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On the Geometric Algebra and Homotopy

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Abstract

This paper aims at evaluating geometric algebra's applications and characteristics, as well as assessing the new characteristic known as pinched homotopy. We elaborate a new approach to the pinched tensor and present the impact of such tensor on the homotopy. We also substitute the ordinary tensor product with the pinched one. Motivated by the different properties of the two tensors, we study several cases on which the difference is shown with providing counterexamples. The proof of some theorems ensure this approach.

Keywords: geometric algebra, null homotopy, pinched

1. Introduction

The geometric algebras were invented by William Kingdon Clifford (1845 − 1879). In 1876, the result was announced for the first time in a talk, which was posthumously published in 1882. The first publication pertaining to the invention was already printed in another paper in 1878. The statement 'there is much new to say regarding the subject pertaining to geometric algebra' may come as a surprise to few. This can be explained by the fact that mathematicians worldwide have known for many years how geometric algebra is associated with a given quadratic form. By the end of the sixties, a detailed study regarding algebraic properties had been done. This work aims at classifying all geometric algebras with regards to matrix algebras into one of the three associative division algebras: the real, quaternion and complex algebras. However, geometric algebra is much more than just a category of as Clifford algebra. Geometric algebra offers the grammar through which construction of geometric algebra is done. One can arrive at a true geometric algebra only when augmentation of this grammar is done with a number of secondary definitions as well as concepts. In fact, it is very easy to understand the geometric algebra's properties, as they are planes, Euclidean vectors and higher-dimensional (hyper) surfaces. It is the combination of computational power along with manipulation of these objects which makes geometric algebra even more interesting to study. This computational power does not lean on developing explicit matrix representations, as much attention has not been given to the matrix representations pertaining to the employed algebras. Thus, little common ground exists between this research work versus earlier work done regarding the study and classification of geometric algebras.

The point of this paper is to continue the investigation of the properties of the special case of tensor product. The main tool here is the pinched tensor product of complexes, introduced by Y. Alkhezi & D. Jorgensen (2019) and Y. Alkhezi (2020). Many other researches have studied the ordinary tensor product as example see A.Charlier et al (1992), or see J. Adams (1995). On the other hand, geometric algebra is applied to different fields of research and applications by Y. Benhadid & Y. Alkhezi (2017). Following the definition of the pinched tensor product elaborated in Y.Alkhezi (2020), we apply this new tensor to some properties from the general tensor and see which still hold.

In this paper, we present a general overview about the geometric algebra and homotopy. Our goals is to elucidates the homotopy and presents the key results for a comparison between the ordinary tensor and the pinched. The preservation of null homotopy regarding to some special cases is elaborated.

2. Geometric Algebra

Geometric algebra can be defined as a finite dimensional associative algebra in a commutative ring, which is represented as a quadratic function. $\mathbb{R}$ is regarded as the scalars pertaining to geometric algebra. They also confer to familiar laws pertaining to subtraction, addition and multiplication. Both commutative and associative characteristics are associated with the multiplication and addition of scalars. Addition of vectors in to each other in geometric algebra can be done as well as multiplication by scalars in the usual manner. The addition of vectors results in commutative and associative nature. Distribution occurs due to multiplication by scalars over addition of vectors. Here, any element pertaining to geometric algebra can be referred to a clif, geometric extent in two directions or bivectors visualised as a surface patch and with grade 2.

The 4 elements, 1 scalar, e_1, e_2 vectors and e_{12} bivector from a basis for the Geometric algebra Cl_2 of the vector plane $\mathbb{R}^2$, that is, an arbitrary element

$$u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12} \quad \text{in} \quad Cl_2$$

is a linear combination of a scalar u_0, a vector $u_1 e_1 + u_2 e_2$ and a bivector $u_{12} e_{12}$. (The Clifford algebra Cl_n of $\mathbb{R}^n$ contains 0-vectors (or scalars), 1-vectors (or just vectors), 2-vectors, $\cdots$, n-vectors. The aggregates of k vectors give the linear space Cl_n a multivector structure $Cl_n = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \cdots \oplus \bigwedge^n \mathbb{R}^n)$.
The Geometric algebra Cl_2 is a 4-dimensional real linear space with basis elements $1, e_1, e_2, e_{12}$ which have the multiplication table.

Table 1. Multiplication table in Cl_2

	e_1	e_2	e_{12}
e_1	1	e_{12}	e_2
e_2	$-e_{12}$	1	$-e_1$
e_{12}	$-e_2$	$-e_1$	-1

Extracting the scalar and bivector parts of the Geometric product we have as products of two vectors $a_1 e_1 + a_2 e_2$ and $b = b_1 e_1 + b_2 e_2$

$$a \cdot b = a_1 b_1 + a_2 b_2, \quad \text{the scalar product 'a dot b'}, \tag{1}$$

$$a \wedge b = (a_1 b_2 - a_2 b_1) e_{12}, \quad \text{the exterior product 'a wedge b'} . \tag{2}$$

For more details about geometric algebra, you may refer to (Y. Alkhezi, 2020); (Y. Benhadid & Y. Alkhezi, 2017); (A. Charlier et al, 1992) and (P. Lounesto, 2001).

3. Homotopy

A chain homotopy can be defined as a type of homotopy that falls under a category of chain complexes in terms of the standard interval object associated with chain complexes, observations as well as advancing of the development in the current concept (readers can also see (Adam et al, 1995); (Gelfand et al, 2013); (Hilton et al, 2012) and (Keller & Bernhard, 1998)).

In an abelian category A, a cochain complex C can be defined as a sequence of objects in A.

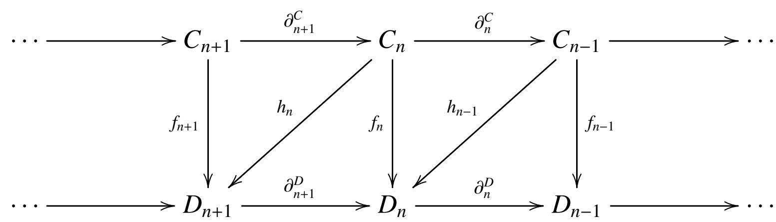
$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots , \qquad \partial^2 = 0.$$

In general, such type of a sequence can be indexed by $\mathbb{Z}$, but for several applications, one employs complexes that are bounded above, bounded below or bounded on both sides.

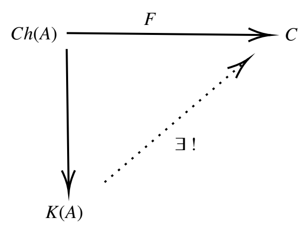
Between the two cochain complexes, a chain map $f : C \to D$ can be defined as a sequence of $f_n : C_n \to D_n$ of morphisms, in a way that the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow{\partial_{n+2}} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots
\end{array}$$

Definition 1 We say that a chain map $f : C \to D$ is *null homotopic* if there are maps $h_n : C_n \to D_{n+1}$ such that $f_n = h_{n-1} \partial_n^C + \partial_{n+1}^D h_n$ for all n. In this case we write $f \sim 0$



A chain map $f : C \to D$ can be referred to a quasi-isomorphism if isomorphisms are induced on all cohomologies (thus, each chain homotopy equivalence can be defined as a quasi-isomorphism). Let us assume A to be an abelian category. All chain equivalences pertaining to the category $\mathcal{K}(A)$ of chain complexes are inverted to get the homotopy category A. More accurately, there is A if chain equivalences are sent by a natural functor $Ch(A) \to \mathcal{K}(A)$ which has the following universal property $F : Ch(A) \to C$



As per the above definition, if 'chain equivalence' is substituted by 'quasi-isomorphism', we arrive at the definition of the derived category $\mathcal{D}(A)$. Alternatively, the homotopy category $\mathcal{K}(A)$ can be defined as the category whose objects are the ones of $Ch(A)$ and whose morphisms are defined by

$$Hom_{\mathcal{K}(A)}(C, D) = Hom_{Ch(A)}(C, D)/N$$

where N stands for the group of null homotopic maps. For more details you may see (J. Rotman, 2009) and (A. Weibel, 1994).

We now recall the main tool of this paper, a variant of the tensor product of complexes, introduced in (Y. Alkhezi & D. Jorgensen, 2019) and called the pinched tensor product. We first recall the definition of the ordinary tensor product of complexes, and most of the following definitions may found (Y. Alkhezi, 2019) and (Y. Alkhezi, 2020). Throughout this section, R is a commutative ring.

Definition 2 *Let C and D be complexes of R-modules. The tensor product $C \otimes_R D$ of C and D is specified by letting*

$$(C \otimes_R D)_n = \coprod_{i \in \mathbb{Z}} C_i \otimes_R D_{n-i}.$$

The differential is defined by,

$$\partial_n^{C \otimes_R D}(c \otimes d) = \partial_i^C(c) \otimes d + (-1)^i c \otimes \partial_{n-i}^D(d).$$

For $c \in C_i$ and $d \in D_{n-i}$. The sign $(-1)^i$ ensures that $\partial_n^{C \otimes_R D} \partial_{n+1}^{C \otimes_R D} = 0$ for all n.

Definition 3 *Let C be a complex of R-modules. Then, the* Shift of C, *ΣC is complex of R-modules defined by $(\Sigma C)_n = C_{n-1}$, and $\partial_n^{\Sigma C} = -\sigma_{n-2} \partial_{n-1}^C \sigma_{n-1}^{-1}$ for all n. Also,* the canonical map $\sigma : C \to \Sigma C$, *is obtained by shifting degrees of elements, specifically, if $c \in C$. Then $|\sigma(e)| = |c| + 1$.*

Definition 4 *Let C and D be complexes of R-modules. We define the pinched tensor product $C \otimes_R^{\bowtie} D$ of C and D by,*

$$(C \otimes_R^{\bowtie} D)_n = \begin{cases} (C_{\geq 0} \otimes_R D_{\geq 0})_n & \text{for} \quad n \geq 0 \\ (C_{\leq -1} \otimes_R (\sigma D)_{\leq 0})_n & \text{for} \quad n \leq -1 \end{cases}$$

with the differential defined by,

$$\partial_n^{C \otimes_R^{\bowtie} D} = \begin{cases} \partial_n^{C_{\geq 0} \otimes_R D_{\geq 0}} & \text{for} \quad n \geq 1 \\ \partial_0^C \otimes_R \sigma_{-1}^D \circ \partial_0^D & \text{for} \quad n = 0 \\ \partial_n^{C_{\leq -1} \otimes_R (\sum D)_{\leq 0}} & \text{for} \quad n \leq -1. \end{cases}$$

It is clear from the definition that $\partial_{n-1}^{C\otimes_R^{\bowtie} D}\partial_n^{C\otimes_R^{\bowtie} D} = 0$ for all n.

The following Proposition is very important and the full proof can be found in (J. Rotman, 2009).

Proposition 1 *Let M be a right R-module. Then,*

$$M \otimes_R R \cong M.$$

Proposition 2 *Let C be a complex. Then,*

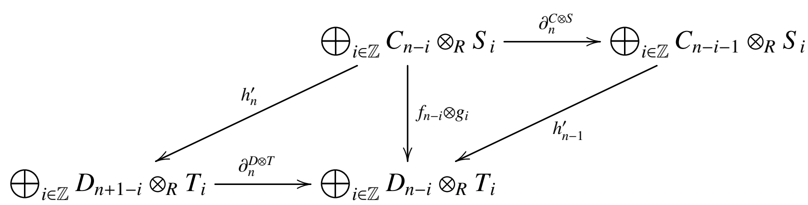
$$C \otimes_R R \cong C.$$

Proof The proof follow easily from Proposition 1.

Theorem 1 (J. Rotman, 2009) *Let $f : C \to D$ and $g : S \to T$ be morphisms of complexes of R-modules with $f \sim 0$. Then,*

$$f \otimes g \sim 0.$$

Proof Consider the diagram



Define $h'_n = (h_{n-i} \otimes g_i)_{i\in Z}$. Choose an element $c \otimes s \in C_{n-i} \otimes_R S_i$. Then,

$$\begin{aligned}
&\partial_{n+1}^{D\otimes_R^{\bowtie} T}(h'_n)(c \otimes s) + (h'_{n-1})\partial_n^{C\otimes_R^{\bowtie} S}(c \otimes s)\\
&= \partial_{n+1}^{D\otimes_R^{\bowtie} T}(h'_n)_{i\in\mathbb{Z}}(c \otimes s) + (h'_{n-1})_{i\in\mathbb{Z}}\Big(\partial_{n-i}^C(c) \otimes s + (-1)^{n-i} c \otimes \partial_i^S(s)\Big)\\
&= \partial_{n+1}^{D\otimes_R^{\bowtie} T}\Big(h_n(c) \otimes g_i(s)\Big) + h_{n-1}(\partial_{n-i}^C(c)) \otimes g_i(s) + (-1)^{n-i} h_n(c) \otimes g_{i-1}(\partial_i^S(s))\\
&= \Big(\partial_{n-i+1}^D\Big(h_n(c)\Big) \otimes g_i(s) + (-1)^{n-i+1} h_n(c) \otimes \partial_i^T(g_i(s))\Big)\\
&\qquad + h_{n-1}(\partial_{n-i}^C(c)) \otimes g_i(s) + (-1)^{n-i} h_n(c) \otimes g_{i-1}\Big(\partial_i^S(s)\Big) \qquad (1)\\
&= \partial_{n-i+1}^D\Big(h_n(c)\Big) \otimes g_i(s) + h_{n-1}\Big(\partial_{n-i}^C(c)\Big) \otimes g_i(s)\\
&= \Big(\partial_{n-i+1}^D(h_n(c)) + h_{n-1}(\partial_{n-i}^C(c))\Big) \otimes g_i(s)\\
&= f_{n-i}(c) \otimes g_i(s)\\
&= (f \otimes g)_n(c \otimes s).
\end{aligned}$$

Where in (1) we use the fact that $\partial_i^T g_i = g_{i-1}\partial_i^S$. Therefore, $f \otimes g \sim 0$. □

Corollary 1 *Let $f : C \to D$ and $\mathrm{Id}_S : S \to S$ with $f \sim 0$. Then,*

$$f \otimes \mathrm{Id}_S \sim 0.$$

Proof Repeat the same work as in Theorem 1 by replacing g with the identity map.

Theorem 2 *Let C be a complex and $S = 0 \to R \to R \to 0$ where R sits in degrees 0 and -1. Then,*

$$C \otimes_R^{\bowtie} S \cong C.$$

Proof We want to show that $(C \otimes_R^{\bowtie} S)_n \cong C$. Consider three cases: $n \geqslant 0$, $n \leqslant -1$ and $n = 0$.

$n \geqslant 0$:

$(C \otimes_R^{\bowtie} S)_{\geqslant 0} = C_{\geqslant 0} \otimes_R S_{\geqslant 0} = C_{\geqslant 0} \otimes_R R \cong C$, by Proposition 2.

$n \leqslant -1$:

$(C \otimes_R^{\bowtie} S)_{\leqslant -1} = C_{\leqslant -1} \otimes_R (\Sigma S)_{\leqslant 0} = C_{\leqslant -1} \otimes_R S_{\leqslant -1} = C_{\leqslant -1} \otimes_R R \cong C$, by Proposition 2.

$n = 0$:

In this case the diagram is

$$\begin{array}{ccc} C_0 \otimes_R S_0 & \longrightarrow & C_{-1} \otimes_R (\Sigma S)_0 \\ \downarrow \cong & & \downarrow \cong \\ C_0 & \longrightarrow & C_{-1} \end{array}$$

Which is equal to

$$\begin{array}{ccc} C_0 \otimes_R R & \longrightarrow & C_{-1} \otimes_R R \\ \downarrow \cong & & \downarrow \cong \\ C_0 & \longrightarrow & C_{-1} \end{array}$$

Therefore, $C \otimes_R^{\bowtie} S \cong C$.

□

Remark 1 In Theorem 1 if we replace the ordinary tensor product with the pinched tensor product, then there is no map going from $(C \otimes_R^{\bowtie} S)_{-1} = C_{-1} \otimes_R (\Sigma S)_0 \to (D \otimes_R^{\bowtie} T)_0 = D_0 \otimes_R T_0$. That means that the pinched tensor product is a functor from the category of complexes to the complexes, but we cannot extend to it a functor on the homotopy categories, as we can see in next example.

Remark 2 If $f : C \to D$ and $g : S \to T$ represent morphisms pertaining to complexes of R-modules along with f homotopic to 0, then $f \otimes g$ is homotopic to 0, and this property is not true for the pinched tensor product as we can see in the next example.

Example 1 *Let C and D be complexes and* $f : C \to D$ *such that* $f \nsim 0$, *and let* $S = 0 \to R \to R \to 0$ *where the degree of R are* 0 *and* -1 *respectively, and* $g : S \to S$, *the identity map, then clearly we see that* $g \sim 0$

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{1} & R & \longrightarrow & 0 \\ & & \downarrow 1 & \swarrow 1 & \downarrow 1 & & \\ 0 & \longrightarrow & R & \xrightarrow{1} & R & \longrightarrow & 0 \end{array}$$

Also, by Theorem 2 , we have $C \otimes_R^{\bowtie} S \cong C$ and $D \otimes_R^{\bowtie} S \cong D$. Then, $f \otimes^{\bowtie} g = f \otimes^{\bowtie} \mathrm{Id}_S : C \otimes_R^{\bowtie} S \to D \otimes_R^{\bowtie} S$. But we know

$$\begin{array}{ccc} C \otimes_R^{\bowtie} S & \xrightarrow{f \otimes^{\bowtie} \mathrm{Id}_S} & D \otimes_R^{\bowtie} S \\ \downarrow \cong & & \downarrow \cong \\ C & \xrightarrow{f} & D \end{array}$$

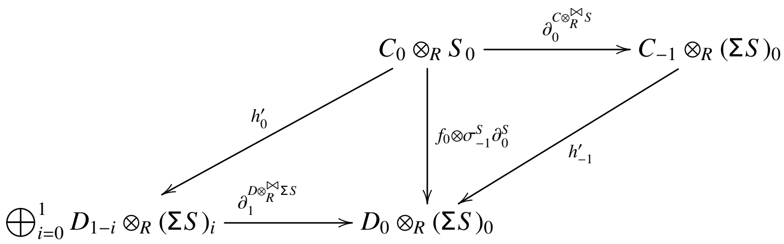
Therefore, $f \otimes^{\bowtie} g \nsim 0$.

Theorem 3 Let $f : C \to D$ with $f \sim 0$ and $\sigma_{n-1}\partial_n^S : S_n \to (\Sigma S)_n$ be morphisms of complexes of R-modules. Then, $f \otimes^{\bowtie} \sigma_{n-1}\partial_n^S : C \otimes_R^{\bowtie} S \to D \otimes_R^{\bowtie} \Sigma S$ satisfies $f \otimes \sigma_{n-1}\partial_n^S \sim 0$.

Proof We consider three cases: $n = 0, n \geqslant 1$ and $n \leqslant -1$.

$n = 0$:

In this case the diagram is

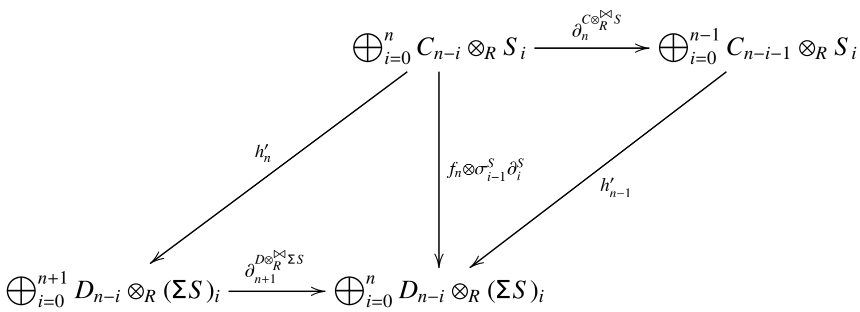


Define $h'_0 = h_0 \otimes \sigma_{-1}^S \partial_0^S$ and $h'_{-1} = h_{-1} \otimes (\Sigma S)_0$. Choose an element $c \otimes s \in C_0 \otimes_R S_0$. Then,

$$\begin{aligned}
&\partial_1^{D\otimes_R^{\bowtie}\Sigma S} h'_0(c \otimes s) + h'_{-1}\partial_0^{C\otimes_R^{\bowtie} S}(c \otimes s) \\
&= \partial_1^{D\otimes_R^{\bowtie}\Sigma S}(h_0 \otimes \sigma_{-1}^S\partial_0^S)(c \otimes s) + h'_{-1}(\partial_0^C \otimes \sigma_{-1}^S\partial_0^S)(c \otimes s) \\
&= \partial_1^{D\otimes_R^{\bowtie}\Sigma S}(h_0(c) \otimes \sigma_{-1}^S\partial_0^S(s)) + h'_{-1}(\partial_0^C(c) \otimes \sigma_{-1}^S\partial_0^S(s)) \\
&= (\partial_1^D \otimes (\Sigma S)_0)(h_0(c) \otimes \sigma_{-1}^S\partial_0^S(s)) + (h_{-1} \otimes (\Sigma S)_0)(\partial_0^C(c) \otimes \sigma_{-1}^S\partial_0^S(s)) \\
&= \partial_1^D h_0(c) \otimes \sigma_{-1}^S\partial_0^S(s) + h_{-1}\partial_0^C(c) \otimes \sigma_{-1}^S\partial_0^S(s) \\
&= [\partial_1^D h_0(c) \otimes + h_{-1}\partial_0^C(c)] \otimes \sigma_{-1}^S\partial_0^S(s) \\
&= f_0(c) \otimes \sigma_{-1}^S\partial_0^S(s).
\end{aligned}$$

$n \geqslant 1$:

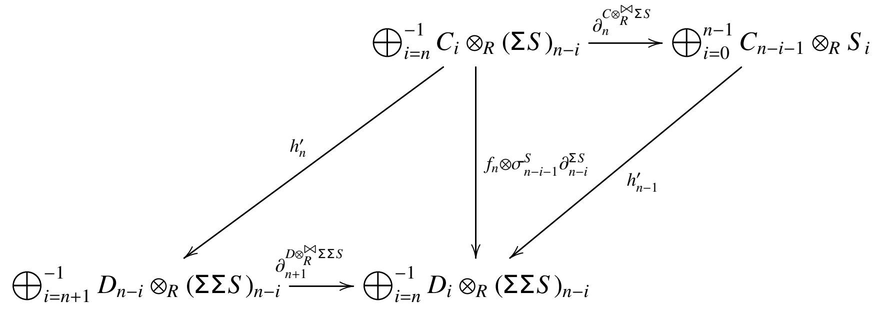
In this case the diagram is



Define $h'_n = h_n \otimes \sigma_{i-1}^S \partial_i^S$ and choose an element $c \otimes s \in C_{n-i} \otimes_R S_i$. Then,

$$\begin{aligned}
&\partial_{n+1}^{D\otimes_R^{\bowtie}\Sigma S} h'_n(c \otimes s) + h'_{n-1}\partial_n^{C\otimes_R^{\bowtie} S}(c \otimes s) \\
&= \partial_{n+1}^{D\otimes_R^{\bowtie}\Sigma S}(h_n \otimes \sigma_{i-1}^S\partial_i^S)(c \otimes s) + h'_{n-1}(\partial_n^C \otimes S_i)(c \otimes s) \\
&= \partial_{n+1}^{D\otimes_R^{\bowtie}\Sigma S}(h_n(c) \otimes \sigma_{i-1}^S\partial_i^S(s)) + h'_{n-1}(\partial_n^C(c) \otimes s) \\
&= (\partial_{n+1}^D \otimes (\Sigma S)_i)(h_n(c) \otimes \sigma_{i-1}^S\partial_i^S(s)) + (h_{n-1} \otimes \sigma_{i-1}^S\partial_i^S)(\partial_n^C(c) \otimes s) \\
&= \partial_{n+1}^D h_n(c) \otimes \sigma_{i-1}^S\partial_i^S(s) + h_{n-1}\partial_n^C(c) \otimes \sigma_{i-1}^S\partial_i^S(s) \\
&= [\partial_{n+1}^D h_n(c) \otimes + h_{n-1}\partial_n^C(c)] \otimes \sigma_{i-1}^S\partial_i^S(s) \\
&= f_n(c) \otimes \sigma_{i-1}^S\partial_i^S(s).
\end{aligned}$$

$n \leqslant -1$:
In this case the diagram is



Define $h'_n = h_n \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}$ and choose an element $c \otimes s \in C_i \otimes_R (\Sigma S)_{n-i}$. Then,

$$\begin{aligned}
&\partial_{n+1}^{D\otimes_R^{\bowtie}\Sigma\Sigma S} h'_n(c\otimes s) + h'_{n-1}\partial_n^{C\otimes_R^{\bowtie}\Sigma S}(c\otimes s)\\
&= \partial_{n+1}^{D\otimes_R^{\bowtie}\Sigma\Sigma S}(h_n \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i})(c\otimes s) + h'_{n-1}(\partial_n^C \otimes \Sigma S_i)(c\otimes s)\\
&= \partial_{n+1}^{D\otimes_R^{\bowtie}\Sigma\Sigma S}(h_n(c) \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}(s)) + h'_{n-1}(\partial_n^C(c) \otimes s)\\
&= (\partial^D_{n+1} \otimes (\Sigma\Sigma S)_i)(h_n(c) \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}(s)) + (h_{n-1} \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}(s))(\partial_n^C(c) \otimes s)\\
&= \partial^D_{n+1}h_n(c) \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}(s) + h_{n-1}\partial^C_n(c) \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}(s)\\
&= [\partial^D_{n+1}h_n(c) \otimes + h_{n-1}\partial^C_n(c)] \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}(s)\\
&= f_n(c) \otimes \sigma^{\Sigma S}_{n-i-1}\partial^{\Sigma S}_{n-i}(s).
\end{aligned}$$

□

4. Conclusion

In this paper, we have proved some theorems related to the geometric algebra and homotopy. The main result is that the pinched tensor does not preserve null homotopy and a positive result for some special cases. This suggests that the pinched tensor product acts as a functor that is categorised as complex, but extending it as a functor for the homotopy categories was not possible. In the future, we can investigate and evaluate more properties such as the mapping cone, which will be considered in my future work. Thus, more applications can be studied pertaining to these properties that would allow the resolving of numerous issues.

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