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The Damped Harmonic Oscillator at the Classical Limit of the Snyder-de Sitter Space

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Abstract

Valtancoli in his paper entitled (P. Valtancoli, *Canonical transformations and minimal length*, J. Math. Phys. 56, 122107 2015) has shown how the deformation of the canonical transformations can be made compatible with the deformed Poisson brackets. Based on this work and through an appropriate canonical transformation, we solve the problem of one dimensional (1D) damped harmonic oscillator at the classical limit of the Snyder-de Sitter (SdS) space. We show that the equations of the motion can be described by trigonometric functions with frequency and period depending on the deformed and the damped parameters. We eventually discuss the influences of these parameters on the motion of the system.

Keywords: deformed Heisenberg algebra, damped harmonic oscillator, deformed poisson bracket, deformed canonical transformation, non-commutative geometry, Snyder-de Sitter space

1. Introduction

The search of quantum gravity is one the active field of research that has attracted much attention in the last decades. Among all the candidate theories (Amati, Ciafaloni, & Veneziano, 1989; Scardigli,1999; Rovelli, & Smolin, 1995) to address this problem, the non-commutative geometry seems to be the promising approach to quantum gravity. In this sense, more recently, Lawson has proposed a model of quantum non-commutative geometry (Lawson, 2020; Lawson 2021) that may describe the space-time at Planck scale. The interesting physical result obtained in this theory which differs from similar theories (Fring, Gouba,& Scholz, 2010; Lawson, Gouba, & Avossevou, 2017) is that, this non-commutative space-time leads to minimal and maximal lengths of graviton for simultaneous measurement. The existence of this maximal length which is the basic difference from the one of the minimal length scenario, brings a lot of new features in the representation of this space and could be the approach candidate to the measurement of quantum gravity with energies currently accessible in a laboratory.

However, more general models of non-commutative spaces exist and describe the structure of spacetime at short distance and preserve the Lorentz invariance. The best known is the Snyder model (Snyder, 1947), which is the first attempt of introducing a fundamental length scale and is invariant under the Lorentz group. This model is generalized to a spacetime background of constant curvature namely the Snyder-de Sitter (SdS) model (Kowalski-Glikman,& Smolin, 2004). It is an example of non-commutative spacetime admitting three fundamental parameters and is also called Triply Special Relativity (TSR) and is invariant under the action of the de Sitter group (Kowalski-Glikman,& Smolin, 2004). Some applications of this work have been done in (Mignemi, 2009) and the model of a free particle and of an oscillator have been solved in this framework.

A characteristic of non-commutative spaces is that the corresponding classical phase space is not canonical i.e. the Poisson brackets do not have the usual form. Thus, at the classical limit of the SdS space the solutions of free particle and harmonic oscillator systems had been also obtained by substituting the generalized commutations with the deformed Poisson brackets (Mignemi, 2012). Hence, in this paper we are interesting in the study of a one dimensional (1D) damped harmonic oscillator (Kanai, 1948; Caldirola, 1914), in the deformed Poisson brackets. Since this system is explicitly time-dependent, to determine its solutions of motion, we first make the Hamiltonian simpler by means of a suitable canonical transformation. Then we solve the equations of motion for the new canonical variables in the deformed Poisson brackets. We provide the solutions of the motion in terms of trigonometric functions where the frequency and the period of the motion depend on the deformed and the damped parameters. We show that when the damped parameter is less than a certain value of the deformed parameter, the gravity induces faster motion of system but when it is greater than this value the dissipation slows down the motion of the system.

This paper is organized as follows. In section (2), we review some properties of quantum SdS model and classical limit of SdS model in 1D. We extend in section (3), the classical procedure on the canonical transformation. Based on this study, we explicitly solve in section (4) the 1D damped harmonic oscillator in the classical limit of SdS model. We provide the solutions of the motion in terms of trigonometric functions where we discuss the influence of the deformed parameters and the friction parameters on the motion of the system. We present our conclusion in section (5).

2. Classical Snyder-de Sitter Space

The non relativistic quantum SdS model i.e the Snyder model restricted to a three-dimensional sphere is generated by the positions operators $\hat{x}_\mu$, the momentum operators $\hat{p}_\nu$ and the Lorentz generators $\hat{J}_{\mu\nu}$ such as

$$\begin{aligned} [\hat{x}_\mu, \hat{x}_\nu] &= i\beta^2 \hat{J}_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = i\alpha^2 \hat{J}_{\mu\nu}, \quad \mu, \nu = 0, ...3 \\ [\hat{x}_\mu, \hat{p}_\nu] &= i\left(\eta_{\mu\nu} + \alpha^2 \hat{x}_\mu \hat{x}_\nu + \beta^2 \hat{p}_\mu \hat{p}_\nu + \alpha\beta(\hat{x}_\mu \hat{p}_\nu + \hat{p}_\mu \hat{x}_\nu - \hat{J}_{\mu\nu})\right). \end{aligned} \tag{1}$$

where $\eta = diag(-1, 1, 1, 1)$ is the flat metric and the coupling constants β, α ($\alpha\beta \ll 1$) have dimension of inverse length and inverse mass, respectively. They are usually identified with the square root of the cosmological constant $\alpha = 10^{-24} cm^{-1}$ and with the inverse of the Planck mass, $\beta = 10^5 g^{-1}$ (Mignemi, 2009) . The Lorentz generator with their standard action on the position and momentum operators $\hat{x}_\mu$ nd $\hat{p}_\nu$ satisfy the usual commutation relations such as

$$\begin{aligned} [\hat{J}_{\mu\nu}, \hat{p}_\mu] &= i(\eta_{\nu\lambda}\hat{p}_\mu - \eta_{\nu\lambda}\hat{p}_\nu), \quad [\hat{J}_{\mu\nu}, \hat{p}_\mu] = i(\eta_{\nu\lambda}\hat{x}_\mu - \eta_{\mu\lambda}\hat{x}_\nu), \\ [\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] &= i(\eta_{\nu\rho}\hat{J}_{\nu\sigma} - \eta_{\sigma\mu}\hat{J}_{\rho\nu} - \eta_{\sigma\nu}\hat{J}_{\rho\mu}), \qquad \hbar = 1. \end{aligned} \tag{2}$$

The limit $\alpha \to 0$ the SdS space (1) gives the flats Snyder space (Snyder, 1947)

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 \hat{J}_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{x}_\mu, \hat{p}_\nu] = i\left(\eta_{\mu\nu} + \beta^2 \hat{p}_\mu \hat{p}_\nu\right), \tag{3}$$

while the limit $\beta \to 0$ yields the Heisenberg algebra of quantum mechanics in a de Sitter background endowed with projective coordinates (Mignemi, 2012).

In one-dimensional case, the algebra (1) is reduced into

$$[\hat{x}, \hat{p}] = i\left(1 + \alpha^2 \hat{x}^2 + \beta^2 \hat{p}^2 + \alpha\beta(\hat{x}\hat{p} + \hat{p}\hat{x})\right), \quad [\hat{x}, \hat{x}] = 0 = [\hat{p}, \hat{p}]. \tag{4}$$

and for the simple case $\langle \hat{x} \rangle = 0 = \langle \hat{p} \rangle$, the uncertainty relation is given by

$$\Delta x \Delta p \geq \frac{1}{2} \frac{\alpha^2 \Delta x + \beta^2 \Delta p}{1 + \beta\alpha}. \tag{5}$$

If $\alpha, \beta > 0$, they imply the existence of both minimal position and momentum uncertainties, given by

$$\Delta x = \frac{\beta}{\sqrt{1 + 2\alpha\beta}} = \beta(1 - \alpha\beta), \quad \Delta p = \frac{\alpha}{\sqrt{1 + 2\alpha\beta}} = \alpha(1 - \alpha\beta). \tag{6}$$

For $\alpha, \beta < 0$, no minimal uncertainties emerge (Mignemi, 2012).

In one dimensional classical limit, the commutator (4) is replaced by the deformed Poisson bracket

$$\{x, p\} = 1 + (\alpha x + \beta p)^2, \quad \{x, x\} = 0 = \{p, p\} \tag{7}$$

The equations of motion governed by the classical Hamiltonian $H(x, p)$ are given by

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial p}\{x, p\} = \frac{\partial H}{\partial p}\left(1 + (\alpha x + \beta p)^2\right), \tag{8}$$

$$\dot{p} = \{p, H\} = -\frac{\partial H}{\partial x}\{x, p\} = -\frac{\partial H}{\partial x}\left(1 + (\alpha x + \beta p)^2\right). \tag{9}$$

3. Deformed Canonical Transformation

It is well known in the formulation of classical mechanics that, the transition from the canonical variables x and p to new arbitrary canonical variables X and P which is called canonical transformation lets the physics of the system invariant. Therefore, the canonical transformation of the variables x and p into the variables X and P defines a new Hamiltonian $K(X, P)$. It is defined by the map

$$x, p, H(p, x) \longmapsto X, P, K(X, P), \tag{10}$$

and the fundamental Poisson brackets read

$$\begin{aligned} \{X, P\}_{X,P} &= \{X, P\}_{x,p} = \{x, p\}_{X,P} = 1, & (11) \\ \{X, X\}_{X,P} &= \{P, P\}_{X,P} = 0. & (12) \end{aligned}$$

The equations of motion are given by

$$\dot{X} = \{X, K\}, \quad \dot{P} = \{P, K\}. \tag{13}$$

Moreover, under this transformation the old Hamiltonian $H(P, X)$ is transformed into the new Hamiltonian as follows

$$K = H + \frac{\partial F}{\partial t}, \tag{14}$$

where F is the generating function. Now, if we are convinced that the Poisson brackets are invariant under the canonical transformation, then the deformation of the Poisson brackets must be also invariant under this transformation (Valtancoli, 2015), i.e

$$\{x, p\} = 1 + (\alpha x + \beta p)^2 \longrightarrow \{X, P\} = 1 + (\alpha X + \beta P)^2. \tag{15}$$

The time evolution of the coordinate X and the momentum P are given by

$$\begin{aligned} \dot{X} &= \{Q, K\} = \frac{\partial K}{\partial P}\left(1 + (\alpha X + \beta P)^2\right), & (16) \\ \dot{P} &= \{P, K\} = -\frac{\partial K}{\partial X}\left(1 + (\alpha X + \beta P)^2\right). & (17) \end{aligned}$$

We are now in position to apply all of these aspects on the damped harmonic oscillator in order to study its equations of motion.

4. Damped Harmonic Oscillator in SdS Space

The damped harmonic oscillator is one of the most fascinating systems that have remained over years a constant source of inspiration in quantum physics (Kanai, 1948; Caldirola, 1941). It has attracted much attention in the literature (Pal, Nandi, & Chakraborty, 2018; Um, Yeon, & George, 2002; Pedrosa,& de Lima, 2014; Jannussis, & Bartzis, 1988; Pedrosa,1987; Lawson, & Avossevou, 2018; Lawson, Sodoga,& Avossevou, 2021; Shang, 2009), since the problem related to this system is far from having a satisfactory solution. In fact the quantization of dissipative systems well known in the literature as Caldirola and Kanai system (Kanai, 1948; Caldirola, 1941) has been criticized for violating certain laws of quantum theory. Recently, a simple and complete solution has been provided (Lawson, Avossevou, & Gouba, 2018) to this problem using the Lewis-Riesenfeld procedure (Lewis, & Riesenfeld 1969). Therefore, in the present situation we are interested in the classical motion of this system in the minimal length scenario. In one dimension, its Hamiltonian is given by

$$H(p, x) = e^{-\gamma t}\frac{p^2}{2} + \frac{\omega_0^2}{2}x^2 e^{\gamma t}, \tag{18}$$

where we set $m = 1$, γ is the constant coefficient of friction and ω_0 is the time-independent harmonic frequency. Since this Hamiltonian is explicitly time-dependent, it does not represent a conserved quantity. So, we achieve its energy conservation through the time-dependent canonical transformation given by the generating function of the second kind (Pedrosa, & de Lima 2014)

$$F(x, p, t) = xpe^{\frac{\gamma t}{2}} - \frac{\gamma}{4}x^2 e^{\gamma t}. \tag{19}$$

Perfoming the following transformation equations

$$X = \frac{\partial F}{\partial p}, \;\; P = \frac{\partial F}{\partial x}, \tag{20}$$

we obtain new canonical variables

$$X = xe^{\frac{\gamma}{2}t}, \quad P = pe^{-\frac{\gamma t}{2}} + \frac{\gamma}{2}xe^{\frac{\gamma t}{2}}. \tag{21}$$

The relations between old (p, x) and new (P, X) coordinates can be written as

$$\begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} e^{\frac{\gamma}{2}t} & -\frac{\gamma}{2}e^{\gamma t} \\ 0 & e^{-\frac{\gamma}{2}t} \end{pmatrix}\begin{pmatrix} P \\ X \end{pmatrix}, \quad \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} e^{-\frac{\gamma}{2}t} & \gamma e^{\frac{\gamma}{2}t} \\ 0 & e^{\frac{\gamma}{2}t} \end{pmatrix}\begin{pmatrix} p \\ x \end{pmatrix}. \tag{22}$$

Their Poisson brakets are canonical

$$\{X, P\} = 1 = \{x, p\}. \tag{23}$$

The transformed Hamiltonian is given by

$$K(X, P, t) = \frac{P^2}{2} + \frac{\omega^2}{2}X^2, \tag{24}$$

where

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}, \tag{25}$$

is the modified frequency. Since the new Hamiltonian is time independent, its conserved energy is

$$E = \frac{P^2}{2} + \frac{\omega^2}{2}X^2. \tag{26}$$

The deformed Poisson brackets between the transformed canonical variables are

$$\{X, P\} = 1 + (\alpha X + \beta P)^2, \quad \{X, X\} = 0 = \{P, P\}. \tag{27}$$

The equations of motion read as

$$\dot{X} = \left(1 + (\alpha X + \beta P)^2\right)P, \tag{28}$$

$$\dot{P} = -\omega^2\left(1 + (\alpha X + \beta P)^2\right)X. \tag{29}$$

After some computations, the solutions of the above equations are given by

$$X = A(t)\left(\frac{\alpha}{\omega}\sin\varpi t - \beta\sqrt{1 + 2\kappa E}\cos\varpi t\right) \tag{30}$$

$$P = B(t)\left(\beta\sin\varpi t + \frac{\alpha}{\omega}\sqrt{1 + 2\kappa E}\cos\varpi t\right) \tag{31}$$

where

$$\begin{aligned} \kappa &= \beta^2 + \frac{\alpha^2}{\omega^2}, \quad \varpi = \omega\sqrt{1 + 2\kappa E}, \\ \omega A(t) &= B(t) = \sqrt{\frac{2E}{\kappa(1 + 2\kappa E\cos^2\varpi t)}}. \end{aligned} \tag{32}$$

The solutions are periodic, but not sinusoidal, and the frequency now depends on the energy of the oscillator. In the limit $\alpha \to 0$, one recovers the flat Snyder oscillator (Snyder, 1947).

Returning to the original variables $x(t)$ and $p(t)$ of the Hamiltonian (18) and with the help of the relation (21) we have:

$$x(t) = C(t)\left(\frac{\alpha\sin\Omega t}{\sqrt{\omega_0^2 - \gamma^2/4}} - \beta\sqrt{1 + 2\kappa E(t)}\cos\Omega t\right) \tag{33}$$

$$\begin{aligned} p(t) &= D(t)\left(\beta\sin\Omega t + \frac{\alpha\sqrt{1 + 2\kappa E(t)}\cos\Omega t}{\sqrt{\omega_0^2 - \gamma^2/4}}\right) \\ &\quad -G(t)\left(\frac{\alpha\sin\Omega t}{\sqrt{\omega_0^2 - \gamma^2/4}} - \beta\sqrt{1 + 2\kappa E(t)}\cos\Omega t\right) \end{aligned} \tag{34}$$

where

$$\Omega(t) = \sqrt{(\omega_0^2 - \gamma^2/4)(1 + 2\kappa E(t))} \quad (35)$$

$$C(t) = e^{-\frac{\gamma t}{2}} A(t) = \frac{e^{-\frac{\gamma t}{2}}}{(\omega_0^2 - \gamma^2/4)} \sqrt{\frac{2E(t)}{\kappa(1 + 2\kappa E(t)\cos^2 \Omega t)}} \quad (36)$$

$$D(t) = e^{\frac{\gamma t}{2}} B(t) = e^{\frac{\gamma t}{2}} \sqrt{\frac{2E(t)}{\kappa(1 + 2\kappa E(t)\cos^2 \Omega t)}} \quad (37)$$

$$G(t) = \frac{\gamma e^{\gamma t}}{2} C(t). \quad (38)$$

In first-order of the deformed parameter κ and the damped parameter γ, the frequency Ω and the period T of the motion are given by

$$\Omega = \omega_0 \left(1 - \frac{\gamma^2}{8\omega_0^2} + \kappa E(t)\right), \quad (39)$$

$$T = \frac{2\pi}{\omega_0} \left(1 + \frac{\gamma^2}{8\omega_0^2} - \kappa E(t)\right), \quad (40)$$

where we neglected the term of order $\beta\gamma^2$. To zero-order in κ and γ i.e for $\kappa = 0 = \gamma$, we recover the ordinary solutions of harmonic oscillator with frequency ω_0 and period $T_0 = \frac{2\pi}{\omega_0}$. If $\gamma < 2\omega_0 \sqrt{2\kappa E(t)}$, the period of the motion is shorter than the ordinary one $T < T_0$, i.e the deformation of Poisson bracket which is the manifestation of an effect of gravity induces a faster motion of the oscillator. But for $\gamma > 2\omega_0 \sqrt{2\kappa E(t)}$, the period of the motion increase which means that the dissipation of the system slows down the motion of the oscillator.

5. Conclusion

In this article, we first studied the properties of the 1D deformed Poisson brackets which are regarded as the classical limit of the generalized Heisenberg commutators. Then, we reviewed the innovative concept of β-canonical transformation introduced by Valtancoli (Valtancoli; 2015) appointed in this work as the deformed canonical transformation. This concept allowed us to maintain the invariance of the deformation of the Poisson brackets in the reparametrization of the phase space variables. We applied all these theories to the well-known one dimensional damped harmonic oscillator. With an appropriate canonical transformation, we transformed the time-dependent Hamiltonian into the time-independent Hamiltonian. Based on (Valtancoli; 2015) we have shown that the solutions of the equations of motion can be expressed in terms of trigonometric functions with the frequency and period depending on the deformed and the damped parameters of the system. Finally, we discussed the influences these parameters on the motion. We have shown that when the deformed parameter is greater than the damped parameter the period of the motion is shorter, conversely the period increases when the damped parameter is greater than the deformed parameter.

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