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Another Proof of Existence of Global Weak Solutions to 1D Pollutant Transport Model

Brahima ROAMBA

Correspondence: Laboratoire de Mathmatiques, d'Informatique et Applications (LaMIA), Universit Nazi BONI, 01 BP 1091 Bobo-Dioulasso, Burkina Faso; Institut Universitaire de Technologie (IUT), Universit Nazi BONI, 01 BP 1091 Bobo-Dioulasso, Burkina Faso.

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Abstract

This paper is devoted to the study of pollutant transport model by water in dimension one. The model studied extend the results obtained in (Roamba, Zabsonré & Zongo, 2017) . However, our model does not take into account cold pressure term and the quadratic friction term as in (Roamba, Zabsonré & Zongo, 2017) which are considered regularizing terms to show the existence of global weak solutions of your model. Without these regularizing terms, we show the existence of global weak solutions in time with a periodic domain.

Keywords: shallow water equations, pollutant, viscosity, friction

1. Introduction

In this paper, we are interested to the study of a pollutant transport model in one dimension.

As a reminder, several studies are being done on pollutant transport models. The authors in (Fernandez-Nieto, Narbona-Reina & Zabsonré, 2013) were the pawns in the formal derivation of a bilayer model coupling shallow water and Reynolds lubrication equations. From this derivation, the authors prove that our model verify a dissapative entropy inequality up to a second'order term. They compare the numerical results with the viscous bilayer shallow water model proposed in (Narbona-Reina, Zabsonré, Fernandez-Nieto & Bresch).

The authors in (Roamba, Zabsonré & Zongo), have proven the existence of global weak solution of a similar model derived in (Fernandez-Nieto, Narbona-Reina & Zabsonré, 2013). To achieve this, the authors have made a technical hypothesis on the height of water, namely the water layer is more important than the layer of the pollutant in the form

$$h_2 \le h_1, \tag{1}$$

where h_1, h_2 represent respectively the pollutant and the water height.

And to overcome this condition, they have resorted to the addition of regularizing terms such as Van Der waals force in (Kitavsev, Laurençot & Niethammer (2011)) and laminar friction term see (Marche (2005); Mellet & Vasseur (2007); Roamba, Zabsonré & Traoré (2016)) to show the existence of global weak solutions of the models considered. The model studied in (Roamba, Zabsonré & Zango (2017)) are read:

$$\partial_t h_1 + \partial_x(h_1 u) = 0, \tag{2}$$

$$\partial_t(h_1 u) + \partial_x(h_1 u^2) + \frac{1}{2} g \partial_x h_1^2 - 4\nu \partial_x(h_1 \partial_x u) + \frac{u_1}{\beta} - h_1 \partial_x(\sigma \partial_x^2 h_1 - V(h_1))$$

$$rgh_1 \partial_x h_2 + rgh_2 \partial_x(h_1 + h_2) + r_1 h_1 |u|^2 u = 0. \tag{3}$$

$$\partial_t h_2 + \partial_x(h_2 u_1) - \varepsilon \partial_x^2 h_2 - \partial_x\Big((a{h_2}^2 + c{h_2}^3)\partial_x p_2\Big) = 0. \tag{4}$$

with

$$\partial_x p_2 = \rho_2 g \partial_x(h_1 + h_2) \quad \text{and} \quad V(h_1) = \frac{1}{h_1^3} - \frac{\alpha}{h_1^4} \quad (\alpha > 0), \tag{5}$$

where $(t, x) \in (0, T)\times]0, 1[$,

and h_1, h_2 are respectively, the water and the pollutant heights, u is the water velocity. The ratio of densities is denoted $r = \dfrac{\rho_2}{\rho_1}$ where ρ_1 and ρ_2 are respectively the densities of the water and the pollutant. ν is the kinematic viscosity; g is the constant gravity.

The coefficients σ, r_1 and β are respectively the coefficients of the interfaz tension, quadratic friction and positive slip length parameters. a and c are constants. α and ε are positive constants.

The model studied in this paper does not take into account cold pressure term (Van Der Waals force) and the quadratic friction term as in (Roamba, Zabsonré & Zango (2017)). It is written as follows:

$$\partial_t h_1 + \partial_x(h_1 u) = 0, \tag{6}$$

$$\partial_t(h_1 u) + \partial_x(h_1 u^2) - 4\nu\partial_x(h_1\partial_x u) + rg\partial_x[h_1(h_2 + \frac{1}{2r}h_1)] + \frac{1}{2}rg\partial_x h_2^2 - \sigma h_1\partial_x^3 h_1 + au = 0, \tag{7}$$

$$\partial_t h_2 + \partial_x(h_2 u) - \varepsilon\partial_x^2 h_2 = 0. \tag{8}$$

where $(t, x) \in (0, T)\times]0, 1[$.

There are many results on the existence of solutions of the one-dimensional Navier Stokes equations. In (Bresch & Desjardins (2003)), the authors proved the existence of global weak solutions for 2D viscous Shallow Water equations and convergence to quasi-geotrophic model. In the paper, the authors shown the control of the vaccum thanks to an entropy named BD-entropy, which was introduced firstly in (Bresch, Desjardins & Lin (2003)). We note that the authors in (Bresch, Desjardins & Gérard-Varet (2007); Toumbou, Roux & Sene (2007)) have used this BD-entropy to get existence result of global weak solutions for Shallow-Water and viscous compressible Navier-Stokes equations. We have used this entropy in our work.

The authors in (Haspot (2018)) have proved a result of global strond solutions to the Navier-Stokes system with degenerate viscosity coefficient. This work has been developed in (Kang & Vasseur (2020)).

We integrate their ideas to limit the water height.

We draw on the work done in (Constantin, Drivas, Nguyen & Pasqualotto (2020); Haspot (2018); Kang & Vasseur (2020)) to improve the results obtained in (Roamba, Zabsonré & Zango (2017)), by showing global existence of weak solutions of one-dimensional pollutant transport model without resorting to cold pressure term and regularizing terms.

The rest of paper is organized as follows: In the **Section** 2, we give firstly the definition of global weak solutions, secondly we establish a classical energy inequality and the "mathematical BD entropy", which give some regularities on the unknowns. We also give an existence theorem of global weak solutions in the same section. The **Section** 3 contains the proof of the energies estimates and main existence result theorem.

We complete the system studied with the initial conditions

$$h_1(0, x) = h_{1_0}(x), \quad h_2(0, x) = h_{2_0}(x), \quad (h_1 u)(0, x) = \mathbf{m}_0(x) \quad \text{in }]0, 1[. \tag{9}$$

$$\begin{gathered} h_{1_0} \in L^2(0, 1), \quad |h_{1_0} + h_{2_0}| \in L^2(0, 1), \quad \partial_x(h_{1_0}) \in L^2(0, 1), \\ \partial_x\mathbf{m}_0 \in L^1(0, 1), \quad \mathbf{m}_0 = 0 \quad \text{if } h_{1_0} = 0, \\ \frac{|\mathbf{m}_0|^2}{h_{1_0}} \in L^1(0, 1), \quad \log h_{1_0} \in L^1(0, 1). \end{gathered} \tag{10}$$

2. Mains Results

Definition 1. *We say that (h_1, h_2, u) is a weak solution of (6)-(8), with the initial conditions (9)-(10), verifying the entropy inequalities (14) and (16) if for all smooth test function $\phi = \phi(t, x)$ with $\phi(T, .) = 0$, we have:*

$$h_{1_0}\phi(0, .) - \int_0^T\int_0^1 h_1\partial_t\phi - \int_0^T\int_0^1 h_1 u\partial_x\phi = 0, \tag{11}$$

$$-h_{2_0}\phi(0, .) - \int_0^T\int_0^1 h_2\partial_t\phi - \int_0^T\int_0^1 h_2 u\partial_x\phi + \varepsilon\int_0^T\int_0^1 \partial_x h_2\partial_x\phi = 0, \tag{12}$$

$$h_{1_0}u_0\phi(0, .) - \int_0^T\int_0^1 h_1 u\partial_t\phi - \int_0^T\int_0^1 h_1 u^2\partial_x\phi + 4\nu\int_0^T\int_0^1 h_1\partial_x u\partial_x\phi$$

$$+\sigma\int_0^T\int_0^1(\partial_x^2h_1\partial_xh_1)\phi+\sigma\int_0^T\int_0^1 h_1\partial_x^2h_1\partial_x\phi+a\int_0^T\int_0^1 u\phi$$
$$-rg\int_0^T\int_0^1 h_1h_2\partial_x\phi-\frac{1}{2}g\int_0^T\int_0^1 h_1^2\partial_x\phi-\frac{1}{2}rg\int_0^T\int_0^1 h_2^2\partial_x\phi \tag{13}$$

Lemma 1. *(Energy inequality) For classical solutions of the system* (6) − (8)*, the following inequality holds:*

$$\frac{d}{dt}\int_0^1\left[\frac{1}{2}h_1|u|^2+\frac{1}{2}g(1-r)|h_1|^2+\frac{1}{2}rg|h_1+h_2|^2+\frac{1}{2}\sigma|\partial_xh_1|^2\right]+4\nu\int_0^1 h_1|\partial_xu|^2$$
$$+a\int_0^1|u|^2+\frac{1}{2}gr\varepsilon\int_0^1|\partial_xh_2|^2\leq\frac{1}{2}rg\varepsilon\int_0^1|\partial_xh_1|^2. \tag{14}$$

Remark 1. *Notice that the term in the right of* (14) *can be controlled using Gronwall's lemma.*

Corollary 1. *Let* (h_1,h_2,u) *be a solution of model* (6) − (8)*. Then, thanks to Lemma* 1 *we have:*

$$\sqrt{h_1}u \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(0,1)), \quad \sqrt{h_1}\partial_xu \quad \textit{is bounded in} \quad L^2(0,T;L^2(0,1)),$$
$$u \quad \textit{is bounded in} \quad L^2(0,T;L^2(0,1)), \quad (h_1+h_2) \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(0,1)),$$
$$\partial_xh_1 \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(0,1), \quad \partial_xh_2 \quad \textit{is bounded in} \quad L^2(0,T;L^2(0,1)),$$
$$h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(0,1)), \quad h_2 \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(0,1)).$$

Corollary 2. *(see(Haspot (2018)))*

There exists a constant $C>0$ *such as*

$$h_1\geq C. \tag{15}$$

We will need in the following some additional regularity on h_1 and this will be achieved through an additional BD entropy inequality presented in the next **lemma**

Lemma 2. *(BD-entropy) For smooth solutions* (h_1,h_2,u) *of model* (6) − (8) *satisfying the classical energy equality of the* ***Lemma*** *1, we have the following mathematical BD entropy inequality:*

$$\frac{d}{dt}\int_0^1\left[\frac{1}{2}h|u+4\nu\partial_x\log(h_1)|^2-4\nu a\log(h_1)+\frac{1}{2}g(1-r)|h_1|^2+\frac{1}{2}rg|h_1+h_2|^2+\frac{1}{2}\sigma|\partial_xh_1|^2\right]$$
$$+a\int_0^1|u|^2+4\nu\int_0^1(g+gr\frac{h_2}{h_1})|\partial_xh_1|^2+rg\int_0^1(\varepsilon+4\nu\frac{h_2}{h_1})\partial_xh\partial_xh_2$$
$$+4\sigma\nu\int_0^1|\partial_x^2h_1|^2+gr\varepsilon\int_0^1|\partial_xh_2|^2\leq\frac{1}{2}rg\varepsilon\int_0^1|\partial_xh_1|^2. \tag{16}$$

While waiting to give the proof of the **Lemma** 2, we show how to control the term $\int_0^T\int_0^1(\varepsilon+4\nu\frac{h_2}{h_1})\partial_xh\partial_xh_2$, then all the others are good sign.

Indeed,

We have, $\frac{h_2}{h_1}\partial_xh_1\partial_xh_2=(\partial_x\log h_1)h_2\partial_xh_2$, we can write:

$$\int_0^T\int_0^1|\frac{h_2}{h_1}\partial_xh_1\partial_xh_2|\leq\frac{1}{2}\int_0^T\int_0^1|\partial_x\log(h_1)|^2+\frac{1}{2}\int_0^T\int_0^1|h_2\partial_xh_2|^2.$$

We will now look at the two terms to the right of the above inequality separately. For the first one, we have:

$$\begin{aligned}\int_0^T\int_0^1|\partial_x\log(h_1)|^2&=\int_0^T\int_0^1\frac{|\partial_xh_1|^2}{h_1^2}\\&\leq\frac{1}{C^2}\int_0^T\int_0^1|\partial_xh_1|^2 \quad (\text{ see equation (15)}).\end{aligned}$$

So

$$\partial_x(\log h_1) \text{ is in } L^2(0,T;L^2(0,1)).$$

For the second one, since $h_2 \in L^\infty(0,T;H^1(0,1))$ and $\partial_x h_2 \in L^2(0,T;L^2(0,1))$ then, $h_2\partial_x h_2 \in L^2(0,T;L^2(0,1))$ (By Sobolev embedding (see (Marche (2005)) for instance)) which completes the proof.

Corollary 3. *Let* (h_1, h_2, u) *be a solution of model* (6) − (8).

Then, thanks to ***Lemma*** 2*, we have:*

$$\sqrt{h_1},\quad \partial_x\sqrt{h_1}\quad \text{are bounded in}\quad L^\infty(0,T;L^2(0,1))\quad \text{and}\quad \partial_x^2 h_1\quad \text{is bounded in}\quad L^2(0,T;L^2(0,1))$$

Remark 2. *In the* ***Corollary*** *1, the estimate*

$$\sqrt{h_1}u\quad \text{is bounded in}\quad L^\infty(0,T;L^2(0,1)),$$

implies,

$$h_1 u\quad \text{is bounded in}\quad L^\infty(0,T;L^2(0,1)),$$

this lead us

$$\partial_t h_1\quad \text{is bounded in}\quad L^\infty(0,T;W^{-1,2}(0,1)),$$

$$h_1 \text{ and } \sqrt{h_1} \text{ are bounded in } L^\infty(0,T;L^\infty(0,1)).$$

Corollary 4. *The* ***Remark*** *2 and the* ***Corollary*** *2, allows us to state the following result, there exists constants* $0 < C$ *and* R*, such as:*

$$0 < C \le h_1 \le R.$$

Remark 3. *We have the following additional regularities:*

1. h_2 *is bounded in* $L^2(0,T;L^\infty(0,1)) \cap L^\infty(0,T;L^2(0,1))$,
2. u *is bounded in* $L^2(0,T;L^\infty(0,1)) \cap L^\infty(0,T;L^2(0,1))$.

Theorem 1. *There exists a global weak solutions to the system (6)-(8) with initial data (9)-(10), and satisfying energy enequalities (14) and (16).*

3. Proof of the Energies Inequalities and Theorem 1

In this section we give proof of some results.

3.1 Proof of ***Lemma*** *1*

First, we multiply the momentum equation by u and we integrate from 0 to 1.

$$\int_0^1 \frac{1}{2}\partial_t(h_1u^2) + \frac{1}{2}\int_0^1 gu\partial_x h_1^2 - 4\int_0^1 u\partial_x(\nu h_1\partial_x u) + rg\int_0^1 h_1u\partial_x h_2$$
$$+rg\int_0^1 h_2u\partial_x(h_1+h_2) + a\int_0^1 |u|^2 = 0. \tag{17}$$

We use the mass conservation equation (6) for simplification. Then, we obtain:

- $-4\displaystyle\int_0^1 u\partial_x(\nu h_1\partial_x u) = 4\nu\int_0^1 h_1(\partial_x u)^2$
- $\begin{aligned}-\sigma\int_0^1 h_1u\partial_x^3 h_1 &= \sigma\int_0^1 \partial_x(h_1u)\partial_x^2 h_1\\ &= -\sigma\int_0^1 \partial_t h_1\partial_x^2 h_1\\ &= \sigma\int_0^1 \partial_{xt}h_1\partial_x h_1\\ &= \frac{1}{2}\sigma\int_0^1 \partial_t|\partial_x h_1|^2.\end{aligned}$

- $rg\int_0^1 h_1 u\partial_x h_2 = -rg\int_0^1 h_2\partial_x(h_1 u) = rg\int_0^1 h_2\partial_t h_1.$
- $\frac{1}{2}g\int_0^1 u\partial_x h_1^2 = \frac{1}{2}g\int_0^1 \partial_t|h_1|^2.$
- $rg\int_0^1 h_2\partial_x(h_1+h_2)u = -rg\int_0^1 (h_1+h_2)\partial_x(h_2 u).$

The equation for the thin film flow give us : $\partial_x(h_2 u) = -\partial_t h_2 + \varepsilon\partial_x^2 h_2$ and we have:

- $rg\int_0^1 h_2\partial_x(h_1+h_2)u = rg\varepsilon\int_0^1 \partial_x h_1\partial_x h_2 + rg\varepsilon\int_0^1 |\partial_x h_2|^2 + \frac{1}{2}rg\frac{d}{dt}\int_0^1 |h_2|^2 + rg\int_0^1 h_1\partial_t h_2$

Substituting all these terms in (17), we get (14) by integrating under 0 to T.

3.2 Proof of ***Lemma*** *2*

Let us multiply (7) by $4\nu\partial_x\log(h)$, integrate with respect to x and use an integration by parts. Thanks to the equation (6) and the initial conditions (9) – (10), we have:

$$4\nu\int_0^1(\partial_t u + u\partial_x u)\partial_x h_1 + 4\nu g\int_0^1|\partial_x h_1|^2 + 16\nu^2\int_0^1 h_1\partial_x u\partial_x\Big(\frac{\partial_x h_1}{h_1}\Big) + 4a\nu\int_0^1\frac{u\partial_x h_1}{h_1} + 4\sigma\nu\int_0^1|\partial_x^2 h_1|^2 +$$
$$+4\nu_1 rg\int_0^1\partial_x h_2\partial_x h_1 + 4\nu_1 rg\int_0^1\frac{h_2}{h_1}|\partial_x h|^2 + 4\nu rg\int_0^1\frac{h_2}{h_1}\partial_x h_2\partial_x h_1 = 0. \tag{18}$$

On the one hand, a further integration by parts of the first term in (18), equation (6), and the energy inequality (14) give:

$$\begin{aligned}
&4\nu\int_0^1(\partial_t u + u\partial_x u)\partial_x h_1\\
&= 4\nu\Big(\frac{d}{dt}\int_0^1 u\partial_x h_1 - \int_0^1 u\partial_{xt}h_1 + \int_0^1 u\partial_x u\partial_x h_1\Big)\\
&= 4\nu\Big(\frac{d}{dt}\int_0^1 u\partial_x h_1 - \int_0^1 \partial_x u\partial_x(h_1 u) + \int_0^1 u\partial_x u\partial_x h_1\Big)\\
&= 4\nu\Big(\frac{d}{dt}\int_0^1 u\partial_x h_1 - \int_0^1 h_1(\partial_x u)^2\Big)\\
&= \frac{d}{dt}\int_0^1\Big[4\nu u\partial_x h_1 + \frac{1}{2}h_1|u|^2 + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}\sigma|\partial_x h_1|^2\Big]\\
&+ a\int_0^1|u|^2 + rg\varepsilon\int_0^1\partial_x h_1\partial_x h_2 + rg\varepsilon\int_0^1|\partial_x h_2|^2.
\end{aligned}\tag{19}$$

We can write the third and the fourth term in (18) as follow:

- $16\nu^2\int_0^1\partial_x\Big(\frac{\partial_x h_1}{h_1}\Big)\partial_x u h_1 = \frac{1}{2}\frac{d}{dt}\int_0^1 h_1|4\nu\log(h_1)|^2$
- $4a\nu\int_0^1\frac{u\partial_x h_1}{h_1} = 4a\nu\int_0^1\frac{\partial_x(uh_1)}{h_1} - 4a\nu\int_0^1\partial_x u$
  $= -4a\nu\frac{d}{dt}\int_0^1\log(h_1)$ (See ([**?**]) .

Substituting finally the last three identities into (18), we obtain (16).

3.3 Proof of ***Theorem*** *1*

This section is devoted to the prove of **Theorem** 1. Let (h_1^k, h_2^k, u^k) be a sequence of weak solutions with initial data

$$h^k_{1|t=0} = h^k_{1_0},\quad h^k_{2|t=0} = h^k_{2_0},\quad (h_1^k u^k)_{|t=0} = m_0^k$$

such that

$$h^k_{1_0}\longrightarrow h_{1_0}\text{ in }L^1(\Omega),\quad h^k_{2_0}\longrightarrow h_{2_0}\text{ in }L^1(\Omega),\quad m_0^k\longrightarrow m_0\text{ in }(L^1(\Omega))^2,$$

and satisfies

$$4\nu\int_0^1 \partial_x \log(h^k_{1_0}) + \int_0^1 \left[h^k_{1_0}\left|u^k_0\right|^2 + \frac{1}{2}g(1-r)\left|h^k_{1_0}\right|^2 + \frac{1}{2}rg\left|h^k_{1_0} + h^k_{2_0}\right|^2 + \frac{1}{2}\sigma\left|\partial_x h^k_{1_0}\right|^2\right] \leq C.$$

Such approximate solutions can be built by a regularization of capillary effect.

3.3.1 Strong Convergence of $\sqrt{h_1^k}$, h_1^k and h_2^k

We first give the spaces in which $\sqrt{h_1^k}$ is bounded.

By integrating the mass equation, we obtain : $\sqrt{h_1^k}$ in $L^\infty(0,T;L^2(0,1))$.

As **Remark 3** gives us $\partial_x\sqrt{h_1^k}$ in $L^\infty(0,T;L^2(0,1))$,
so

$$\sqrt{h_1^k} \quad \text{is bounded in } L^\infty(0,T;L^\infty(0,1)). \tag{20}$$

Moreover, still using the mass equation, we obtain the following equality:

$$\begin{aligned}\partial_t\sqrt{h_1^k} &= \frac{1}{2}\sqrt{h_1^k}\partial_x u^k - \partial_x(\sqrt{h_1^k}u^k),\\ &= \frac{1}{2}\sqrt{h_1^k}\partial_x u^k - u^k\partial_x\sqrt{h_1^k} - \sqrt{h_1^k}\partial_x u^k\end{aligned}$$

which gives that $\partial_t\sqrt{h_1^k}$ is bounded in $L^2(0,T;L^2(0,1))$.
Applying Aubin-Simon lemma (Lions (1969); Simon (1987)), we can extract a subsequence, still denoted $(h_1^k)_{1\leq k}$, such that

$$\sqrt{h_1^k} \quad \text{strongly converges to} \quad \sqrt{h_1} \text{ in } L^2(0,T;L^2(0,1)).$$

According to the **Corollary** 4, we show that

$$\left|h_1^k - h_1\right| \leq \sqrt{c_2}\left|\sqrt{h_1^k} - \sqrt{h_1}\right| \Rightarrow \left|h_1^k - h_1\right|^2 \leq c_2\left|\sqrt{h_1^k} - \sqrt{h_1}\right|^2.$$

This ensure

$$h_1^k \quad \text{strongly converges to} \quad h_1 \text{ in } L^2(0,T;L^2(0,1)).$$

We have $h_2^k \in L^2(0,T;L^\infty(0,1))$. Moreover, we have $\partial_t {h_2}^k = -\partial_x(h_2^h u^k) + \varepsilon\partial_x^2 h_2$.

We have $h_2^k \in L^\infty(0,1)$ and $u^k \in L^2(0,1)$, so $h_1^k u^k \in L^2(0,1)$, according to the Sobolev embeddings, we show that the first term is in $W^{-1,2}(0,1)$. By analogy we prove that the last term is in the same space and we also get $\partial_t h_2^k$ in this space. Thanks to the Aubin-Simon lemma, we find:

$$h_2^k \quad \text{strongly Converges to} \quad h_2 \quad \text{in} \quad L^2(0,T;W^{-1,2}(0,1)).$$

3.3.2 Strong Convergence of $h^k u_1^k$ and u^k

We have h_1^k that's bounded in $L^\infty(0,T;L^\infty(0,1))$ and u^k that's bounded in $L^2(0,T;L^\infty(0,1))$, What gives us $h_1^k u^k$ is bounded in $L^2(0,T;L^\infty(0,1))$.

Let's look now $\partial_x(h_1^k u^k)$. We have:

$$\partial_x(h_1^k u^k) = h_1^k\partial_x u^k + u^k\partial_x h_1^k,$$

based on the estimates obtained on h_1^k in **Remark** 3, we get:

$$(h_1^k u_1^k)_k \text{ bounded in } L^2(0,T;W^{1,2}(0,1)).$$

Moreover, the momentum equation (7) enables us to write the time derivation of the water discharge:

$$\partial_t(h_1^k u^k) = -\partial_x(h_1^k {u_1^k}^2) - \frac{1}{2}g\partial_x {h_1^k}^2 + 4\nu_1\partial_x(h_1^k\partial_x u) - au^k + \sigma h_1^k\partial_x^3 h_1^k - rgh_1^k\partial_x h_2^k - rgh_2^k\partial_x(h_1^k + h_2^k)$$

we then study each term:

• $\partial_x(h_1^k(u^k)^2) = \partial_x(h_1^k u^k(u^k))$ which is in $L^2(0,T;W^{-1,2}(0,1))$.

• as h_1^k is in $L^\infty(0,T;L^\infty(0,1))$, and $\partial_x h_1^k$ is in $L^2(0,T;L^2(0,1))$ and we can write the following relation :

• $\partial_x[(h_1^k)^2]$ is bounded in $L^2(0,T;L^2(0,1))$.

• $\partial_x(h_1^k\partial_x u^k)$ is bounded in $L^2(0,T;W^{-1,2}(0,1))$.

• The last four terms are bounded in $L^\infty(0,T;W^{-1,2}(0,1))$.

Then, applying **Aubin-Simon** lemma, we obtain,

$$(h_1^k u^k)_k \text{ stongly Converges to } h_1 u \text{ in } C^0(0,T;W^{-1,2}(0,1)).$$

3.3.3 Strong Convergence of u^k, $h_1^k\partial_x u^k$ and $\sqrt{h^k}u^k$

Thanks to **Corollary** 4 and **Remark** 3, we have u^k and $\partial_x u^k$ are bounded in $L^2(0,T,L^2(0,1))$. In order to obtain new estimates on u^k, we are going to control the right hand side of the following equation:

$$\partial_t u^k = -u^k\partial_x u^k + 4\nu\partial_x \log h_1^k \partial_x u^k + 4\nu\partial_x^2 u^k - g\partial_x h_1^k - rg\partial_x h_2^k - rgh_2^k\partial_x \log h_1^k - rg\frac{h_2}{h_1}\partial_x h_2^k + \sigma\partial_x^3 h_1^k.$$

Thanks to the estimates obtained on h_1^k, h_2^k and u^k, all the terms to the right of equality except the last term are in $L^2(0,T,L^2(0,1))$.

On the other hand, $\partial_x^2 h_1^k$ is bounded in $L^2(0,T,L^2(0,1))$, this lead us to $\partial_x^3 h_1^k$ is bounded in $L^2(0,T,W^{-1,2}(0,1))$ Aubin Simon's lemma leads us to the following result:

$$(u^k)_k \text{ stongly converges to } u \text{ in } L^2(0,T;W^{-1,2}(0,1)).$$

However, the function $(h_1^k,\partial_x u^k) \longmapsto h_1^k\partial_x u^k$ is continous in $L^\infty(0,T;L^\infty 1(0,1))\times L^2(0,T;L^2(0,1))$ to $L^2(0,T;L^2(0,1))$.

So,

$$h_1^k\partial_x u^k \quad \text{weakly converges to} \quad h_1\partial_x u \quad \text{in } L^2(0,T;L^2(0,1)).$$

Thanks to the **Corollary** 4, we say that's exists constants $0<\alpha$ and $\beta<+\infty$ such as $\alpha \le h_1^k \le \beta$.

For all constant $k>\beta$, we have the following norm:

$$\int_0^T \int_0^1 \left|\sqrt{h_1^k}u^k - \sqrt{h_1}u\right|^2 \le \kappa \int_0^T |u^k-u|^2 \longrightarrow 0.$$

So,

$$\sqrt{h_1^k}u^k \quad \text{strongly converges to} \quad \sqrt{h_1}u \qquad \text{in} \quad L^2(0,T;(L^2(\Omega))^2).$$

3.3.4 Strong Convergence of $\partial_x h_1^k$, $h_2^k\partial_x h_1^k$, $\partial_x^2 h_1^k$, $h_1^k\partial_x^2 h_1$ and $\partial_x h_1^k\partial_x^2 h_1^k$

• We have $\partial_x h_1^k$ bounded in $L^2(0,T;H^1(0,1))$ and $\partial_t\partial_x h_1^k$ is bounded in $L^\infty(0,T;H^{-2}(0,1))$ since $\partial_t h_1^k$ is bounded in $L^\infty(0,T;H^{-1}(0,1))$. Thanks to compact injection of $H^1(0,1)$ in $L^2(0,1)$ in one dimension, we have:

$$\partial_x h_1^k \quad \text{strongly converges to} \quad \partial_x h \quad \text{in } L^2(0,T;L^2(0,1))$$

• The bound of $\partial_x^2 h_1^k$ in $L^2(0,T;L^2(0,1))$ and $\partial_x h_2^k$ in $L^2(0,T;L^2(0,1))$ gives us:

$$\partial_x^2 h_1^k \quad \text{weakly converges to} \quad \partial_x^2 h_1 \quad \text{in } L^2(0,T;L^2(0,1)),$$

$$\partial_x h_2^k \quad \text{weakly converges to} \quad \partial_x h_2 \quad \text{in } L^2(0,T;L^2(0,1)).$$

• Thanks to the strong convergence of h_1^k, h_2^k, $\partial_x h_1^k$ and the weak convergence of $\partial_x^2 h_1^k$, we have:

$$h_2^k\partial_x h_1^k \quad \text{strongly converges to} \quad h_2\partial_x h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$
\begin{aligned}
h_1^k\partial_x^2 h_1^k &\quad \text{strongly converges to} \quad h_1\partial_x^2 h \quad \text{in } L^1(0,T;L^1(0,1)),\\
\partial_x h_1^k\partial_x^2 h_1^k &\quad \text{strongly converges to} \quad \partial_x h_1\partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),\\
h_1^k\partial_x h_2^k &\quad \text{strongly converges to} \quad h_1\partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),\\
h_2^k\partial_x h_2^k &\quad \text{strongly converges to} \quad h_2\partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),\\
(h_1^k)^2 &\quad \text{strongly converges to} \quad {h_1}^2 \quad \text{in } L^1(0,T;L^1(0,1)).
\end{aligned}
$$

3.3.5 Convergences of $h_2^k u^k$ and $\partial_x^2 h^k$

We know that $\partial_x h_2^k$ is bounded in $L^2(0,T;L^2(0,1))$ this implies $\partial_x^2 h_2^k$ is in $L^1(0,T;W^{-1,2}(0,1))$.

So,

$$\partial_x^2 h_2^k \quad \text{weakly converges to} \quad \partial_x^2 h_2 \in L^1(0,T;W^{-1,2}(0,1))$$

To end we have u^k weakly converges to u in $L^2(0,T;L^2(0,1))$ and the strong convergence of h_2^k to h_2, gives us:

$$h_2^k u^k \quad \text{weakly converges to} \quad h_2 u \quad \text{in } L^1(0,T;L^1(0,1)).$$

4. Conclusion

In this paper, we show the existence of global weak solutions of a 1D pollutant transport model. We note that, for the model studied in this paper we did not take into account regularizing terms (cold pressure and quadratic friction terms) as in (Roamba, Zabsonré & Zongo, 2017) and we considered a better transport equation than that used in (Roamba, Zabsonré & Zongo, 2017).

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