (-1)^{|r||s|} [\eta r, [s, t]] = [[s, r], \eta t] - (-1)^{|r||t|} [[s, t], \eta r]$$

which implies relation (3). □

Definition 2.4. *The Hom Leibniz superalgebra $(\mathcal{L}, [.,.], \eta)$ is called :*

- *multiplicative if $\eta([r, s]) = [\eta r, \eta s]$.*
- *regular if η is an automorphism of $\mathcal{L}$.*
- *involutive if it is multiplicative and $\eta^2 = Id_{\mathcal{L}}$.*

Definition 2.5. *Let $(\mathcal{L}, [.,.], \eta)$ and $(\mathcal{L}', [.,.], \eta')$ be two Hom Leibniz superalgebras. A linear map $f : \mathcal{L} \to \mathcal{L}'$ is a morphism of Hom Leibniz superalgebras if $f([r, s]) = [f(r), f(s)]$ for all $r, s \in \mathcal{L}$ and $f \circ \eta = \eta' \circ f$.*

Given a left Hom Leibniz superalgebra $(\mathcal{L}, [.,.], \eta)$ and $r \in \mathcal{L}_{|r|}$ we define the left (resp. right) multiplication L_r (resp. R_r) by $L_r(s) = [r, s]$ and $R_r(s) = (-1)^{|r||s|}[s, r]$. We have the following property :

Proposition 2.6. *Let $(\mathcal{L}, [.,.], \eta)$ be a left Hom Leibniz superalgebra. Then we have the following relations :*

1. $L_{[r,s]} \circ \eta = L_{\eta r} \circ L_s - (-1)^{|r||s|} L_{\eta s} \circ L_r$

2. $R_{[r,s]} \circ \eta = L_{\eta r} \circ R_s + (-1)^{|r||s|} R_{\eta s} \circ R_r$

3. $R_{[r,s]} \circ \eta = L_{\eta r} \circ R_s - (-1)^{|r||s|} R_{\eta s} \circ L_r$

4. $R_{\eta s} \circ R_r = -R_{\eta s} \circ L_r$

for all $r \in \mathcal{L}_{|r|}$ *and* $s \in \mathcal{L}_{|s|}$.

Proof. The proof is straightfoward, 1) and 2) are etablished by applying the relation (1) to the triplet (r, s, t) and (r, t, s) with the variable t used as argument. The relation 3) is obtained by direct calculation and relation 4) follows from a combinaison of 2) and 3). □

Definition 2.7. *Let* $(\mathcal{L}, [.,.], \eta)$ *be a left Hom Leibniz superalgebra and* I *be a* $\mathbb{Z}_2-$*graded subspace of* $\mathcal{L}$.

- *I is a Hom Leibniz sub superalgebra if* $[I, I] \subseteq I$ *and* $\eta(I) \subseteq I$.
- *I is a left(resp. right) ideal if it is an Hom Leibniz sub superalgebra such that* $[I, \mathcal{L}] \subseteq I$ *(resp.* $[\mathcal{L}, I] \subseteq I$*).*

We define the left (resp. right) center of $\mathcal{L}$ *by* $Z^l(\mathcal{L}) = \{x \in \mathcal{L}\,; [x, \mathcal{L}] = 0\}$ *(resp.* $Z^r(\mathcal{L}) = \{x \in \mathcal{L}\,; [\mathcal{L}, x] = 0\}$*). The center of* $\mathcal{L}$ *is defined by* $Z(\mathcal{L}) = Z^r(\mathcal{L}) \cap Z^l(\mathcal{L})$. *Denote by* $ker(\mathcal{L})$ *the subspace of* $\mathcal{L}$ *generated by elements of the form* $[r, s] + (-1)^{|r||s|}[s, r]$ *with* $r, s \in \mathcal{L}$.

Remark 2.8. *The relation 4) of proposition (2.6) implies that* $[ker(\mathcal{L}), Im(\eta)] = \{0\}$. *If* $ker(\mathcal{L}) = \{0\}$, *then* $(\mathcal{L}, [.,.], \eta)$ *is a Hom Lie superalgebra.*

Definition 2.9. *Let* $(\mathcal{L}, [.,.], \eta)$ *be a Hom Leibniz superalgebra and B a bilinear form on* $\mathcal{L}$. *Then B is said to be :*

- *supersymmetric if for all* $r \in \mathcal{L}_{|r|}, s \in \mathcal{L}_{|s|}$, *we have* $B(r, s) = (-1)^{|r||s|} B(s, r)$,
- *invariant if for all* $r, s, t \in \mathcal{L}$, *we have* $B([r, s], t) = B(r, [s, t])$,
- *non degenerate if* $B(r, s) = 0$ *for all* $s \in \mathcal{L}$, *then* $r = 0$.

Definition 2.10. *Let* $(\mathcal{L}, [.,.], \eta)$ *be a Hom Leibniz superalgebra. Then the set of all bilinear forms on* $\mathcal{L}$ *denoted by* $\mathfrak{B}$ *is a graded space in the following way* $\mathfrak{B} = \mathfrak{B}_{\bar{0}} \oplus \mathfrak{B}_{\bar{1}}$ *where*

$$\mathfrak{B}_\gamma = \{B \in \mathfrak{B}; B(\mathcal{L}_\alpha, \mathcal{L}_\beta) \subseteq \mathbb{K}_{\alpha+\beta+\gamma} \quad \forall \alpha, \beta \in \mathbb{Z}_2\}$$

with $\mathbb{K}_{\bar{0}} = \mathbb{K}$ *and* $\mathbb{K}_{\bar{1}} = \{0\}$.

Definition 2.11. *A Hom Leibniz superalgebra* $(\mathcal{L}, [.,.], \eta)$ *is said to be even (resp. odd) quadratic if it is endowed with an invariant, supersymmetric and non degenerate even (resp. odd) bilinear form B such that*

$$B(\eta r, s) = B(r, \eta s).$$

for all $r, s \in \mathcal{L}$.

Definition 2.12. *Let* $(\mathcal{L}, [.,.], \eta, B)$ *be a quadratic Hom Leibniz superalgebra and I a subspace of* $\mathcal{L}$. *The subspace* $I^\perp = \{r \in \mathcal{L}, B(r, I) = 0\}$ *is called the orthogonal space of I. An ideal I is said to be totally isotrop if,* $B(I, I) = 0$.

Lemma 2.13. *Let* $(\mathcal{L}, [.,.], \eta, B)$ *be a quadratic Hom Leibniz superalgebra and I an ideal of* $\mathcal{L}$. *Then* $I^\perp$ *is also an ideal of* $\mathcal{L}$.

Proof. The proof is similar to the one for Hom Lie algebra, see lemma 2.6 of (Benayadi, S. & Makhlouf, A.(2014)). □

In the sequel all Hom Leibniz superalgebras considered are multiplicative.

Lemma 2.14. *Let* $(\mathcal{L}, [.,.], \eta, B)$ *be a left or right quadratic Hom Leibniz superalgebra. Then* $\mathcal{L}$ *is symmetric.*

Proof. Let $(\mathcal{L}, [.,.], \eta, B)$ be a left quadratic Hom Leibniz superalgebra. Let $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$, $t \in \mathcal{L}_{|t|}$ and $v \in \mathcal{L}_{|v|}$. We have

$$\begin{aligned}
B\Big([\eta r,[s,t]] + (-1)^{|r|(|s|+|t|)}[[s,t],\eta r], v\Big) &= B([\eta r,[s,t]], v) + (-1)^{|r|(|s|+|t|)} B([[s,t],\eta r], v) \\
&= B(\eta r, [[s,t], v]) + (-1)^{|r|(|s|+|t|)} B([s,t],[\eta r, v]) \\
&= B(r, [[\eta s, \eta t], \eta v]) + (-1)^{|v|(|s|+|t|)} B(\eta r, [v,[s,t]]) \\
&= (-1)^{|r|(|s|+|t|+|v|)} B([\eta s,\eta t],[\eta v, r]) + (-1)^{|v|(|s|+|t|)} B([r,\eta v],[\eta s,\eta t]) \\
&= (-1)^{|r||v|+|v|(|s|+|t|)} B([[\eta v, r],\eta s],\eta t) + (-1)^{|v|(|s|+|t|)} B([[r,\eta v],\eta s],\eta t) \\
&= (-1)^{|v|(|s|+|t|)} B\Big([[r,\eta v] + (-1)^{|r||v|}[\eta v, r],\eta s],\eta t\Big) = 0
\end{aligned}$$

because from remark 2.8, we have $[[r,\eta v] + (-1)^{|r||v|}[\eta v, r], \eta s] = 0$. Since the bilinear form B is non degenerate, then

$$[\eta r,[s,t]] = -(-1)^{|r|(|s|+|t|)}[[s,t],\eta r]$$

Hence $\mathcal{L}$ is symmetric. □

3. Representations of Hom Leibniz Superalgebras

In this section we introduce representations of Hom Leibniz superalgebras and study the adjoint and the dual representation.

Definition 3.1. *Let $(\mathcal{L}, [.,.], \eta)$ be a Hom Leibniz superalgebra. A representation of $\mathcal{L}$ in a $\mathbb{Z}_2$−graded vector space $V := V_{\bar{0}} \oplus V_{\bar{1}}$ is given by a triplet (λ, ϕ, χ) where $\lambda, \phi : \mathcal{L} \to (End(V))_{\bar{0}}$ are linear maps and $\chi \in (End(V))_{\bar{0}}$ such that*

$$\phi_{[r,s]} \circ \chi = \phi_{\eta r} \circ \phi_s - (-1)^{|r||s|}\phi_{\eta s} \circ \phi_r$$

$$\lambda_{[r,s]} \circ \chi = \phi_{\eta r} \circ \lambda_s - (-1)^{|r||s|}\lambda_{\eta s} \circ \phi_r$$

$$\lambda_{[r,s]} \circ \chi = \phi_{\eta r} \circ \lambda_s + (-1)^{|r||s|}\lambda_{\eta s} \circ \lambda_r$$

for all $r \in \mathcal{L}_{|r|}$ and $s \in \mathcal{L}_{|s|}$.

We said that (λ, ϕ) is a representation of $\mathcal{L}$ over V with respect to χ. We will use often the notation (V, λ, ϕ, χ) to denote such representation.

Example 3.2. *Let $(\mathcal{L}, [.,.], \eta)$ be a Hom Leibniz superalgebra. Then according to proposition (2.6), (R, L) is a representation of $\mathcal{L}$ over $\mathcal{L}$ with respect to η. This representation is called the adjoint representation of $\mathcal{L}$.*

Definition 3.3. *Let $(\mathcal{L}, [.,.], \eta)$ be a Hom Leibniz superalgebra, V, V' two $\mathbb{Z}_2$−graded vector spaces and χ(resp. χ') be even endomorphism of V(resp. V'). A representation (ϕ, λ) of $\mathcal{L}$ over V with respect to χ is said to be isomorph to a representation (ϕ', λ') of $\mathcal{L}$ over V' with respect to χ', if there exists an isomorphism $\Phi : V \to V'$ such that*

$$\Phi \circ \phi_r = \phi'_r \circ \Phi, \quad \Phi \circ \lambda_r = \lambda'_r \circ \Phi \quad \textit{and} \quad \chi' \circ \Phi = \Phi \circ \chi \qquad \forall r \in \mathcal{L}.$$

Given a representation (V, λ, ϕ, χ) of a Hom Leibniz superalgebra $\mathcal{L}$, we define $\lambda^*, \phi^* : \mathcal{L} \to (End(V^*))_{\bar{0}}$ and $\chi^* \in (End(V^*))_{\bar{0}}$ by

$$\lambda^*_r(f) = (-1)^{|r||f|} f \circ \phi_r, \qquad \phi^*_r(f) = (-1)^{|r||f|} f \circ \lambda_r \quad \text{and} \quad \chi^*(f) = f \circ \chi$$

Proposition 3.4. *The quadruplet $(V^*, \lambda^*, \phi^*, \chi^*)$ defined above is a representation of $\mathcal{L}$ over V^* if and only if we have the following relations :*

1. $\chi \circ \lambda_{[r,s]} = (-1)^{|r||s|}\lambda_s \circ \lambda_{\eta r} - \lambda_r \circ \lambda_{\eta s}$
2. $\chi \circ \phi_{[r,s]} = (-1)^{|r||s|}\phi_s \circ \lambda_{\eta r} - \lambda_r \circ \phi_{\eta s}$
3. $\chi \circ \phi_{[r,s]} = (-1)^{|r||s|}\phi_s \circ \lambda_{\eta r} + \phi_r \circ \phi_{\eta s}$

for all $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$

Proof. $(V^*, \lambda^*, \phi^*, \chi^*)$ is a representation if and only if

$$\phi^*_{[r,s]} \circ \chi^* = \phi^*_{\eta r} \circ \phi^*_s - (-1)^{|r||s|}\phi^*_{\eta s} \circ \phi^*_r$$

$$\lambda^*_{[r,s]} \circ \chi^* = \phi^*_{\eta r} \circ \lambda^*_s - (-1)^{|r||s|}\lambda^*_{\eta s} \circ \phi^*_r$$

$$\lambda^*_{[r,s]} \circ \chi^* = \phi^*_{\eta r} \circ \lambda^*_s + (-1)^{|r||s|}\lambda^*_{\eta s} \circ \lambda^*_r$$

for all $r \in \mathcal{L}_{|r|}$ and $s \in \mathcal{L}_{|s|}$. The first relation implies that for all $f \in V^*$ we have

$$\phi^*_{[r,s]} \circ \chi^*(f) = \phi^*_{\eta r} \circ \phi^*_s(f) - (-1)^{|r||s|}\phi^*_{\eta s} \circ \phi^*_r(f)$$

on one side

$$\phi^*_{[r,s]} \circ \chi^*(f) = \phi^*_{[r,s]}(f \circ \chi) = (-1)^{|f|(|r|+|s|)} f \circ \chi \circ \lambda_{[r,s]}$$

On the other side we have

$$\begin{aligned}\phi^*_{\eta r} \circ \phi^*_s(f) - (-1)^{|r||s|}\phi^*_{\eta s} \circ \phi^*_r(f) &= (-1)^{|s||f|}\phi^*_{\eta r}(f \circ \lambda_s) - (-1)^{|r|(|s||f|)}\phi^*_{\eta s}(f \circ \lambda_r)\\ &= (-1)^{|s||f|+|r|(|f|+|s|)} f \circ \lambda_s \circ \lambda_{\eta r} - (-1)^{|f|(|r|+|s|)} f \circ \lambda_r \circ \lambda_{\eta s}\\ &= (-1)^{|f|(|r|+|s|)}\left[f\left((-1)^{|r||s|}\lambda_s \circ \lambda_{\eta r} - \lambda_r \circ \lambda_{\eta s}\right)\right]\end{aligned}$$

Therefore we have the relation 1). The other relations are obtained with a similar computation. □

Proposition 3.5. *Let $(\mathcal{L}, [.,.], \eta, B)$ be a quadratic Hom Leibniz superalgebra. Then we have the following relations :*

1. $\eta \circ L_{[r,s]} = -R_r \circ L_{\eta s} + (-1)^{|r||s|} L_s \circ R_{\eta r}$
2. $\eta \circ R_{[r,s]} = -R_r \circ R_{\eta s} + (-1)^{|r||s|} R_s \circ R_{\eta r}$
3. $\eta \circ L_{[r,s]} = L_r \circ L_{\eta s} + (-1)^{|r||s|} L_s \circ R_{\eta r}$

Proof. For relation 1), let $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$, $t \in \mathcal{L}_{|t|}$ and $z \in \mathcal{L}_{|z|}$, we have

$$\begin{aligned}& B\left(\eta \circ L_{[r,s]}(t) + R_r \circ L_{\eta s}(t) - (-1)^{|r||s|} L_s \circ R_{\eta r}(t), z\right)\\ =\ & B\left(\eta([[r,s],t]), z\right) + (-1)^{|r|(|s|+|t|)} B\left([[\eta s,t],r], z\right) - (-1)^{|r|(|s|+|t|)} B\left([s,[t,\eta r]], z\right)\\ =\ & B\left([[r,s],t], \eta z\right) + (-1)^{|r|(|s|+|t|)} B\left([\eta s,t],[r,z]\right) - (-1)^{|r|(|s|+|t|)+|z|(|r|+|s|+|t|)} B\left([z,s],[t,\eta r]\right)\\ =\ & (-1)^{|z|(|r|+|s|+|t|)} B\left([\eta z,[r,s]], t\right) + (-1)^{|r|(|s|+|t|)+|s|(|r|+|z|)+|t|(|r|+|z|)} B\left([[r,z],\eta s], t\right)\\ & -(-1)^{|r|(|s|+|t|)+|z||s|+|s|(|r|+|t|)} B\left(t,[\eta r,[z,s]]\right)\\ =\ & (-1)^{|z|(|r|+|s|+|t|)} B\left([\eta z,[r,s]], t\right) + (-1)^{|z|(|s|+|t|)} B\left([[r,z],\eta s], t\right) - (-1)^{|z|(|s|+|t|)} B\left([\eta r,[z,s]], t\right)\\ =\ & (-1)^{|z|(|s|+|t|)} B\left((-1)^{|r||z|}[\eta z,[r,s]] + [[r,z],\eta s] - [\eta r,[z,s]], t\right) = 0\end{aligned}$$

since B is non degenerate then $\eta \circ L_{[r,s]} = -R_r \circ L_{\eta s} + (-1)^{|r||s|} L_s \circ R_{\eta r}$. The two other relations are proved in a similar way. □

Remark 3.6. *The proposition 3.5 yields for all Hom Leibniz superalgebra $(\mathcal{L}, [.,.], \eta)$ such that $Im(\eta^2 - Id_{\mathcal{L}}) \subseteq Z(\mathcal{L})$.*

Define by

$$Ł^*_r(f) = (-1)^{|r||f|} f \circ R_r, \qquad R^*_r(f) = (-1)^{|r||f|} f \circ L_r \quad \text{and} \quad \eta^*(f) = f \circ \eta$$

for all $r \in \mathcal{L}_{|r|}$. Then we obtain the following result

Proposition 3.7. *Let $(\mathcal{L}, [.,.], \eta)$ be a quadratic Hom Leibniz superalgebra. Then $(\mathcal{L}^*, L^*, R^*, \eta^*)$ is a representation of $\mathcal{L}$.*

Proof. The proof is straightfoward, it suffices to use the relations 1),2) and 3) of proposition 3.5 and the proposition 3.4. □

The representation $(\mathcal{L}^*, L^*, R^*, \eta^*)$ is called the dual representation of the adjoint representation.

Theorem 3.8. *A Hom Leibniz superalgebra* $(\mathcal{L}, [.,.], \eta)$ *is quadratic if and only if the adjoint representation and its dual representation are equivalent.*

Proof. Let us assume that $\mathcal{L}$ is quadratic with bilinear form B. Define $\Phi : \mathcal{L} \to \mathcal{L}^*$ by $\Phi(r)(s) = (-1)^{|s||B|}B(r, s)$ for all $r \in \mathcal{L}$ and $s \in \mathcal{L}_{|s|}$. We shall check that

$$L_r^* \circ \Phi = \Phi \circ L_r, \quad R_r^* \circ \Phi = \Phi \circ R_r \quad \text{and} \quad \Phi \circ \eta = \eta^* \circ \Phi.$$

Let r, s and t be homogeneous elements of $\mathcal{L}$, we have

$$\begin{aligned}
(L_r^* \circ \Phi(s))(t) &= (-1)^{|r|(|s|+|t|)}(\Phi(s) \circ R_r)(t) \\
&= (-1)^{|r|(|s|+|\Phi|)+|r||t|}\Phi(s)([t, r]) \\
&= (-1)^{|r|(|s|+|\Phi|+|t|)+|B|(|r|+|t|)}B(s, [t, r]) \\
&= (-1)^{|r|(|s|+|\Phi|)+|B||t|}B(s, [t, r]) \quad \text{since} \quad |\Phi| = |B| \\
&= (-1)^{|t|(|r|+|B|)+|s||t|}B(t, [r, s]) = (-1)^{|t||B|}B([r, s], t) \\
&= (-1)^{|t||B|}B(L_r(s), t) = (\Phi \circ L_r(s))(t)
\end{aligned}$$

Hence, $L_r^* \circ \Phi = \Phi \circ L_r$ for all $r \in \mathcal{L}_{|r|}$. The relation $R_r^* \circ \Phi = \Phi \circ R_r$ is etablished by a similar calculation.

$$\begin{aligned}
(\Phi \circ \eta r)(s) &= \Phi(\eta r)(s) = (-1)^{|s||B|}B(\eta r, s) \\
&= (-1)^{|s||B|}B(r, \eta s) = \Phi(r)(\eta s) \\
&= (\Phi(r) \circ \eta)(s) = ({}^t\eta(\Phi(r)))(s) \\
&= (\eta^* \circ \Phi(r))(s)
\end{aligned}$$

therefore, we have $\Phi \circ \eta = \eta^* \circ \Phi$. Since B is non degenerate, then Φ is an isomorphism. Thus the adjoint representation and its dual are isomorph.

Conversely let us assume that the adjoint representation and the co-adjoint representation are equivalent and let us show that $\mathcal{L}$ is quadratic. Since the adjoint representation and its dual representation are equivalent, then there exists an isomorphism $\Phi : \mathcal{L} \to \mathcal{L}^*$ such that

$$L_r^* \circ \Phi = \Phi \circ L_r, \quad R_r^* \circ \Phi = \Phi \circ R_r \quad \text{and} \quad \Phi \circ \eta = \eta^* \circ \Phi.$$

Set

$$B : (r, s) \in \mathcal{L} \times \mathcal{L} \to B(r, s) = (-1)^{|\Phi||s|}\Phi(r)(s)$$

Let $r, s, t \in \mathcal{L}$. We have

$$\begin{aligned}
B([r, s], t) &= (-1)^{|\Phi||t|}\Phi([r, s])(t) = (-1)^{|\Phi||t|+|r||s|}(\Phi \circ R_s(r))(t) \\
&= (-1)^{|\Phi||t|+|r||s|}R_s^* \circ \Phi(r)(t) \quad \text{because} \quad \Phi \circ R_s = R_s^* \circ \Phi \\
&= (-1)^{|\Phi||t|+|r||s|+|\Phi(r)||s|}(\Phi(r) \circ L_s)(t) \\
&= (-1)^{|\Phi||t|+|\Phi||s|+|\Phi(r)|(|s|+|t|)}B(r, [s, t]) = B(r, [s, t]).
\end{aligned}$$

Then B is an invariant bilinear form. Since B is not necessary super-symmetric then we will construct another one from B which is super-symmetric. For that, let us consider the supersymmetric part of B denoted by B_{Sym} and the skew-supersymmetric part denoted by B_{Skew} defined by

$$B_{Sym}(r, s) = \frac{1}{2}\left(B(r, s) + (-1)^{|r||s|}B(s, r)\right) \quad \text{and} \quad B_{Skew}(r, s) = \frac{1}{2}\left(B(r, s) - (-1)^{|r||s|}B(s, r)\right)$$

Let us show that B_{Sym} and B_{Skew} are invariant if B is invariant. According to the isomorphism of the representations, we have $(\Phi \circ L_r(s))(t) = (L_r^* \circ \Phi(s))(t) = (-1)^{|r|(|\Phi|+|s|)}\Phi(s) \circ R_r(t)$ which implies that

$$\begin{aligned}
B([r, s], t) &= (-1)^{|\Phi||t|}\Phi \circ L_r(s)(t) = (-1)^{|\Phi||t|+|r|(|\Phi|+|s|)}\Phi(s) \circ R_r(t) \\
&= (-1)^{|\Phi||t|+|r||\Phi|+|\Phi||r|+|\Phi||t|+|r||s|}B(s, R_r(t)) = (-1)^{|r|(|s|+|t|)}B(s, [t, r])
\end{aligned}$$

then for all $r, s, t \in \mathcal{L}$, we have

$$B([r,s],t) = (-1)^{|r|(|s|+|t|)} B(s,[t,r]). \tag{4}$$

Let $r, s, t \in \mathcal{L}$. We have

$$\begin{aligned} B_{Sym}([r,s],t) &= \frac{1}{2}\Big(B([r,s],t) + (-1)^{|t|(|r|+|s|)} B(t,[r,s])\Big) \\ &= \frac{1}{2}\Big(B([r,s],t) + (-1)^{|t|(|r|+|s|)+|s|(|r|+|t|)} B([s,t],r)\Big) \quad \text{according to} \quad (4) \\ &= \frac{1}{2}\Big(B([r,s],t) + (-1)^{|r|(|t|+|s|)} B([s,t],r)\Big) = B_{Sym}(r,[s,t]). \end{aligned}$$

Therefore if B is invariant then B_{Sym} is so. In analogous way, we show that B_{Skew} is also invariant. Now let us put

$$\mathcal{L}_{Sym} = \{x \in \mathcal{L}, B_{Sym}(x,y) = 0, \forall y \in \mathcal{L}\} \quad \text{and} \quad \mathcal{L}_{Skew} = \{x \in \mathcal{L}, B_{Skew}(x,y) = 0, \forall y \in \mathcal{L}\}$$

Since $B = B_{Sym} + B_{Skew}$ and B is non degenerate then we have $\mathcal{L}_{Sym} \cap \mathcal{L}_{Skew} = \{0\}$. Let us show that $[\mathcal{L}_{Sym}, \mathcal{L}_{Sym}] = \{0\}$. Let $r, s, t \in \mathcal{L}$ we have

$$\begin{aligned} B_{Skew}([r,s],t) &= B_{Skew}(r,[s,t]) = -(-1)^{|r|(|s|+|t|)} B_{Skew}([s,t],r) = -(-1)^{|r|(|s|+|t|)} B_{Skew}(s,[t,r]) \\ &= (-1)^{|t|(|s|+|r|)} B_{Skew}([t,r],s) = (-1)^{|t|(|s|+|r|)} B_{Skew}(t,[r,s]) \\ &= -B_{Skew}([r,s],t) \end{aligned}$$

then $B_{Skew}([r,s],t) = 0$. Which implies that $[\mathcal{L},\mathcal{L}] \subseteq \mathcal{L}_{Skew}$, therefore $[\mathcal{L}_{Sym}, \mathcal{L}_{Sym}] \subseteq \mathcal{L}_{Skew}$. And since $\mathcal{L}_{Sym}$ is an ideal of $\mathcal{L}$ then $[\mathcal{L}_{Sym}, \mathcal{L}_{Sym}] \subseteq \mathcal{L}_{Sym}$. Then $[\mathcal{L}_{Sym}, \mathcal{L}_{Sym}] \subseteq \mathcal{L}_{Sym} \cap \mathcal{L}_{Skew} = \{0\}$. Which gives us $[\mathcal{L}_{Sym}, \mathcal{L}_{Sym}] = \{0\}$. Since $\mathcal{L}_{Sym} \cap \mathcal{L}_{Skew} = \{0\}$ and $\mathcal{L}_{Sym}$ is a $\mathbb{Z}_2-$graded sub vector space then $\mathcal{L}_{Sym}$ admits at least a complement H in $\mathcal{L}$ containing $\mathcal{L}_{Skew}$ (ie $\mathcal{L} = H \oplus \mathcal{L}_{Sym}$ and $\mathcal{L}_{Skew} \subseteq H$.) Let γ be a supersymmetric and non degenerate bilinear form on $\mathcal{L}_{Sym}$, since $[\mathcal{L}_{Sym}, \mathcal{L}_{Sym}] = 0$ then γ is invariant. Define $B^1 = \mathcal{L} \times \mathcal{L} \to \mathbb{K}$ as follow

$$B^1 \mid_{H\times H} = B_{Sym} \mid_{H\times H}; \quad B^1 \mid_{\mathcal{L}_{Sym}\times\mathcal{L}_{Sym}} = \gamma; \quad B^1 \mid_{H\times\mathcal{L}_{Sym}} = B^1 \mid_{\mathcal{L}_{Sym}\times H} = 0,$$

then we obtain a quadratic Hom Leibniz superalgebra $(\mathcal{L}, [.,.], \eta, B^1)$. □

4. T^*-extention of Hom Leibniz Superalgebras

In this section we extend the notion of T^*-extention introduced by Bordemann, M. to the case of Hom Leibniz superalgebras.

Definition 4.1. *Let $(\mathcal{L}, [.,.], \eta)$ be a symmetric Hom Leibniz superalgebra. The bilinear map $\psi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}^*$ is called bi-cocycle if we have*

$$\psi(\eta r,[s,t]) - \psi([r,s],\eta t) - (-1)^{|r||s|}\psi(\eta s,[r,t]) + (-1)^{|r|(|s|+|t|)}\psi(s,t)\circ R_{\eta r} - \psi(r,s)\circ L_{\eta t} - (-1)^{|s||t|}\psi(r,t)\circ R_{\eta s} = 0 \tag{5}$$

for all $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$ and $t \in \mathcal{L}_{|t|}$.

Theorem 4.2. *Let $(\mathcal{L}, [.,.], \eta)$ be a symmetric Hom Leibniz superalgebra and $\psi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}^*$a bi-cocycle of $\mathcal{L}$. Then if $Im(Id_{\mathcal{L}} - \eta^2) \subseteq Z(\mathcal{L})$ we have $\tilde{\mathcal{L}} := \mathcal{L} \oplus \mathcal{L}^*$ endowed with the product*

$$[r+f, s+g] = [r,s] + \psi(r,s) + f\circ L_s + (-1)^{|r||s|} g\circ R_r \tag{6}$$

for all $r+f \in \tilde{\mathcal{L}}_{|r|}$ and $s+g \in \tilde{\mathcal{L}}_{|s|}$, is a Hom Leibniz superalgebra with twisted map $\xi := \eta +^t \eta$.

Moreover if we define the non degenerate, supersymmetric bilinear form B over $\tilde{\mathcal{L}}$ by

$$B(r+f, s+g) = f(s) + (-1)^{|r||s|} g(r) \tag{7}$$

then B is invariant if and only if

$$\psi(r,s)(t) = (-1)^{|r|(|s|+|t|)}\psi(s,t)(r) \tag{8}$$

If the relation (8) is satisfied, then $(\tilde{\mathcal{L}} := \mathcal{L} \oplus \mathcal{L}^*, [\ ,\ ], \xi, B)$ is a quadratic Hom Leibniz superalgebra.

Proof. Let us first show that $(\tilde{\mathcal{L}}, [\ ,\ ], \xi)$ is a Hom Leibniz superalgebra. For that let $(r+f, s+g, t+h) \in \tilde{\mathcal{L}}_{|r|} \otimes \tilde{\mathcal{L}}_{|s|} \otimes \tilde{\mathcal{L}}_{|t|}$,

$$[\xi(r+f), [s+g, t+h]] = \Big[\eta r + f\circ\eta, [s,t] + \psi(s,t) + g\circ L_t + (-1)^{|s||t|} h\circ R_s\Big]$$

$$= [\eta r, [s,t]] + \psi(\eta r, [s,t]) + f\circ\eta\circ L_{[s,t]} + (-1)^{|r|(|s|+|t|)}\Big(\psi(s,t) + g\circ L_t + (-1)^{|s||t|} h\circ R_s\Big)\circ R_{\eta r}$$

$$= [\eta r, [s,t]] + \psi(\eta r, [s,t]) + f\circ\eta\circ L_{[s,t]} + (-1)^{|r|(|s|+|t|)}\psi(s,t)\circ R_{\eta r} + (-1)^{|r|(|s|+|t|)} g\circ L_t\circ R_{\eta r}$$

$$+(-1)^{|r|(|s|+|t|)+|s||t|} h\circ R_s\circ R_{\eta r}$$

On the other hand we have

$$[[r+f, s+g], \xi(t+h)] + (-1)^{|r||s|}[\xi(s+g), [r+f, t+h]]$$

$$= \psi([r,s], \eta t) + (-1)^{|r||s|}\psi(\eta s, [r,t]) + \psi(r,s)\circ L_{\eta t} + (-1)^{|s||t|}\psi(r,t)\circ R_{\eta s}$$

$$+ f\circ\Big(L_s\circ L_{\eta t} + (-1)^{|s||t|} L_t\circ R_{\eta s}\Big) + (-1)^{|r||s|} g\circ\Big(\eta\circ L_{[r,t]} + R_r\circ L_{\eta t}\Big)$$

$$+(-1)^{|t|(|r|+|s|)} h\circ\Big(\eta\circ R_{[r,s]} + R_r\circ R_{\eta s}\Big)$$

by using the proposition 3.5, remark 3.6 and the fact that ψ is a bi-cocycle, we conclude that

$$[\xi(r+f), [s+g, t+h]] = [[r+f, s+g], \xi(t+h)] + (-1)^{|r||s|}[\xi(s+g), [r+f, t+h]].$$

Let us show that $B(\xi(r+f), s+g) = B(r+f, \xi(s+g))$

$$\begin{aligned} B(\xi(r+f), s+g) &= B\Big((\eta + {}^t\eta)(r+f), s+g\Big) \\ &= B(\eta r + f\circ\eta, s+g) \\ &= f\circ\eta(s) + (-1)^{|r||s|} g\circ\eta(r) \\ &= B(r+f, \eta(s) + g\circ\eta) \\ &= B(r+f, \xi(s+g)) \end{aligned}$$

With a straightfoward calculation, one can see that B is invariant if and only if the relation 8 is fulfilled. □

Definition 4.3. *The quadratic Hom Leibniz superalgebra $(\tilde{\mathcal{L}}, [.,.], \xi, B)$ is called T^*-extension of $(\mathcal{L}, [.,.], \eta)$ by means of ψ and it is denoted by $T^*_\psi\mathcal{L}$. We said that ψ is cyclic if the relation (8) is fulfilled.*

Since the notion of T^*-extension of a Hom Leibniz superalgebra $(\mathcal{L}, [.,.], \eta)$ is related to some bi-cocycle $\psi : \mathcal{L}\times\mathcal{L} \to \mathcal{L}^*$, then it is natural to study when two diferent bi-cocycles induce the same T^*-extension up to isomorphism.

Definition 4.4 (Martin, B.(1997))**.** *Let $(\mathcal{L}, [.,.], \eta)$ be a symmetric Hom Leibniz superalgebra and $\psi_1, \psi_2 : \mathcal{L}\times\mathcal{L} \to \mathcal{L}^*$ two bi-cocycles of $\mathcal{L}$. Let $T^*_{\psi_1}\mathcal{L}$ (resp. $T^*_{\psi_2}\mathcal{L}$) the T^*-extension of $\mathcal{L}$ by means of ψ_1(resp. ψ_2). We say that $T^*_{\psi_1}\mathcal{L}$ and $T^*_{\psi_2}\mathcal{L}$ are equivalent or isomorph if there exists an isomorphism of Hom Leibniz superalgebras $\Phi : T^*_{\psi_1}\mathcal{L} \to T^*_{\psi_2}\mathcal{L}$ such that $\Phi|_{\mathcal{L}^*} = Id_{\mathcal{L}^*}$ and Φ induce on the quotient space $T^*_{\psi_1}\mathcal{L}|_{\mathcal{L}^*} = T^*_{\psi_2}\mathcal{L}|_{\mathcal{L}^*} = \mathcal{L}$ the identity map. $T^*_{\psi_1}\mathcal{L}$ and $T^*_{\psi_2}\mathcal{L}$ are isometrically equivalent if they are equivalent and moreover Φ is an isometry.*

The following result gives a characterization of equivalence of two T^*-extensions of Hom Leibniz superalgebra.

Proposition 4.5. *Let $(\mathcal{L}, [.,.], \eta)$ be a symmetric Hom Leibniz superalgebra and $\psi_1, \psi_2 : \mathcal{L}\times\mathcal{L} \to \mathcal{L}^*$ two bi-cocycles of $\mathcal{L}$. Then $T^*_{\psi_1}\mathcal{L}$ and $T^*_{\psi_2}\mathcal{L}$ are equivalent if and only if there exists a linear map $\delta : \mathcal{L} \to \mathcal{L}^*$ such that :*

1. $\delta(r)\cdot\eta = \delta(\eta r) \quad \forall r \in \mathcal{L}$
2. $\psi_1(r,s) - \psi_2(r,s) = \delta(r)\cdot L_s + (-1)^{|r||s|}\delta(s)\cdot R_r - \delta([r,s]) \quad \forall r \in \mathcal{L}_{|r|}, s \in \mathcal{L}_{|s|}$

$T^*_{\psi_1}\mathcal{L}$ and $T^*_{\psi_2}\mathcal{L}$ are isometrically equivalent if and only if the conditions (1) and (2) above are fulfilled and $\delta_{Sym} = 0$ where δ_{Sym} is the supersymmetric part of δ defined by

$$\delta_{Sym}(r)(s) = \frac{1}{2}\left(\delta(r)(s) + (-1)^{|r||s|}\delta(s)(r)\right)$$

Proof. Let us assume that $T^*_{\psi_1}\mathcal{L}$ and $T^*_{\psi_2}\mathcal{L}$ are equivalent. Then there exists an isomorphism $\Phi : T^*_{\psi_1}\mathcal{L} \to T^*_{\psi_2}\mathcal{L}$ such that for all $r + f \in T^*_{\psi_1}\mathcal{L}$ we have $\Phi(r + f) = \Phi(r) + f = \Phi_1(r) + \Phi_2(r) + f$ where Φ_1(resp. Φ_2) is the component of Φ that sends $\mathcal{L}$ into $\mathcal{L}$(resp. $\mathcal{L}^*$). Since $r = p(r) = p \cdot \Phi(r) = \Phi_1(r)$ then $\Phi(r + f) = r + \Phi_2(r) + f$. Let us define $\delta = \Phi_2$. Since $\xi \cdot \Phi = \Phi \cdot \xi$ then for all $r \in \mathcal{L}$ we have

$$\begin{aligned}
\xi \cdot \Phi(r) = \Phi \cdot \xi(r) \quad &\Leftrightarrow \quad \left(\eta +^t \eta\right)(r + \delta(r)) = \Phi \cdot \left(\eta +^t \eta\right)(r) \\
&\Leftrightarrow \quad \eta r + \delta(r) \cdot \eta = \eta r + \delta(\eta r) \\
&\Leftrightarrow \quad \delta(r) \cdot \eta = \delta(\eta r)
\end{aligned}$$

then we have 1)

For 2) by using the compatibility of Φ with the bracket we have for all $r + f, s + g \in T^*_{\psi_1}\mathcal{L}$

$$\begin{aligned}
& \Phi([r + f, s + g]) = [\Phi(r + f), \Phi(s + g)] \\
\Leftrightarrow\ & \Phi([r, s]) + \psi_1(r, s) + f \cdot L_s + (-1)^{|r||s|} g \cdot R_r = [r, s] + \psi_2(r, s) + (\delta(r) + f) \cdot L_s + (-1)^{|r||s|}(\delta(s) + g) \cdot R_r \\
\Leftrightarrow\ & [r, s] + \delta([r, s]) + \psi_1(r, s) + f \cdot L_s + (-1)^{|r||s|} g \cdot R_r = [r, s] + \psi_2(r, s) \\
& +\delta(r) \cdot L_s + f \cdot L_s + (-1)^{|r||s|}\delta(s) \cdot R_r + (-1)^{|r||s|} g \cdot R_r \\
\Leftrightarrow\ & \psi_1(r, s) - \psi_2(r, s) = \delta(r) \cdot L_s + (-1)^{|r||s|}\delta(s) \cdot R_r - \delta([r, s]).
\end{aligned}$$

The T^*-extensions $T^*_{\psi_1}(\mathcal{L})$ and $T^*_{\psi_2}(\mathcal{L})$ are isometrically equivalent if they are equivalent and moreover ϕ is an isometry. We have

$$\begin{aligned}
B(\Phi(r + f), \Phi(s + g)) = B(r + f, s + g) \quad &\Leftrightarrow \quad B(r + \delta(r) + f, s + \delta(s) + g) = f(s) + (-1)^{|r||s|} g(r) \\
&\Leftrightarrow \quad (\delta(r) + f)(s) + (-1)^{|r||s|}(\delta(s) + g)(r) = f(s) + (-1)^{|r||s|} g(r) \\
&\Leftrightarrow \quad \delta(r)(s) + f(s) + (-1)^{|r||s|}\delta(s)(r) + (-1)^{|r||s|} g(r) = f(s) + (-1)^{|r||s|} g(r) \\
&\Leftrightarrow \quad \delta(r)(s) + (-1)^{|r||s|}\delta(s)(r) = 0.
\end{aligned}$$

Therefore $\delta_{Sym} = 0$. □

Definition 4.6. *Let $(\mathcal{L}, [.,.], \eta, B), (\mathcal{L}', [.,.], \eta', B')$ be two quadratic Hom Leibniz superalgebras and $m : \mathcal{L} \to \mathcal{L}'$. We define a bilinear form $m^*B' : (x, y) \in \mathcal{L} \otimes \mathcal{L} \mapsto m^*B'(x, y) = B'(mx, my)$.*

Proposition 4.7. *Let $(\mathcal{L}, [.,.], \eta, B), (\mathcal{L}', [.,.], \eta', B')$ be two quadratic Hom Leibniz superalgebras and $m : \mathcal{L} \to \mathcal{L}'$. Then*

1. *If I is an ideal of $\mathcal{L}$ then its orthogonal $I^\perp$ is also an ideal and $[I, I^\perp] = 0$*
2. *m^*B' is an invariant bilinear form if and only if B' is invariant*
3. *Any isotropic $\mathbb{Z}_2-$graded subspace of $\mathcal{L}$ of dimension $\frac{\dim(\mathcal{L})}{2}$ is an ideal if and only if I is abelian.*

Proof. The proof is straightfoward and similar to the one in the case of non associative algebra see (Martin, B.(1997)). □

The following result gives necessary and sufficient conditions for a quadratic Hom Leibniz superalgebra to be isomorph to a T^*-extension.

Theorem 4.8. *Let $(\mathcal{L}, [.,.], \eta, B)$ be a quadratic Hom Leibniz superalgebra. Then $\mathcal{L}$ is isomorph to a T^*-extension $T^*_\psi A$ if and only if $\mathcal{L}$ contains an isotropic ideal I of dimension $\dim(\mathcal{L})/2$ and in such case $\mathcal{L}/I \cong A$.*

Proof. Let us assume that $\mathcal{L}$ is isomorph to a T^*-extension $T^*_\psi A : (A \oplus A^*, [.,.], \eta_A, B_A)$. It is clear that $\dim(A^*) = \dim(T^*_\psi A)/2$. Since for all $r + f \in (T^*_\psi A)_{|r|}$ and $s + g \in (T^*_\psi A)_{|s|}$, $B_A(r + f, s + g) = f(s) + (-1)^{|r||s|} g(r)$ then $B_A(A^*, A^*) = 0$. Hence A^* is an isotropic ideal of dimension $\dim(T^*_\psi A)/2$. Therefore $\mathcal{L}$ contains an isotropic ideal of dimension $\dim(\mathcal{L})/2$.

Conversely let us assume that $(\mathcal{L}, [.,.], \eta, B)$ contains an isotropic ideal I of dimension $\dim(\mathcal{L})/2$. We define by $\mathcal{H} := \mathcal{L}/I$ and $p : \mathcal{L} \to \mathcal{H}$ the canonical projection. Since I is an ideal, $\eta(I) \subseteq I$ which implies that η induce a twisted map $\tilde{\eta}$ over $\mathcal{H}$ such that $p \cdot \eta = \tilde{\eta} \cdot p$. The space $\mathcal{L}$ is a finite dimensional $\mathbb{Z}_2-$graded vector space over a field of characteristic different from 2 then I admits at least a supplement $\mathcal{H}_0$ in $\mathcal{L}$ such that $\mathcal{H}_0 = \mathcal{H}_0^{\perp}$. Put $p_0 : \mathcal{U} \longrightarrow H_0$ the orthogonal projection of $\mathcal{U}$ in H_0 in the direction of I and p_1 the one of $\mathcal{U}$ in I in the direction of H_0. Consider the application $B^\sharp$ define by

$$B^\sharp : \begin{array}{ccc} I & \longrightarrow & \mathcal{H}^* \\ i & \mapsto & B^\sharp(i) \end{array} \quad \text{with} \quad B^\sharp(i) : \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathbb{K} \\ pr & \mapsto & B(i,r) \end{array}$$

It is clear that $B^\sharp$ is well define and moreover it's an isomorphism : indeed let $i \in I$ such that $B^\sharp(i) = 0$. Then $B(i,r) = 0$ for all $r \in \mathcal{L}$ and the fact that B is non degenerate implies $i = 0$. Hence $B^\sharp$ is injective and since $I = I^\perp$ then $\dim(B^\sharp(I)) = \dim(\mathcal{L}/I) = \dim(\mathcal{H}) = \dim(\mathcal{H}^*)$. Therefore $B^\sharp(I) = \mathcal{H}^*$.

Let $r \in \mathcal{L}$ and $i \in I$,

$$B^\sharp(i) \cdot \tilde{\eta}(pr) = B^\sharp(i)(p\eta r) = B(i, \eta r) = B(\eta i, r) = B^\sharp(\eta i)(pr)$$

then $B^\sharp(i) \cdot \tilde{\eta} = B^\sharp(\eta i)$. Let's give now the expressions $B^\sharp([r,i])$ and $B^\sharp([i,r])$ for all $r \in \mathcal{U}$ and $i \in I$. Let $s \in \mathcal{L}$ we have

$$\begin{aligned}
B^\sharp([r,i])(ps) &= B([r,i],s) = (-1)^{|s|(|r|+|i|)} B(s,[r,i]) \\
&= (-1)^{|s|(|r|+|i|)} B([s,r],i) = (-1)^{|s|(|r|+|i|)+|i|(|r|+|s|)} B(i,[s,r]) \\
&= (-1)^{|r|(|s|+|i|)} B^\sharp(i)(p([s,r])) = (-1)^{|r|(|s|+|i|)} B^\sharp(i)([ps,pr]) \\
&= (-1)^{|r|(|s|+|i|)+|r||s|} B^\sharp(i) \cdot R_{pr}(ps) = (-1)^{|r||i|} B^\sharp(i) \cdot R_{pr}(ps) \\
&= (-1)^{|r||i|+|B^\sharp(i)||r|} \left(L_{pr}^* \cdot B^\sharp(i)\right)(ps) = (-1)^{|r||i|+(|B|+|i|)|r|} \left(L_{pr}^* \cdot B^\sharp(i)\right)(ps) \\
&= (-1)^{|r||B|} \left(L_{pr}^* \cdot B^\sharp(i)\right)(ps)
\end{aligned}$$

Then for all $r \in \mathcal{L}$ and $i \in I$ we have $B^\sharp([r,i]) = (-1)^{|r||B|} \left(L_{pr}^* \cdot B^\sharp(i)\right)$.

$$\begin{aligned}
B^\sharp([i,r])(ps) &= B([i,r],s) = B(i,[r,s]) \\
&= B^\sharp(i)(p([r,s])) = B^\sharp(i)([pr,ps])) \\
&= \left(B^\sharp(i) \cdot L_{pr}\right)(ps) = (-1)^{|B^\sharp(i)||r|} \left(R_{ps}^* \cdot B^\sharp(i)\right)(ps) \\
&= (-1)^{|r|(|i|+|B|)} \left(R_{pr}^* \cdot B^\sharp(i)\right)(ps)
\end{aligned}$$

Therefore we have $B^\sharp([i,r]) = (-1)^{|r|(|i|+|B|)} \left(R_{pr}^* \cdot B^\sharp(i)\right)$. Let us consider now the bilinear application $\psi : (ph_0, ph_0') \in \mathcal{H} \times \mathcal{H} \mapsto B^\sharp\left(p_1([h_0, h_0'])\right)$ for $h_0, h_0' \in \mathcal{H}_0$ and the application $m : h_0 + i \in \mathcal{L} \mapsto ph_0 + B^\sharp(i) \in \tilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}^*$. The restriction of p on $\mathcal{H}_0$ is an isomorphism moreover $B^\sharp(i)$ is an isomorphism, then m is also an isomorphism. Assuming for the moment that ψ is a bi-cocycle of $\mathcal{H}$ and cyclic, let's show that m is an isomorphism of Hom Leibniz superalgebra between $\mathcal{L}$ and $T_\psi^*\mathcal{H}$, for that we have to check $m\left([h_0 + i, h_0' + i']\right) = [m(h_0 + i), m(h_0' + i')]$ and $m \cdot \eta = \tilde{\xi} \cdot m$ where

$\tilde{\xi} := \tilde{\eta} +^t \tilde{\eta}$ is the twisted map of $\tilde{\mathcal{H}}$. Let $h_0 + i, h_0^{'} + i^{'} \in \mathcal{L}$ then

$$\begin{aligned}
m\left([h_0 + i, h_0^{'} + i^{'}]\right) &= m\left([h_0, h_0^{'}] + [h_0, i^{'}] + [i, h_0^{'}] + [i, i^{'}]\right) \\
&= m\left(p_0([h_0, h_0^{'}]) + p_1([h_0, h_0^{'}]) + [h_0, i^{'}] + [i, h_0^{'}]\right) \quad \text{because} \quad [I, I] = 0 \quad \text{from} \quad (4.7) \\
&= p\left(p_0([h_0, h_0^{'}])\right) + B^{\sharp}\left(p_1([h_0, h_0^{'}]) + [h_0, i^{'}] + [i, h_0^{'}]\right) \\
&= p\left([h_0, h_0^{'}]\right) + B^{\sharp}\left(p_1([h_0, h_0^{'}])\right) + B^{\sharp}\left([h_0, i^{'}]\right) + B^{\sharp}\left([i, h_0^{'}]\right) \\
&= [ph_0, ph_0^{'}] + \psi\left(ph_0, ph_0^{'}\right) + (-1)^{|h_0||B|}\left(L^*_{ph_0} \cdot B^{\sharp}(i^{'})\right) \\
&\quad +(-1)^{|h_0^{'}|(|i|+|B|)}\left(R^*_{ph_0^{'}} \cdot B^{\sharp}(i)\right) \\
&= [ph_0, ph_0^{'}] + \psi\left(ph_0, ph_0^{'}\right) + B^{\sharp}(i) \cdot L_{ph_0^{'}} + (-1)^{|h_0||i^{'}|} B^{\sharp}(i^{'}) \cdot R_{ph_0} \\
&= \left[ph_0 + B^{\sharp}(i), ph_0^{'} + B^{\sharp}(i^{'})\right] = \left[m(h_0 + i), m(h_0^{'} + i^{'})\right]
\end{aligned}$$

Let us show the second relation

$$\begin{aligned}
m \cdot \eta(h_0 + i) &= m\left(\eta h_0 + \eta i\right) = p\eta h_0 + B^{\sharp}\left(\eta i\right) \\
&= \tilde{\eta} \cdot ph_0 + B^{\sharp}(\eta i) \quad \text{because} \quad \tilde{\eta} \cdot p = p \cdot \eta \\
&= \left(\tilde{\eta} +^t \tilde{\eta}\right)\left(ph_0 + B^{\sharp}(i)\right) \\
&= \tilde{\xi} \cdot m(h_0 + i)
\end{aligned}$$

then $m \cdot \eta = \tilde{\xi} \cdot m$. This means that m is an isomorphism of Hom Leibniz superalgebras between $\mathcal{L}$ and $T^*_{\psi}\mathcal{H}$.

Now let's show that ψ is cyclic. Let $h_0 + i, h_0^{'} + i^{'} \in \mathcal{L}$ we have

$$\begin{aligned}
m^* B_{\mathcal{H}}(h_0 + i, h_0^{'} + i^{'}) &= B_{\mathcal{H}}\left(m(h_0 + i), m(h_0^{'} + i^{'})\right) \\
&= B_{\mathcal{H}}\left(ph_0 + B^{\sharp}(i), ph_0^{'} + B^{\sharp}(i^{'})\right) \\
&= B^{\sharp}(i)(ph_0^{'}) + (-1)^{|h_0||h_0^{'}|} B^{\sharp}(i^{'})(ph_0) \\
&= B(i, h_0^{'}) + (-1)^{|h_0||h_0^{'}|} B(i^{'}, h_0) \\
&= B(i, h_0^{'}) + (-1)^{|h_0||h_0^{'}|+|i^{'}||h_0|} B(h_0, i^{'}) \\
&= B(i, h_0^{'}) + (-1)^{2|h_0||h_0^{'}|} B(h_0, i^{'}) \quad \text{because} \quad |\, i^{'} \,| = |\, h_0^{'} \,| \\
&= B(i, h_0^{'}) + B(h_0, i^{'}) = B(h_0 + i, h_0^{'} + i^{'})
\end{aligned}$$

because I and $\mathcal{H}_0$ are isotropic. Then $m^* B_{\mathcal{H}} = B$ which implies according to the proposition (4.7) that $B_{\mathcal{H}}$ is invariant and thus ψ is cyclic.

Now let us show that ψ is a bi-cocycle of $\mathcal{H}$. Let $h_0 \in (\mathcal{H}_0)_{|h_0|}, h_0^{'} \in (\mathcal{H}_0)_{|h_0^{'}|}, h_0^{''} \in (\mathcal{H}_0)_{|h_0^{''}|}$ and $h_0^1 \in (\mathcal{H}_0)_{|h_0^1|}$.

Let us first compute the following expressions

$$\psi(ph_0^{'}, ph_0^{''}) \cdot R_{\tilde{\eta} ph_0} \quad \text{and} \quad \psi(ph_0, ph_0^{'}) \cdot L_{\tilde{\eta} ph_0^{''}} \tag{9}$$

$$\begin{aligned}
\psi(ph_0^{'}, ph_0^{''}) \cdot R_{\tilde{\eta} ph_0}(ph_0^1) &= (-1)^{|h_0||h_0^1|} \psi(ph_0^{'}, ph_0^{''})\left([ph_0^1, \tilde{\eta} ph_0]\right) \\
&= (-1)^{|h_0||h_0^1|} \psi(ph_0^{'}, ph_0^{''})\left([ph_0^1, p\eta h_0]\right) \quad \text{because} \quad \tilde{\eta} \cdot p = p \cdot \eta \\
&= (-1)^{|h_0||h_0^1|} B^{\sharp}\left(p_1[h_0^{'}, h_0^{''}]\right)\left(p[h_0^1, \eta h_0]\right) \\
&= (-1)^{|h_0||h_0^1|} B\left(p_1[h_0^{'}, h_0^{''}], [h_0^1, \eta h_0]\right) \\
&= (-1)^{|h_0||h_0^1|} B\left([p_1[h_0^{'}, h_0^{''}], h_0^1], \eta h_0\right) \quad \text{invariance of} \quad B \\
&= (-1)^{|h_0||h_0^1|+|h_0||h_0^{'}|+|h_0||h_0^{''}|+|h_0||h_0^1|} B\left(\eta h_0, [p_1[h_0^{'}, h_0^{''}], h_0^1]\right) \quad \text{supersymmetric of} \quad B \\
&= (-1)^{|h_0|(|h_0^{'}|+|h_0^{''}|)} B\left([\eta h_0, [p_1[h_0^{'}, h_0^{''}]], h_0^1\right) \quad \text{invariance of} \quad B
\end{aligned}$$

With a similar calculation, we show that

$$\psi(ph_0, ph_0^{'}) \cdot L_{\tilde{\eta}ph_0^{''}}(ph_0^1) = B\left([p_1[h_0, h_0^{'}], \eta h_0^{''}], h_0^1\right)$$

A straightfoward computation and the help of relation (1) show that

$$\psi\left(\tilde{\eta}ph_0, [ph_0^{'}, ph_0^{''}]\right) - \psi\left([ph_0, ph_0^{'}], \tilde{\eta}ph_0^{''}\right) - (-1)^{|h_0||h_0^{'}|}\psi\left(\tilde{\eta}ph_0^{'}, [ph_0, ph_0^{''}]\right) = 0 \quad (10)$$

On the other side we have

$$(-1)^{|h_0|(|h_0^{'}|+|h_0^{''}|)}\psi(ph_0^{'}, ph_0^{''}) \cdot R_{\tilde{\eta}ph_0}(ph_0^1) - \psi(ph_0, ph_0^{'}) \cdot L_{\tilde{\eta}ph_0^{''}}(ph_0^1) - (-1)^{|h_0^{'}||h_0^{''}|}\psi(ph_0, ph_0^{''}) \cdot R_{\tilde{\eta}ph_0^{'}}(ph_0^1)$$

$$= B\left([\eta h_0, p_1[h_0^{'}, h_0^{''}]], h_0^1\right) - B\left([p_1[h_0, h_0^{'}], \eta h_0^{''}], h_0^1\right) - (-1)^{|h_0||h_0^{'}|}B\left([\eta h_0^{'}, p_1[h_0, h_0^{''}]], h_0^1\right) \quad \text{according to (9)}$$

$$= B\left([\eta h_0, [h_0^{'}, h_0^{''}]], h_0^1\right) - B\left([\eta h_0, p_0[h_0^{'}, h_0^{''}]], h_0^1\right) - B\left([[h_0, h_0^{'}], \eta h_0^{''}], h_0^1\right) + B\left([p_0[h_0, h_0^{'}], \eta h_0^{''}], h_0^1\right)$$

$$-(-1)^{|h_0||h_0^{'}|}B\left([\eta h_0^{'}, [h_0, h_0^{''}]], h_0^1\right) + (-1)^{|h_0||h_0^{'}|}B\left([\eta h_0^{'}, p_0[h_0, h_0^{''}]], h_0^1\right) \quad \text{since} \quad p_1[h_0, h_0^{'}] = [h_0, h_0^{'}] - p_0[h_0, h_0^{'}]$$

$$= B\left([\eta h_0, [h_0^{'}, h_0^{''}]] - [[h_0, h_0^{'}], \eta h_0^{''}] - (-1)^{|h_0||h_0^{'}|}[\eta h_0^{'}, [h_0, h_0^{''}]], h_0^1\right) - B\left([\eta h_0, p_0[h_0^{'}, h_0^{''}]], h_0^1\right) + B\left([p_0[h_0, h_0^{'}], \eta h_0^{''}], h_0^1\right)$$

$$+(-1)^{|h_0||h_0^{'}|}B\left([\eta h_0^{'}, p_0[h_0, h_0^{''}]], h_0^1\right) = 0 \quad \text{because} \quad \mathcal{H}_0 = \mathcal{H}_0^{\perp}.$$

Therefore

$$\begin{aligned}&\psi\left(\tilde{\eta}ph_0, [ph_0^{'}, ph_0^{''}]\right) - \psi\left([ph_0, ph_0^{'}], \tilde{\eta}ph_0^{''}\right) - (-1)^{|h_0||h_0^{'}|}\psi\left(\tilde{\eta}ph_0^{'}, [ph_0, ph_0^{''}]\right)\\ &+(-1)^{|h_0|(|h_0^{'}|+|h_0^{''}|)}\psi(ph_0^{'}, ph_0^{''}) \cdot R_{\tilde{\eta}ph_0} - \psi(ph_0, ph_0^{'}) \cdot L_{\tilde{\eta}ph_0^{''}} - (-1)^{|h_0^{'}||h_0^{''}|}\psi(ph_0, ph_0^{''}) \cdot R_{\tilde{\eta}ph_0^{'}} = 0\end{aligned}$$

Which means that ψ is a bi-cocycle of $\mathcal{H}$. Finalely we conclude that $\mathcal{L}$ is isomorph to the T^*-extension $T_{\psi}^*\mathcal{H}$. □

5. Inductive Description of Regular Hom Leibniz Superalgebras

This section is devoted to introduce double extension of Hom Leibniz superalgebras and give an inductive description of odd quadratic regular Hom Leibniz superalgebras.

Lemma 5.1. *Let $(\mathcal{L}, [.,.], \eta)$ be a Hom Leibniz superalgebra, $\mathcal{H}$ vector space and $\psi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{H}^*$ a bilinear map such that*

$$\psi(\eta r, [s, t]) = \psi([r, s], \eta t) + (-1)^{|r||s|}\psi(\eta s, [r, t]) \quad (11)$$

for all $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$ and $t \in \mathcal{L}_{|t|}$. Then $\tilde{\mathcal{L}} := \mathcal{L} \oplus \mathcal{H}^*$ endowed with the bracket

$$[r + f, s + g] = [r, s] + \psi(r, s) \quad \forall r + f \in \tilde{\mathcal{L}}_{|r|}, \forall s + g \in \tilde{\mathcal{L}}_{|s|} \quad (12)$$

is a Hom Leibniz superalgebra where the twisted map is defined by $\tilde{\eta} := \eta + Id_{\mathcal{H}^*}$

Proof. Let $r + f \in \tilde{\mathcal{L}}_{|r|}$, $s + g \in \tilde{\mathcal{L}}_{|s|}$ and $t + h \in \tilde{\mathcal{L}}_{|t|}$, we have

$$[\tilde{\eta}(r + f), [s + g, t + h]] = [\eta r + f, [s, t] + \psi(s, t)] = [\eta r, [s, t]] + \psi(\eta r, [s, t])$$

On the other hand we have

$$\begin{aligned}[[r + f, s + g], \tilde{\eta}(t + h)] + (-1)^{|r||s|}[\tilde{\eta}(s + g), [r + f, t + h]] &= [[r, s] + \psi(r, s), \eta t + h] + (-1)^{|r||s|}[\eta s + g, [r, t] + \psi(r, t)]\\ &= [[r, s], \eta t] + \psi([r, s], \eta t) + (-1)^{|r||s|}[\eta s, [r, t]] + (-1)^{|r||s|}\psi(\eta s, [r, t])\end{aligned}$$

then according to the Hom Leibniz identity (1) and relation (11), we have

$$[\tilde{\eta}(r + f), [s + g, t + h]] = [[r + f, s + g], \tilde{\eta}(t + h)] + (-1)^{|r||s|}[\tilde{\eta}(s + g), [r + f, t + h]]$$

Therefore $(\tilde{\mathcal{L}}, [.,.], \tilde{\eta})$ is a Hom Leibniz superalgebra. □

Lemma 5.2. *Let $(\mathcal{L}, [.,.], \eta)$ be a Hom Leibniz superalgebra, V a $\mathbb{Z}_2-$graded vector space and $\chi \in (End(V))_{\bar{0}}$. If (ϕ, λ) is a representation of $\mathcal{L}$ over V with respect to χ. Then, $\mathcal{L}_1 := \mathcal{L} \oplus V$ endowed with the bracket*

$$[r + u, s + v] = [r, s] + \phi_r(v) - (-1)^{|r||s|}\lambda_s(u) \quad \forall r + u \in (\mathcal{L}_1)_{|r|}, \forall s + v \in (\mathcal{L}_1)_{|s|} \tag{13}$$

is a Hom Leibniz superalgebra with twisted map $\eta_1 := \eta + \chi$.

Proof. Let $r + u \in (\mathcal{L}_1)_{|r|}$, $s + v \in (\mathcal{L}_1)_{|s|}$ and $t + w \in (\mathcal{L}_1)_{|t|}$, we have

$$[\eta_1(r + u), [s + v, t + w]] = [\eta r + \chi(u), [s, t] + \phi_s(w) - (-1)^{|s||t|}\lambda_t(v)]$$

$$= [\eta r, [s, t]] + \phi_{\eta r} \circ \phi_s(w) - (-1)^{|s||t|}\phi_{\eta r} \circ \lambda_t(v) - (-1)^{|r|(|s|+|t|)}\lambda_{[s,t]} \circ \chi(u)$$

On the other hand, we have

$$[[r + u, s + v], \eta_1(t + w)] + (-1)^{|r||s|}[\eta_1(s + v), [r + u, t + w]]$$

$$= [[r, s] + \phi_r(v) - (-1)^{|r||s|}\lambda_s(u), \eta t + \chi(w)] + (-1)^{|r||s|}[\eta s + \chi(v), [r, t] + \phi_r(w) - (-1)^{|r||t|}\lambda_t(u)]$$

$$= [[r, s], \eta t] + \phi_{[r,s]} \circ \chi(w) - (-1)^{|t|(|r|+|s|)}\lambda_{\eta t} \circ \phi_r(v) + (-1)^{|t|(|r|+|s|)+|r||s|}\lambda_{\eta t} \circ \lambda_s(u)$$

$$+(-1)^{|r||s|}[\eta s, [r, t]] + (-1)^{|r||s|}\phi_{\eta s} \circ \phi_r(w) - (-1)^{|r|(|s|+|t|)}\phi_{\eta s} \circ \lambda_t(u) - (-1)^{|s||t|}\lambda_{[r,t]} \circ \chi(v)$$

$$= [[r, s], \eta t] + (-1)^{|r||s|}[\eta s, [r, t]] + \left(\phi_{[r,s]} \circ \chi + (-1)^{|r||s|}\phi_{\eta s} \circ \phi_r\right)(w)$$

$$+(-1)^{|t|(|r|+|s|)+|r||s|}\lambda_{\eta t} \circ \lambda_s(u) - (-1)^{|r|(|s|+|t|)}\phi_{\eta s} \circ \lambda_t(u)$$

$$-(-1)^{|s||t|}\lambda_{[r,t]} \circ \chi(v) - (-1)^{|t|(|r|+|s|)}\lambda_{\eta t} \circ \phi_r(v)$$

$$= [\eta r, [s, t]] + \phi_{\eta r} \circ \phi_s(w) - (-1)^{|r|(|s|+|t|)}\lambda_{[s,t]} \circ \chi(u) - (-1)^{|s||t|}\phi_{\eta r} \circ \lambda_t(v)$$

$$= [\eta_1(r + u), [s + v, t + w]]$$

Hence, $(\mathcal{L}_1, [.,.], \eta_1)$ is a Hom Leibniz superalgebra. □

Theorem 5.3. *Let $(\mathcal{L}, [.,.], \eta, B)$ be a quadratic Hom Leibniz superalgebra, $\mathcal{H} = \mathbb{K}e$ an one dimensional vector space , $\chi \in End(\mathcal{H})_{\bar{0}}$, (ϕ, λ) a representation of $\mathcal{L}$ on $\mathcal{H}$ with respect to χ and let $\Gamma \in End(\mathcal{L})_{\bar{0}}$ such that*

$$\eta \circ \Gamma([r, s]) = [\Gamma \circ \eta(r), s] - (-1)^{|r||s|}[\Gamma \circ \eta(s), r] \quad \forall r \in \mathcal{L}_{|r|}, \forall s \in \mathcal{L}_{|s|} \tag{14}$$

Then, the space $\hat{\mathcal{L}} := \mathcal{H} \oplus \mathcal{L} \oplus \mathcal{H}^*$ endowed with the bracket

$$[h + r + f, h^{'} + s + g] = [r, s] + B(\Gamma r, s)e^* + \phi_r(h^{'}) - (-1)^{|r||s|}\lambda_s(h) \quad \forall\, h + r + f \in \hat{\mathcal{L}}_{|r|} \,\forall\, h^{'} + s + g \in \hat{\mathcal{L}}_{|s|},$$

is a Hom Leibniz superalgebra where the twisted map is defined by $\hat{\xi} := \chi + \eta + Id_{\mathcal{H}^*}$.

Proof. Set $\psi(r, s) = B(\Gamma r, s)e^*$ and let us first show that ψ is a bi-cocycle. Let $r \in \mathcal{L}_{|r|}$, $s \in \mathcal{L}_{|s|}$ and $t \in \mathcal{L}_{|t|}$. Then we have

$$\begin{aligned}
\psi(\eta r, [s, t]) - \psi([r, s], \eta t) - (-1)^{|r||s|}\psi(\eta s, [r, t]) &= B(\Gamma \circ \eta(r), [s, t])\, e^* - B(\Gamma([r, s]), \eta t)\, e^* - (-1)^{|r||s|}B(\Gamma \circ \eta(s), [r, t])\, e^* \\
&= B([\Gamma \circ \eta(r), s], t)\, e^* - B(\eta \circ \Gamma([r, s]), t)\, e^* - (-1)^{|r||s|}B([\Gamma \circ \eta(s), r], t)\, e^* \\
&= B\left([\Gamma \circ \eta(r), s] - \eta \circ \Gamma([r, s]) - (-1)^{|r||s|}[\Gamma \circ \eta(s), r], t\right) e^* \\
&= 0 \quad \text{according to relation} \quad (14)
\end{aligned}$$

Therefore, ψ is a bi-cocycle. Now according to lemma 5.1, $\tilde{\mathcal{L}} := \mathcal{L} \oplus \mathcal{H}^*$ endowed with the bracket

$$[r + f, s + g] = [r, s] + \psi(r, s) = [r, s] + B(\Gamma r, s)e^*$$

is a Hom Leibniz superalgebra with twisted map $\tilde{\eta} := \eta + Id_{\mathcal{H}^*}$.

Define $\tilde{\phi}, \tilde{\lambda} : \tilde{\mathcal{L}} \to End(\mathcal{H})$ by

$$\tilde{\phi}(r + f) = \phi_r \quad \text{and} \quad \tilde{\lambda}(r + f) = \lambda_r \quad \forall r + f \in \tilde{\mathcal{L}}_{|r|}$$

Since (ϕ, λ) is a representation of $\mathcal{L}$ over $\mathcal{H}$ with respect to χ then , one can see that $(\tilde{\phi}, \tilde{\lambda})$ is a representation of $\tilde{\mathcal{L}}$ over $\mathcal{H}$ with respect to χ. Hence, according to lemma 5.2, $\hat{\mathcal{L}} := \tilde{\mathcal{L}} \oplus \mathcal{H}$ endowed with the bracket

$$\begin{aligned}[r + f + h, s + g + h'] &= [r + f, s + g] + \tilde{\phi}_{r+f}(h') - (-1)^{|r||s|}\tilde{\lambda}_{s+g}(h) \\ &= [r, s] + \psi(r, s) + \phi_r(h') - (-1)^{|r||s|}\lambda_s(h)\end{aligned}$$

is a Hom Leibniz superalgebr where $\hat{\eta} := \eta + Id_{\mathcal{H}^*} + \chi$ define the twisted map. Therefore, $(\hat{\mathcal{L}}, [.,.], \hat{\eta})$ is a Hom Leibniz superalgebra. □

The Hom Leibniz superalgebra $\hat{\mathcal{L}}$ obtained in the above theorem is called double extension of $\mathcal{L}$ by means of the so called context of double extension $(\phi, \lambda, \Gamma, \eta, \chi)$.

Define by $Z_{\mathcal{L}}(\mathrm{Im}(\eta)) = \{x \in \mathcal{L}; [x, \mathrm{Im}(\eta)] = [\mathrm{Im}(\eta), x] = 0\}$.

Lemma 5.4. *$Z_{\mathcal{L}}(Im(\eta))$ is an ideal of $\mathcal{L}$.*

Proof. Let us show that $\eta(Z_{\mathcal{L}}(\mathrm{Im}(\eta))) \subseteq Z_{\mathcal{L}}(\mathrm{Im}(\eta))$. Let $r \in Z_{\mathcal{L}}(\mathrm{Im}(\eta))$ and $s, t \in \mathcal{L}$. Then we have

$$\begin{aligned}B([\eta r, \eta s], t) &= B(\eta r, [\eta s, t]) = B\left(r, [\eta^2 s, \eta t]\right) \\ &= B\left([r, \eta^2 s], \eta t\right) = 0\end{aligned}$$

and since B is non degenerate, $[\eta r, \eta s] = 0$. Hence $\eta(Z_{\mathcal{L}}(\mathrm{Im}(\eta))) \subseteq Z_{\mathcal{L}}(\mathrm{Im}(\eta))$.

We also have for all $z \in \mathcal{L}_{|z|}$,

$$\begin{aligned}B([[r, s], \eta t], z) &= (-1)^{|z|(|r|+|s|+|t|)}B(z, [[r, s], \eta t]) \\ &= (-1)^{|z|(|r|+|s|+|t|)}B([z, [r, s]], \eta t) \\ &= (-1)^{|z|(|r|+|s|+|t|)}B([\eta z, [\eta r, \eta s]], t) = 0 \quad \text{because} \quad \eta r \in Z_{\mathcal{L}}(\mathrm{Im}(\eta))\end{aligned}$$

hence, the fact that B is non degenerate implies $[[r, s], \eta t] = 0$, then $[Z_{\mathcal{L}}(\mathrm{Im}(\eta)), \mathcal{L}] \subseteq Z_{\mathcal{L}}(\mathrm{Im}(\eta))$. In a similar way, we show that $[\mathcal{L}, Z_{\mathcal{L}}(\mathrm{Im}(\eta))] \subseteq Z_{\mathcal{L}}(\mathrm{Im}(\eta))$. Therefore $Z_{\mathcal{L}}(\mathrm{Im}(\eta))$ is an ideal of $\mathcal{L}$. □

Theorem 5.5. *Let $(\mathcal{L}, [.,.], \eta, B)$ be an odd quadratic regular Hom Leibniz superalgebra such that $Ker(\mathcal{L}) \neq \{0\}$. Then $\mathcal{L}$ is isomorph to a double extension of a Hom Leibniz superalgebra by an one dimensional abelian Hom Leibniz superalgebra.*

Proof. Since $\mathcal{L}$ is a multiplicative Hom Leibniz superalgebra, then $\eta(Ker(\mathcal{L})) \subseteq Ker(\mathcal{L})$. The fact that $\mathbb{K}$ is an algebraically closed field implies the existence of $\alpha \in \mathbb{K}$ and $0 \neq e \in Ker(\mathcal{L})_{|e|}$ such that $\eta e = \alpha e$. Since B is an odd non degenerate bilinear form then, there exists $0 \neq d \in \mathcal{L}_{|e|+\bar{1}}$ such that $B(e, d) = 1$. Set $\mathcal{H} = \mathbb{K}e$, $\mathcal{V} = \mathbb{K}d$ and $\mathcal{E} = (\mathcal{H} \oplus \mathcal{V})^{\perp}$. By using the fact that B is odd and non degenerate, we obtain that $B_{(\mathcal{H}\oplus\mathcal{V})\times(\mathcal{H}\oplus\mathcal{V})}$ is non degenerate, thus $\mathcal{L} = \mathcal{E} \oplus \mathcal{E}^{\perp} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{V}$ and $\mathcal{H}^{\perp} = \mathcal{H} \oplus \mathcal{E}$.

Since $\eta \in Aut(\mathcal{L})$ then according to remark 2.8, $Ker(\mathcal{L}) \subseteq Z_{\mathcal{L}}(\mathrm{Im}(\eta))$, hence we have $[\mathcal{H}, \mathcal{L}] = [\mathcal{L}, \mathcal{H}] = 0$. Therefore H is an ideal of $\mathcal{L}$ and so by lemma 2.13, $\mathcal{H}^{\perp}$ is an ideal.

Since $\mathcal{H}^{\perp}$ is an ideal then, there exists $\psi : \mathcal{E} \times \mathcal{E} \to \mathbb{K}$ and $[.,.]_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ such that

$$[r, s] = [r, s]_{\mathcal{E}} + \psi(r, s)e \quad \forall r, s \in \mathcal{E}$$

The fact that $\mathcal{H}^{\perp}$ is an ideal implies that $\eta(\mathcal{E}) \subseteq \mathcal{E} \oplus \mathcal{H}$, hence there exists $\eta_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ and $\eta_{\mathcal{H}} : \mathcal{E} \to \mathcal{H}$ such that

$$\eta r = \eta_{\mathcal{E}} r + \eta_{\mathcal{H}} r \quad \forall r \in \mathcal{E}_{|r|}$$

Since $\mathcal{H} \subseteq Z(\mathcal{L})$ then the Hom Leibniz super identity become

$$[\eta_{\mathcal{E}} r, [s, t]] = [[r, s], \eta_{\mathcal{E}} t] + (-1)^{|r||s|}[\eta_{\mathcal{E}} s, [r, t]] \tag{15}$$

for all homogeneous elements r, s and t of $\mathcal{L}$. This is equivalent after calculation to

$$[\eta_{\mathcal{E}} r, [s,t]_{\mathcal{E}}]_{\mathcal{E}} = [[r,s]_{\mathcal{E}}, \eta_{\mathcal{E}} t]_{\mathcal{E}} + (-1)^{|r||s|}[\eta_{\mathcal{E}} s, [r,t]_{\mathcal{E}}]_{\mathcal{E}} \tag{16}$$

and

$$\psi(\eta_{\mathcal{E}} r, [s,t]_{\mathcal{E}}) = \psi([r,s]_{\mathcal{E}}, \eta_{\mathcal{E}} t) + (-1)^{|r||s|}\psi(\eta_{\mathcal{E}} s, [r,t]_{\mathcal{E}}) \tag{17}$$

Therefore $(\mathcal{E}, [.,.]_{\mathcal{E}}, \eta_{\mathcal{E}}, B_{\mathcal{E}})$ is a quadratic Hom Leibniz superalgebra where $B_{\mathcal{E}} := B|_{\mathcal{E}\times\mathcal{E}}$.

Since $\mathcal{L} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{V}$ and $\mathcal{H}^{\perp} = \mathcal{H} \oplus \mathcal{E}$ is an ideal of $\mathcal{L}$ then the bracket of $\mathcal{L}$ can be define as follow

$$[r,d] = \alpha_r^{\varphi} e + \xi(r) \quad \text{with} \quad \alpha_r^{\varphi} \in \mathbb{K}, \xi \in End(\mathcal{E})$$

$$[d,r] = \alpha_r^{\phi} e + \delta(r) \quad \text{with} \quad \alpha_r^{\phi} \in \mathbb{K}, \delta \in End(\mathcal{E})$$

$$[d,d] = \alpha e + x_0 + \lambda d \quad \text{where} \quad \alpha, \lambda \in \mathbb{K}, x_0 \in \mathcal{E}$$

Define two applications $\varphi, \phi : \mathcal{E} \to End(\mathcal{H})$ by

$$\varphi_r(e) = \alpha_r^{\varphi} e \quad \text{and} \quad \phi_r(e) = \alpha_r^{\phi} e \quad \forall r \in \mathcal{E}_{|r|}$$

Let us show that $(\varphi, \phi, \eta|_{\mathcal{H}})$ is a representation of $\mathcal{E}$ over $\mathcal{H}$. Let $r \in \mathcal{E}$,

$$B([r,d],d) = B\left(\alpha_r^{\varphi} e + \xi(r), d\right) = \alpha_r^{\varphi} B(e,d) = \alpha_r^{\varphi}$$

On the other hand, by using the invariance and the supersymmetry of B and the fact that $|e||d| = \bar{0}$, we have

$$\begin{aligned} B([r,d],d) &= B(r,[d,d]) = (-1)^{|r|(|d||d|)} B([d,d],r) \\ &= B(d,[d,r]) = B(d, \alpha_r^{\phi} e + \delta r) = (-1)^{|d||e|}\alpha_r^{\phi} B(e,d) \\ &= \alpha_r^{\phi} B(e,d) = \alpha_r^{\phi} \end{aligned}$$

hence $\alpha_r^{\phi} = \alpha_r^{\varphi} = B([r,d],d)$.

Since the bilinear form B is odd, then if $r \in \mathcal{E}_{\bar{0}}$ we have $B([r,d],d) = 0$, then

$$\alpha_r^{\phi} = \alpha_r^{\varphi} = 0$$

If $r \in \mathcal{E}_{\bar{1}}$ then $\varphi_r(e) \in \mathcal{H}_{|e|+\bar{1}}$ and $\phi_r(e) \in \mathcal{H}_{|e|+\bar{1}}$. On the other side we have $\varphi_r(e) = \alpha_r^{\varphi} e \in \mathcal{H}_{|e|}$ and $\phi_r(e) = \alpha_r^{\phi} e \in \mathcal{H}_{|e|}$.

Then $\varphi_r(e), \phi_r(e) \in \mathcal{H}_{|e|} \cap \mathcal{H}_{|e|+\bar{1}} = \{0\}$. Hence for all $r \in \mathcal{E}$, we have

$$\varphi_r(e) = \phi_r(e) = 0$$

Therefore $(\varphi, \phi, \eta|_{\mathcal{H}})$ is a trivial representation of $\mathcal{E}$ over $\mathcal{H}$.

We also have for all $r, s \in \mathcal{E}$,

$$\begin{aligned} B([r,s],d) &= B(r,[s,d]) = B(r, \alpha_s^{\varphi} e + \xi(s)) \\ &= B(r, \xi(s)) \end{aligned}$$

On the other hand we have

$$B([r,s],d) = B(\psi(r,s)e + [r,s]_{\mathcal{E}}, d) = \psi(r,s)B(e,d) = \psi(r,s)$$

Hence, $\psi(r,s) = B(r,\xi(s)) = B(\bar{\xi}r, s)$ where $\bar{\xi}$ is the adjoint of ξ with respect to B.

Now let us show that

$$\eta \circ \bar{\xi}([r,s]) = [\bar{\xi} \circ \eta(r), s] - (-1)^{|r||s|}[\bar{\xi} \circ \eta(s), r] \tag{18}$$

Let $z \in \mathcal{E}$,

$$B\Big(\eta \circ \bar{\xi}([r,s]) - [\bar{\xi} \circ \eta(r), s] + (-1)^{|r||s|}[\bar{\xi} \circ \eta(s), r], z\Big)$$

$$= B\Big(\eta \circ \bar{\xi}([r,s]), z\Big) - B\Big([\bar{\xi} \circ \eta(r), s], z\Big) + (-1)^{|r||s|} B\Big([\bar{\xi} \circ \eta(s), r], z\Big)$$

$$= B\Big(\bar{\xi}([r,s]), \eta z\Big) - B\Big(\bar{\xi} \circ \eta(r), [s,z]\Big) + (-1)^{|r||s|} B\Big(\bar{\xi} \circ \eta(s), [r,z]\Big)$$

$$= B\left([r,s], \xi \circ \eta(z)\right) - B\left(\eta r, \xi([s,z])\right) + (-1)^{|r||s|} B\left(\eta s, \xi([r,z])\right)$$

$$= B\Big([r,s], [\eta(z), d] - \alpha^{\varphi}_{\eta t} e\Big) - B\Big(\eta r, [[s,z], d] - \alpha^{\varphi}_{[s,z]} e\Big) + (-1)^{|r||s|} B\Big(\eta s, [[r,z], d] - \alpha^{\varphi}_{[r,z]} e\Big)$$

$$= B\left([r,s], [\eta(z), d]\right) - B\left(\eta r, [[s,z], d]\right) + (-1)^{|r||s|} B\left(\eta s, [[r,z], d]\right)$$

$$= B\left([[r,s], \eta(z)], d\right) - B\left([\eta(r), [s,z]], d\right) + (-1)^{|r||s|} B\left([\eta s, [r,z]], d\right)$$

$$= B\Big(-[\eta r, [s,z]] + [[r,s], \eta(z)] + (-1)^{|r||s|}[\eta(s), [r,z]], d\Big) = 0 \quad \text{according to relation (1)}$$

and since B is non degenerate, we have relation (18). Therefore $(\varphi, \phi, \bar{\xi}, \eta, \eta|_{\mathcal{H}})$ is a context of double extension of $\mathcal{E}$ by $\mathcal{H}$. Hence we obtain the Hom Leibniz superalgebra $\hat{\mathcal{E}} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{H}^*$. One can easily see that $\hat{\mathcal{E}}$ is isomorph to $\mathcal{L} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{V}$. This proves the theorem. □

Corollary 5.6. *Let $(\mathcal{L}, [.,.], \eta, B)$ be an odd quadratic Hom Leibniz superalgebra. Then $\mathcal{L}$ is obtained from a finite number of one dimensional Hom Lie superalgebras and a regular Hom Lie superalgebra via the notion of double extension.*

Proof. We shall proceed by induction over the dimension of $\mathcal{L}$. If $\mathrm{Ker}(\mathcal{L}) = \{0\}$ then $\mathcal{L}$ is a Hom Lie superalgebra. Now let us assume that $\mathrm{Ker}(\mathcal{L}) \neq \{0\}$.

If $\dim(\mathcal{L}) = 2$ then $\mathcal{L}$ can be seen as a double extension of $\{0\}$ by an one dimensional Hom Lie superalgebra.

Let us assume that $\dim(\mathcal{L}) > 2$ and the result yields for all Hom Leibniz superalgebras of dimension less than $\dim(\mathcal{L})$. According to theorem 5.5, there exists a Hom Leibniz superalgebra $\mathcal{E}$ and an one dimensional abelian Hom Lie superalebra $\mathcal{H}$ such that $\mathcal{L} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{H}^*$. Since $\dim(\mathcal{E}) < \dim(\mathcal{L})$, the induction hypothesis implies that $\mathcal{E}$ can be obtained from a finite number of one dimensional Hom Lie superalgebra and a Hom Lie superalgebra. Hence the result yields for $\mathcal{L}$. □

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