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Some New Identities for the Generalized Fibonacci Polynomials by the $Q(x)$ Matrix

Chung-Chuan Chen[1] & Lin-Ling Huang[1]

[1] Department of Mathematics Education, National Taichung University of Education, Taiwan

Correspondence: Chung-Chuan Chen, Department of Mathematics Education, National Taichung University of Education, Taiwan. E-mail: chungchuan@mail.ntcu.edu.tw

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Abstract

We obtain some new identities for the generalized Fibonacci polynomial by a new approach, namely, the $Q(x)$ matrix. These identities including the Cassini type identity and Honsberger type formula can be applied to some polynomial sequences such as Fibonacci polynomials, Lucas polynomials, Pell polynomials, Pell-Lucas polynomials and so on, which generalize the previous results in references.

Keywords: generalized Fibonacci polynomial, $Q(x)$ matrix, Cassini type identity, Honsberger type formula

1. Introduction

A second order polynomial sequence $F_n(x)$ is said to be *the Fibonacci polynomial* if for $n \geq 2$ and $x \in \mathbb{R}$,

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

with $F_0(x) = 0$ and $F_1(x) = 1$. The Fibonacci polynomial and other polynomials attracted a lot of attention over the last several decades (see, for instance, Falcón & Plaza, 2007, 2009; Gould, 1981; Horadam, 1979; Horadam & Mahon, 1985; Wu & Zhang, 2013). Recently, the generalized Fibonacci polynomial is introduced and studied intensely by many authors (André-Jeannin, 1994, 1995; Flórez, Higuita & Mukherjee, 2018; Flórez, McAnally & Mukherjee, 2018), which is a generalization of the Fibonacci polynomial. Indeed, a polynomial sequence $G_n(x)$ in (Flórez, Higuita & Mukherjee, 2018; Flórez, McAnally & Mukherjee, 2018) is called *the generalized Fibonacci polynomial* if for $n \geq 2$,

$$G_n(x) = c(x)G_{n-1}(x) + d(x)G_{n-2}(x)$$

with $G_0(x)$ and $G_1(x)$, where $c(x)$ and $d(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$. It should be noted that there is no unique generalization of Fibonacci polynomials. Following the similar definitions in (Flórez, McAnally & Mukherjee, 2018), in this note, $\mathcal{F}_n(x)$ is said to be *the Fibonacci type polynomial* if for $n \geq 2$,

$$\mathcal{F}_0(x) = 0,\ \mathcal{F}_1(x) = a \text{ and } \mathcal{F}_n(x) = c(x)\mathcal{F}_{n-1}(x) + d(x)\mathcal{F}_{n-2}(x)$$

where $a \in \mathbb{R} \setminus \{0\}$. If for $n \geq 2$,

$$\mathcal{L}_0(x) = q,\ \mathcal{L}_1(x) = b(x) \text{ and } \mathcal{L}_n(x) = c(x)\mathcal{L}_{n-1}(x) + d(x)\mathcal{L}_{n-2}(x),$$

then the polynomial sequence $\mathcal{L}_n(x)$ is called *the Lucas type polynomial*, where $q \in \mathbb{R} \setminus \{0\}$ and $b(x)$ is a fixed non-zero polynomial in $\mathbb{Q}[x]$. Naturally, both $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ are the generalized Fibonacci polynomials. We note that if we assume $\mathcal{F}_1(x) = a = 1$, then $\mathcal{F}_n(x)$ is the Fibonacci type polynomial given in (Flórez, McAnally & Mukherjee, 2018). In addition, the definition of $\mathcal{L}_n(x)$ is the same with that of Flórez et al (Flórez, McAnally & Mukherjee, 2018) if $|q| = 1$ or 2, and $c(x) = \frac{2}{q}b(x)$. In other words, our definitions of $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ are generalizations of those in (Flórez, McAnally & Mukherjee, 2018).

Our goal is to give some new identities for $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ by applying a new approach, namely the $Q(x)$ matrix, rather than using the Binet formulas. We note that in some references, the authors obtain identities for polynomial sequences by using the Binet formulas. Here, we provide a new way to deduce some new identities for $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$. Besides, these results can be applied to some familiar polynomial sequences. Indeed, the polynomial sequences in the upper part of Table 1 below are the Fibonacci type polynomials. On the other hand, those in the lower part of Table 1 are the Lucas type polynomials. Table 1 is the rearrangement of Table 1 (Flórez, McAnally & Mukherjee, 2018).

Table 1. Some examples of the generalized Fibonacci polynomials

Polynomial	Initial value $G_0(x)$	Initial value $G_1(x)$	Recursive Formula $G_n(x) = c(x)G_{n-1}(x) + d(x)G_{n-2}(x)$
Fibonacci	0	1	$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$
Pell	0	1	$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$
Fermat	0	1	$\Phi_n(x) = 3x\Phi_{n-1}(x) - 2\Phi_{n-2}(x)$
Chebyshev second kind	0	1	$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$
Jacobsthal	0	1	$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x)$
Morgan-Voyce	0	1	$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$
Vieta	0	1	$V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$
Lucas	2	x	$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$
Pell-Lucas	2	$2x$	$D_n(x) = 2xD_{n-1}(x) + D_{n-2}(x)$
Pell-Lucas-prime	1	x	$D'_n(x) = 2xD'_{n-1}(x) + D'_{n-2}(x)$
Fermat-Lucas	2	$3x$	$\vartheta_n(x) = 3x\vartheta_{n-1}(x) - 2\vartheta_{n-2}(x)$
Chebyshev first kind	1	x	$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
Jacobsthal-Lucas	1	1	$\Lambda_n(x) = \Lambda_{n-1}(x) + 2x\Lambda_{n-2}(x)$
Morgan-Voyce	2	$x+2$	$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x)$
Vieta-Lucas	2	x	$v_n(x) = xv_{n-1}(x) - v_{n-2}(x)$

2. Fibonacci Type Polynomials

In this section, we will provide and prove some identities for the Fibonacci type polynomial $\mathcal{F}_n(x)$ by applying the Fibonacci type $Q(x)$ matrix. The original Fibonacci Q matrix was introduced by Charles H. King in his master thesis (cf. Koshy, 2001), and given by

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Fibonacci Q matrix is connected to the Fibonacci sequence F_n, which is defined as below

$$F_0 = 1,\ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Indeed, it is noted in (Gould, 1981) that

$$Q^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}.$$

Using this relation above, some familiar identities can be obtained. For instance,

$$\det \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \left(\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^n$$

implies the Cassini identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Also, using this equality $Q^{n+m} = Q^n Q^m$, one can deduce the Honsberger formula.

In the following, we will apply some similar idea of Q matrix from the numerical cases (Lin, 2012) to the Fibonacci type polynomials $\mathcal{F}_n(x)$. For $n \geq 2$ and $x \in \mathbb{R}$, the Fibonacci type polynomial $\mathcal{F}_n(x)$ is defined by

$$\mathcal{F}_0(x) = 0,\ \mathcal{F}_1(x) = a \text{ and } \mathcal{F}_n(x) = c(x)\mathcal{F}_{n-1}(x) + d(x)\mathcal{F}_{n-2}(x) \tag{1}$$

where $a \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{pmatrix} \mathcal{F}_{n+2}(x) \\ \mathcal{F}_{n+1}(x) \end{pmatrix} = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{F}_{n+1}(x) \\ \mathcal{F}_n(x) \end{pmatrix}.$$

Here we define *the Fibonacci type $Q(x)$ matrix* by

$$Q(x) = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}.$$

We note that if $\mathcal{F}_n(x) = P_n(x)$ is the Pell polynomial as defined in Table 1, then

$$Q(x) = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}$$

which appeared in (Horadam & Mahon, 1985). In addition, we observe that

$$\begin{pmatrix}\mathcal{F}_{n+2}(x)\\ \mathcal{F}_{n+1}(x)\end{pmatrix} = \begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}^n \begin{pmatrix}\mathcal{F}_2(x)\\ \mathcal{F}_1(x)\end{pmatrix} = \begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}^n \begin{pmatrix}ac(x)\\ a\end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix}\mathcal{F}_{n+2}(x)\\ \mathcal{F}_{n+1}(x)\end{pmatrix} = \begin{pmatrix}c(x)\mathcal{F}_{n+1}(x) + d(x)\mathcal{F}_n(x)\\ c(x)\mathcal{F}_n(x) + d(x)\mathcal{F}_{n-1}(x)\end{pmatrix} = \begin{pmatrix}\frac{1}{a}\mathcal{F}_{n+1}(x) & \frac{d(x)}{a}\mathcal{F}_n(x)\\ \frac{1}{a}\mathcal{F}_n(x) & \frac{d(x)}{a}\mathcal{F}_{n-1}(x)\end{pmatrix}\begin{pmatrix}ac(x)\\ a\end{pmatrix}.$$

Hence the below result follows.

Theorem 2.1 *Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial as defined in Eq.* (1). *Then for each $n \in \mathbb{N}$,*

$$\begin{pmatrix}\frac{1}{a}\mathcal{F}_{n+1}(x) & \frac{d(x)}{a}\mathcal{F}_n(x)\\ \frac{1}{a}\mathcal{F}_n(x) & \frac{d(x)}{a}\mathcal{F}_{n-1}(x)\end{pmatrix} = \begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}^n = Q^n(x).$$

Proof. Let $n = 1$. Then

$$\begin{pmatrix}\frac{1}{a}\mathcal{F}_2(x) & \frac{d(x)}{a}\mathcal{F}_1(x)\\ \frac{1}{a}\mathcal{F}_1(x) & \frac{d(x)}{a}\mathcal{F}_0(x)\end{pmatrix} = \begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}.$$

Assume the equality holds for $n = k$. Then we have

$$\begin{pmatrix}\frac{1}{a}\mathcal{F}_{k+1}(x) & \frac{d(x)}{a}\mathcal{F}_k(x)\\ \frac{1}{a}\mathcal{F}_k(x) & \frac{d(x)}{a}\mathcal{F}_{k-1}(x)\end{pmatrix} = \begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}^k.$$

If $n = k + 1$, then

$$\begin{pmatrix}\frac{1}{a}\mathcal{F}_{k+2}(x) & \frac{d(x)}{a}\mathcal{F}_{k+1}(x)\\ \frac{1}{a}\mathcal{F}_{k+1}(x) & \frac{d(x)}{a}\mathcal{F}_k(x)\end{pmatrix} = \begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{a}\mathcal{F}_{k+1}(x) & \frac{d(x)}{a}\mathcal{F}_k(x)\\ \frac{1}{a}\mathcal{F}_k(x) & \frac{d(x)}{a}\mathcal{F}_{k-1}(x)\end{pmatrix} = \begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}^{k+1}.$$

By induction, the result follows. □

The Cassini type identity of the Fibonacci type polynomial $\mathcal{F}_n(x)$ can be obtained below by Theorem 2.1.

Corollary 2.2 *Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial. Then for each $n \in \mathbb{N}$,*

$$\mathcal{F}_n^2(x) - \mathcal{F}_{n+1}(x)\mathcal{F}_{n-1}(x) = a^2(-d(x))^{n-1}.$$

Proof. By Theorem 2.1,

$$\det\begin{pmatrix}\frac{1}{a}\mathcal{F}_{n+1}(x) & \frac{d(x)}{a}\mathcal{F}_n(x)\\ \frac{1}{a}\mathcal{F}_n(x) & \frac{d(x)}{a}\mathcal{F}_{n-1}(x)\end{pmatrix} = \left(\det\begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}\right)^n.$$

Hence

$$\mathcal{F}_n^2(x) - \mathcal{F}_{n+1}(x)F_{n-1}(x) = a^2(-d(x))^{n-1}.$$

□

Example 2.3 *Let $a = 1, c(x) = x, d(x) = 1$ in Eq. (1). Then $\mathcal{F}_n(x)$ is the classical Fibonacci polynomial $F_n(x)$. By Corollary 2.2, we recover the Cassini identity in (Falcón & Plaza, 2009),*

$$F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n.$$

Example 2.4 *Let $\mathcal{F}_n(x)$ be the Pell polynomial $P_n(x)$ as defined in Table 1. By Corollary 2.2,*

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n$$

which is the identity (2.5) in (Horadam & Mahon, 1985).

Example 2.5 *Let $a = 1, c(x) = 1, d(x) = 2x$ in Eq. (1). Then $\mathcal{F}_n(x) = J_n(x)$ is the Jacobsthal polynomial as defined in Table 1. By Corollary 2.2, one can obtain the Cassini identity for the Jacobsthal polynomial below*

$$J_n^2(x) - J_{n+1}(x)J_{n-1}(x) = (-2x)^{n-1}.$$

By Corollary 2.2, we have the result below.

Corollary 2.6 *Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial. Then for each $n \in \mathbb{N}$,*

$$\mathcal{F}_n^2(x) - c(x)\mathcal{F}_n(x)\mathcal{F}_{n-1}(x) - d(x)\mathcal{F}_{n-1}^2(x) = a^2(-d(x))^{n-1}.$$

Proof. By

$$\mathcal{F}_n^2(x) - \mathcal{F}_{n+1}(x)\mathcal{F}_{n-1}(x) = a^2(-d(x))^{n-1}.$$

and

$$\mathcal{F}_{n+1}(x) = c(x)\mathcal{F}_n(x) + d(x)\mathcal{F}_{n-1}(x),$$

we have

$$\begin{aligned} a^2(-d(x))^{n-1} &= \mathcal{F}_n^2(x) - \left(c(x)\mathcal{F}_n(x) + d(x)\mathcal{F}_{n-1}(x)\right)\mathcal{F}_{n-1}(x) \\ &= \mathcal{F}_n^2(x) - c(x)\mathcal{F}_n(x)\mathcal{F}_{n-1}(x) - d(x)\mathcal{F}_{n-1}^2(x). \end{aligned}$$

□

By applying $Q^{n+m}(x) = Q^n(x)Q^m(x)$, we give the Honsberger type formula for the Fibonacci type polynomials $\mathcal{F}_n(x)$ below.

Corollary 2.7 *Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial. Then for each $n, m \in \mathbb{N}$,*

$$a\mathcal{F}_{n+m}(x) = \mathcal{F}_n(x)\mathcal{F}_{m+1}(x) + d(x)\mathcal{F}_{n-1}(x)\mathcal{F}_m(x).$$

Proof. By

$$\begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}^{n+m} = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}^{m},$$

we have

$$\begin{pmatrix} \frac{1}{a}\mathcal{F}_{n+m+1}(x) & \frac{d(x)}{a}\mathcal{F}_{n+m}(x) \\ \frac{1}{a}\mathcal{F}_{n+m}(x) & \frac{d(x)}{a}\mathcal{F}_{n+m-1}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{a}\mathcal{F}_{n+1}(x) & \frac{d(x)}{a}\mathcal{F}_{n}(x) \\ \frac{1}{a}\mathcal{F}_{n}(x) & \frac{d(x)}{a}\mathcal{F}_{n-1}(x) \end{pmatrix} \begin{pmatrix} \frac{1}{a}\mathcal{F}_{m+1}(x) & \frac{d(x)}{a}\mathcal{F}_{m}(x) \\ \frac{1}{a}\mathcal{F}_{m}(x) & \frac{d(x)}{a}\mathcal{F}_{m-1}(x) \end{pmatrix}.$$

Hence by the (2, 1) entry of the first matrix in the equality above,

$$a\mathcal{F}_{n+m}(x) = \mathcal{F}_n(x)\mathcal{F}_{m+1}(x) + d(x)\mathcal{F}_{n-1}(x)\mathcal{F}_m(x).$$

□

Remark 2.8

(i) Let $a = 1$ in Corollary 2.7. Then Corollary 2.7 is the same with the first result of Proposition 1 (Flórez, McAnally & Mukherjee, 2018), and a generalization of Proposition 5 (Falcón & Plaza, 2009).

(ii) If $m = n - 1$ in the above corollary, then for each $n \in \mathbb{N}$,

$$a\mathcal{F}_{2n-1}(x) = \mathcal{F}_n^2(x) + d(x)\mathcal{F}_{n-1}^2(x)$$

which generalizes the numerical case of Fibonacci sequences.

Example 2.9 *Let $a = 1, c(x) = x, d(x) = 1$ in Eq. (1). Then $\mathcal{F}_n(x) = F_n(x)$ is the Fibonacci polynomial as defined in Table 1. By Corollary 2.7, we recover the Honsberger formula in Proposition 5 (Falcón & Plaza, 2009),*

$$F_{n+m}(x) = F_n(x)F_{m+1}(x) + F_{n-1}(x)F_m(x).$$

Example 2.10 *Let $a = 1, c(x) = 2x, d(x) = 1$ in Eq. (1). Then $\mathcal{F}_n(x)$ is the Pell polynomial $P_n(x)$. By Corollary 2.7, we have*

$$P_{n+m}(x) = P_n(x)P_{m+1}(x) + P_{n-1}(x)P_m(x)$$

which is the equality (3.14) in (Horadam & Mahon, 1985).

Using $Q^{n-m}(x) = Q^n(x)Q^{-m}(x)$ for $n \geq m$, we next will prove the d'Ocagne type identity for $\mathcal{F}_n(x)$. Here we need to assume $d(x) \neq 0$ for each $x \in \mathbb{R}$ so that $Q(x)$ is invertible. Moreover, note that

$$Q^{-m}(x) = \begin{pmatrix} \frac{1}{a}\mathcal{F}_{m+1}(x) & \frac{d(x)}{a}\mathcal{F}_m(x) \\ \frac{1}{a}\mathcal{F}_m(x) & \frac{d(x)}{a}\mathcal{F}_{m-1}(x) \end{pmatrix}^{-1} = \frac{1}{(-d(x))^m}\begin{pmatrix} \frac{d(x)}{a}\mathcal{F}_{m-1}(x) & -\frac{d(x)}{a}\mathcal{F}_m(x) \\ -\frac{1}{a}\mathcal{F}_m(x) & \frac{1}{a}\mathcal{F}_{m+1}(x) \end{pmatrix}$$

by Theorem 2.1 and Corollary 2.2.

Corollary 2.11 *Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial, and let $d(x) \neq 0$ for each $x \in \mathbb{R}$. Then for $n, m \in \mathbb{N}$ with $n \geq m$,*

$$a(-d(x))^m\mathcal{F}_{n-m}(x) = \mathcal{F}_n(x)\mathcal{F}_{m+1}(x) - \mathcal{F}_{n+1}(x)\mathcal{F}_m(x).$$

Proof. By $Q^{n-m}(x) = Q^n(x)Q^{-m}(x)$, we have

$$\begin{aligned} &\begin{pmatrix} \frac{1}{a}\mathcal{F}_{n-m+1}(x) & \frac{d(x)}{a}\mathcal{F}_{n-m}(x) \\ \frac{1}{a}\mathcal{F}_{n-m}(x) & \frac{d(x)}{a}\mathcal{F}_{n-m-1}(x) \end{pmatrix} \\ = &\begin{pmatrix} \frac{1}{a}\mathcal{F}_{n+1}(x) & \frac{d(x)}{a}\mathcal{F}_n(x) \\ \frac{1}{a}\mathcal{F}_n(x) & \frac{d(x)}{a}\mathcal{F}_{n-1}(x) \end{pmatrix}\frac{1}{(-d(x))^m}\begin{pmatrix} \frac{d(x)}{a}\mathcal{F}_{m-1}(x) & -\frac{d(x)}{a}\mathcal{F}_m(x) \\ -\frac{1}{a}\mathcal{F}_m(x) & \frac{1}{a}\mathcal{F}_{m+1}(x) \end{pmatrix}. \end{aligned}$$

Hence considering the $(1, 2)$ entry of the first matrix in the equality above,

$$a(-d(x))^m\mathcal{F}_{n-m}(x) = \mathcal{F}_n(x)\mathcal{F}_{m+1}(x) - \mathcal{F}_{n+1}(x)\mathcal{F}_m(x).$$

□

Example 2.12 *Let $\mathcal{F}_n(x)$ be the Fibonacci polynomial $F_n(x)$ as in Table 1. By Corollary 2.11,*

$$(-1)^mF_{n-m}(x) = F_n(x)F_{m+1}(x) - F_{n+1}(x)F_m(x)$$

which is the d'Ocagne identity in Corollary 8 (Falcón & Plaza, 2009), and the identity (47) of Proposition 3 (Flórez, McAnally & Mukherjee, 2018).

We note that $Q(x) = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}$ satisfies $Q^2(x) = c(x)Q(x) + d(x)I$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Using this equality, one can obtain the following expression of $\mathcal{F}_n(x)$.

Theorem 2.13 *Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial. Then for each $n, p \in \mathbb{N}$,*

$$\mathcal{F}_{2n+p}(x) = \sum_{j=0}^{n}\binom{n}{j}c^j(x)d^{n-j}(x)\mathcal{F}_{j+p}(x).$$

Proof. Consider

$$
\begin{aligned}
& \begin{pmatrix} \frac{1}{a}\mathcal{F}_{2n+p+1}(x) & \frac{d(x)}{a}\mathcal{F}_{2n+p}(x) \\ \frac{1}{a}\mathcal{F}_{2n+p}(x) & \frac{d(x)}{a}\mathcal{F}_{2n+p-1}(x) \end{pmatrix} \\
= \; & Q^{2n+p}(x) \\
= \; & Q^{p}(x)\left(Q^{2}(x)\right)^{n} \\
= \; & Q^{p}(x)\left(c(x)Q(x)+d(x)I\right)^{n} \\
= \; & Q^{p}(x)\left(\sum_{j=0}^{n}\binom{n}{j}c^{j}(x)d^{n-j}(x)Q^{j}(x)\right) \\
= \; & \begin{pmatrix} \frac{1}{a}\mathcal{F}_{p+1}(x) & \frac{d(x)}{a}\mathcal{F}_{p}(x) \\ \frac{1}{a}\mathcal{F}_{p}(x) & \frac{d(x)}{a}\mathcal{F}_{p-1}(x) \end{pmatrix} \cdot \sum_{j=0}^{n}\binom{n}{j}c^{j}(x)d^{n-j}(x)\begin{pmatrix} \frac{1}{a}\mathcal{F}_{j+1}(x) & \frac{d(x)}{a}\mathcal{F}_{j}(x) \\ \frac{1}{a}\mathcal{F}_{j}(x) & \frac{d(x)}{a}\mathcal{F}_{j-1}(x) \end{pmatrix}.
\end{aligned}
$$

Then by Corollary 2.7 and the $(1,2)$ entry of the first matrix in the above equality, we have

$$
\begin{aligned}
a\mathcal{F}_{2n+p}(x) &= \sum_{j=0}^{n}\binom{n}{j}c^{j}(x)d^{n-j}(x)\left(\mathcal{F}_{p}(x)\mathcal{F}_{j+1}(x)+d(x)\mathcal{F}_{p-1}(x)\mathcal{F}_{j}(x)\right) \\
&= a\sum_{j=0}^{n}\binom{n}{j}c^{j}(x)d^{n-j}(x)\mathcal{F}_{j+p}(x).
\end{aligned}
$$

□

Example 2.14 *Let $\mathcal{F}_n(x)$ be the Fibonacci polynomial $F_n(x)$ in which $a=1, c(x)=x, d(x)=1$ in Eq. (1). By Theorem 2.13, we have*

$$F_{2n+p}(x)=\sum_{j=0}^{n}\binom{n}{j}x^{j}F_{j+p}(x).$$

Given $n=2$ and $p=1$, we have

$$F_5(x)=F_1(x)+2xF_2(x)+x^2F_3(x).$$

Indeed, this equality holds for $F_1(x)=1, F_2(x)=x, F_3(x)=x^2+1$ and $F_5(x)=x^4+3x^2+1$.

3. Lucas Type Polynomials

Based on the results of Fibonacci type $\mathcal{F}_n(x)$, some identities of Lucas type polynomials $\mathcal{L}_n(x)$ will be demonstrated in this section. Throughout this section, we assume $\mathcal{L}_n(x)$ and $\mathcal{F}_n(x)$ have the same recursive formula with $\mathcal{L}_0(x)=\mathcal{F}_1(x)$, that is, for $n \geq 2$,

$$\mathcal{F}_0(x)=0,\ \mathcal{F}_1(x)=a \text{ and } \mathcal{F}_n(x)=c(x)\mathcal{F}_{n-1}(x)+d(x)\mathcal{F}_{n-2}(x),$$

and

$$\mathcal{L}_0(x)=a,\ \mathcal{L}_1(x)=b(x) \text{ and } \mathcal{L}_n(x)=c(x)\mathcal{L}_{n-1}(x)+d(x)\mathcal{L}_{n-2}(x) \tag{2}$$

where $a \in \mathbb{R}\setminus\{0\}$. By applying Theorem 2.1, one can connect $\mathcal{L}_n(x)$ with $\mathcal{F}_n(x)$ below.

Theorem 3.1 *Let $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ be the Fibonacci type polynomial and Lucas type polynomial respectively with $\mathcal{L}_0(x)=\mathcal{F}_1(x)=a$. Then for each $n \in \mathbb{N}$,*

$$\begin{pmatrix} \mathcal{L}_{n+2}(x) & \mathcal{L}_{n+1}(x) \\ \mathcal{L}_{n+1}(x) & \mathcal{L}_{n}(x) \end{pmatrix} = \begin{pmatrix} \frac{b(x)c(x)+ad(x)}{a} & \frac{b(x)d(x)}{a} \\ \frac{b(x)}{a} & d \end{pmatrix}\begin{pmatrix} \mathcal{F}_{n+1}(x) & \mathcal{F}_{n}(x) \\ \mathcal{F}_{n}(x) & \mathcal{F}_{n-1}(x) \end{pmatrix}$$

Proof. First, we will prove $\mathcal{L}_n(x)=\frac{b(x)}{a}\mathcal{F}_n(x)+d(x)\mathcal{F}_{n-1}(x)$ holds for each $n \in \mathbb{N}$. Let $n=1$. Then

$$\mathcal{L}_1(x)=b(x)=\frac{b(x)}{a}\mathcal{F}_1(x)+d(x)\mathcal{F}_0(x).$$

Let $n=2$. Then

$$\mathcal{L}_2(x)=b(x)c(x)+ad(x)=\frac{b(x)}{a}\mathcal{F}_2(x)+d(x)\mathcal{F}_1(x).$$

Assume this equality hods for $n = k-1$ and k. Let $n = k+1$. Then

$$\begin{aligned}\mathcal{L}_{k+1}(x) &= c(x)\mathcal{L}_k(x) + d(x)\mathcal{L}_{k-1}(x)\\ &= c(x)\left(\frac{b(x)}{a}\mathcal{F}_k(x) + d(x)\mathcal{F}_{k-1}(x)\right) + d(x)\left(\frac{b(x)}{a}\mathcal{F}_{k-1}(x) + d(x)\mathcal{F}_{k-2}(x)\right)\\ &= \frac{b(x)}{a}\left(c(x)\mathcal{F}_k(x) + d(x)\mathcal{F}_{k-1}(x)\right) + d(x)\left(c(x)\mathcal{F}_{k-1}(x) + d(x)\mathcal{F}_{k-2}(x)\right)\\ &= \frac{b(x)}{a}\mathcal{F}_{k+1}(x) + d(x)\mathcal{F}_k(x).\end{aligned}$$

By induction, $\mathcal{L}_n(x) = \frac{b(x)}{a}\mathcal{F}_n(x) + d(x)\mathcal{F}_{n-1}(x)$ holds for all $n \in \mathbb{N}$. Also,

$$\begin{aligned}\mathcal{L}_n(x) &= \frac{b(x)}{a}\mathcal{F}_n(x) + d(x)\mathcal{F}_{n-1}(x)\\ &= \frac{b(x)}{a}\left(c(x)\mathcal{F}_{n-1}(x) + d(x)\mathcal{F}_{n-2}(x)\right) + d(x)\mathcal{F}_{n-1}(x)\\ &= \frac{b(x)c(x) + ad(x)}{a}\mathcal{F}_{n-1}(x) + \frac{b(x)d(x)}{a}\mathcal{F}_{n-2}(x).\end{aligned}$$

One has the result by these two equalities

$$\mathcal{L}_n(x) = \frac{b(x)}{a}\mathcal{F}_n(x) + d(x)\mathcal{F}_{n-1}(x)$$

and

$$\mathcal{L}_n(x) = \frac{b(x)c(x) + ad(x)}{a}\mathcal{F}_{n-1}(x) + \frac{b(x)d(x)}{a}\mathcal{F}_{n-2}(x).$$

□

Next, we will demonstrate the relation between Lucas type polynomials and the Fibonacci type $Q(x)$ matrix .

Theorem 3.2 *Let $\mathcal{L}_n(x)$ be the Lucas type polynomial. Then for each $n \in \mathbb{N}$,*

$$\begin{pmatrix}\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x)\\ \mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)\end{pmatrix} = \begin{pmatrix}\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x)\\ \mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)\end{pmatrix} Q^n(x)$$

Proof. By Theorem 2.1 and Theorem 3.1, we have

$$\begin{aligned}&\begin{pmatrix}\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x)\\ \mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)\end{pmatrix}\\ =&\begin{pmatrix}\mathcal{L}_{n+2}(x) & \mathcal{L}_{n+1}(x)\\ \mathcal{L}_{n+1}(x) & \mathcal{L}_n(x)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & d(x)\end{pmatrix}\\ =&\begin{pmatrix}\frac{b(x)c(x)+ad(x)}{a} & \frac{b(x)d(x)}{a}\\ \frac{b(x)}{a} & d\end{pmatrix}\begin{pmatrix}\mathcal{F}_{n+1}(x) & \mathcal{F}_n(x)\\ \mathcal{F}_n(x) & \mathcal{F}_{n-1}(x)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & d(x)\end{pmatrix}\\ =&\begin{pmatrix}b(x)c(x) + ad(x) & b(x)d(x)\\ b(x) & ad(x)\end{pmatrix}\begin{pmatrix}\frac{1}{a}\mathcal{F}_{n+1}(x) & \frac{d(x)}{a}\mathcal{F}_n(x)\\ \frac{1}{a}\mathcal{F}_n(x) & \frac{d(x)}{a}\mathcal{F}_{n-1}(x)\end{pmatrix}\\ =&\begin{pmatrix}\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x)\\ \mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)\end{pmatrix}\begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}^n\end{aligned}$$

for each $n \in \mathbb{N}$. □

Using Theorem 3.2, one has the Cassini type identity for the Lucas type polynomial $\mathcal{L}_n(x)$.

Corollary 3.3 *Let $\mathcal{L}_n(x)$ be the Lucas type polynomial. Then for each $n \in \mathbb{N}$,*

$$\mathcal{L}_{n+2}(x)\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) = \left(\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x)\right)(-d(x))^n.$$

Proof. By Theorem 3.2, we have

$$\det\begin{pmatrix}\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x)\\ \mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)\end{pmatrix} = \det\begin{pmatrix}\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x)\\ \mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)\end{pmatrix}\left(\det\begin{pmatrix}c(x) & d(x)\\ 1 & 0\end{pmatrix}\right)^n.$$

Hence

$$\mathcal{L}_{n+2}(x)\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) = \left(\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x)\right)(-d(x))^n.$$

□

Example 3.4 *Let $a = 2, b(x) = 2x, c(x) = 2x, d(x) = 1$ in Eq. (2). Then $\mathcal{L}_n(x) = D_n(x)$ is the Pell-Lucas polynomial as defined in Table 1. By Corollary 3.3, the Cassini identity for the Pell-Lucas polynomial $D_n(x)$ is*

$$D_{n+2}(x)D_n(x) - D_{n+1}^2(x) = (4x^2 + 4)(-1)^n.$$

By Corollary 3.3, we have the result below.

Corollary 3.5 *Let $\mathcal{L}_n(x)$ be the Lucas type polynomial. Then for each $n \in \mathbb{N}$,*

$$c(x)\mathcal{L}_{n+1}(x)\mathcal{L}_n(x) + d(x)\mathcal{L}_n^2(x) - \mathcal{L}_{n+1}^2(x) = \left(\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x)\right)(-d(x))^n.$$

Proof. By

$$\mathcal{L}_{n+2}(x)\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) = \left(\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x)\right)(-d(x))^n$$

and

$$\mathcal{L}_{n+2}(x) = c(x)\mathcal{L}_{n+1}(x) + d(x)\mathcal{L}_n(x),$$

we have

$$\begin{aligned} & \left(\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x)\right)(-d(x))^n \\ = \;& \left(c(x)\mathcal{L}_{n+1}(x) + d(x)\mathcal{L}_n(x)\right)\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) \\ = \;& c(x)\mathcal{L}_{n+1}(x)\mathcal{L}_n(x) + d(x)\mathcal{L}_n^2(x) - \mathcal{L}_{n+1}^2(x). \end{aligned}$$

□

Using $Q^2(x) = c(x)Q(x) + d(x)I$ again, we have the expression of $\mathcal{L}_n(x)$.

Theorem 3.6 *Let $\mathcal{L}_n(x)$ be the Lucas type polynomial. Then for each $n, p \in \mathbb{N}$,*

$$\mathcal{L}_{2n+p}(x) = \sum_{j=0}^{n}\binom{n}{j}c^j(x)d^{n-j}(x)\mathcal{L}_{p+j}(x).$$

Proof. By Theorem 3.2, we have

$$\begin{aligned} & \begin{pmatrix}\mathcal{L}_{2n+p+2}(x) & d(x)\mathcal{L}_{2n+p+1}(x)\\ \mathcal{L}_{2n+p+1}(x) & d(x)\mathcal{L}_{2n+p}(x)\end{pmatrix} \\ = \;& \begin{pmatrix}\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x)\\ \mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)\end{pmatrix} Q^{2n+p}(x) \\ = \;& \begin{pmatrix}\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x)\\ \mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)\end{pmatrix} Q^p(x)\left(Q^2(x)\right)^n \\ = \;& \begin{pmatrix}\mathcal{L}_{p+2}(x) & d(x)\mathcal{L}_{p+1}(x)\\ \mathcal{L}_{p+1}(x) & d(x)\mathcal{L}_p(x)\end{pmatrix}(c(x)Q(x) + d(x)I)^n \\ = \;& \begin{pmatrix}\mathcal{L}_{p+2}(x) & d(x)\mathcal{L}_{p+1}(x)\\ \mathcal{L}_{p+1}(x) & d(x)\mathcal{L}_p(x)\end{pmatrix}\left(\sum_{j=0}^{n}\binom{n}{j}c^j(x)d^{n-j}(x)Q^j(x)\right) \\ = \;& \sum_{j=0}^{n}\binom{n}{j}c^j(x)d^{n-j}(x)\begin{pmatrix}\mathcal{L}_{p+j+2}(x) & d(x)\mathcal{L}_{p+j+1}(x)\\ \mathcal{L}_{p+j+1}(x) & d(x)\mathcal{L}_{p+j}(x)\end{pmatrix} \end{aligned}$$

By considering the $(2,2)$ entry of the first matrix in the above equality, we have

$$\mathcal{L}_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} c^j(x) d^{n-j}(x) \mathcal{L}_{p+j}(x).$$

□

Example 3.7 *Let $\mathcal{L}_n(x)$ be the Morgan-Voyce polynomial $C_n(x)$ in which $a = 2, b(x) = x + 2, c(x) = x + 2, d(x) = -1$ in Eq. (2). By Theorem 3.6, we have*

$$C_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} (x+2)^j (-1)^{n-j} C_{p+j}(x).$$

Finally, we end up this note by providing two identities in which $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ are involved.

Proposition 3.8 *Let $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ be the Fibonacci type polynomial and Lucas type polynomial respectively with $\mathcal{L}_0(x) = \mathcal{F}_1(x) = a$. Then for each $n, m \in \mathbb{N}$,*

$$a\mathcal{L}_{n+m}(x) = \mathcal{L}_{n+1}(x)\mathcal{F}_m(x) + d(x)\mathcal{L}_n(x)\mathcal{F}_{m-1}(x).$$

Proof. By Theorem 3.2, we have

$$\begin{aligned} &\begin{pmatrix} \mathcal{L}_{n+m+2}(x) & d(x)\mathcal{L}_{n+m+1}(x) \\ \mathcal{L}_{n+m+1}(x) & d(x)\mathcal{L}_{n+m}(x) \end{pmatrix} \\ = &\begin{pmatrix} \mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\ \mathcal{L}_1(x) & d(x)\mathcal{L}_0(x) \end{pmatrix} Q^n(x)Q^m(x) \\ = &\begin{pmatrix} \mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x) \\ \mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x) \end{pmatrix} \begin{pmatrix} \frac{1}{a}\mathcal{F}_{m+1}(x) & \frac{d(x)}{a}\mathcal{F}_m(x) \\ \frac{1}{a}\mathcal{F}_m(x) & \frac{d(x)}{a}\mathcal{F}_{m-1}(x) \end{pmatrix}. \end{aligned}$$

Then by the $(2,2)$ entry of the first matrix in the above equality, we have

$$a\mathcal{L}_{n+m}(x) = \mathcal{L}_{n+1}(x)\mathcal{F}_m(x) + d(x)\mathcal{L}_n(x)\mathcal{F}_{m-1}(x)$$

for each $n, m \in \mathbb{N}$. □

Example 3.9 *Let $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ be the Jacobsthal polynomial $J_n(x)$ and the Jacobsthal-Lucas polynomial $\Lambda_n(x)$ respectively, as defined in Table 1. Then $\Lambda_0(x) = J_1(x) = 1$ which satisfies the condition in Proposition 3.8. Hence we have the following equality for $J_n(x)$ and $\Lambda_n(x)$:*

$$\Lambda_{n+m}(x) = \Lambda_{n+1}(x)J_m(x) + 2x\Lambda_n(x)J_{m-1}(x).$$

Proposition 3.10 *Let $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ be the Fibonacci type polynomial and Lucas type polynomial respectively with $\mathcal{L}_0(x) = \mathcal{F}_1(x) = a$. Let $d(x) \neq 0$ for each $x \in \mathbb{R}$. Then for each $n, m \in \mathbb{N}$ with $n \geq m$,*

$$a(-d(x))^m \mathcal{L}_{n-m}(x) = \mathcal{L}_n(x)\mathcal{F}_{m+1}(x) - \mathcal{L}_{n+1}(x)\mathcal{F}_m(x).$$

Proof. By Theorem 3.2 and $Q^{n-m}(x) = Q^n(x)Q^{-m}(x)$, we have

$$\begin{aligned} &\begin{pmatrix} \mathcal{L}_{n-m+2}(x) & d(x)\mathcal{L}_{n-m+1}(x) \\ \mathcal{L}_{n-m+1}(x) & d(x)\mathcal{L}_{n-m}(x) \end{pmatrix} \\ = &\begin{pmatrix} \mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\ \mathcal{L}_1(x) & d(x)\mathcal{L}_0(x) \end{pmatrix} Q^n(x)Q^{-m}(x) \\ = &\begin{pmatrix} \mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x) \\ \mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x) \end{pmatrix} \frac{1}{(-d(x))^m} \begin{pmatrix} \frac{d(x)}{a}\mathcal{F}_{m-1}(x) & -\frac{d(x)}{a}\mathcal{F}_m(x) \\ -\frac{1}{a}\mathcal{F}_m(x) & \frac{1}{a}\mathcal{F}_{m+1}(x) \end{pmatrix}. \end{aligned}$$

Then considering the $(2,2)$ entry of the first matrix in the above equality, we have

$$a(-d(x))^m \mathcal{L}_{n-m}(x) = \mathcal{L}_n(x)\mathcal{F}_{m+1}(x) - \mathcal{L}_{n+1}(x)\mathcal{F}_m(x).$$

□

Example 3.11 *Let $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ be the Jacobsthal polynomial $J_n(x)$ and the Jacobsthal-Lucas polynomial $\Lambda_n(x)$ respectively. Then $\Lambda_0(x) = J_1(x) = 1$ and*

$$(-2x)^m \Lambda_{n-m}(x) = \Lambda_n(x)J_{m+1}(x) - \Lambda_{n+1}(x)J_m(x).$$

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