To the Question of Gauss's Curvature in *n*-Dimensional Euclidian Space

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Abstract

An alternative way of calculating the Gauss curve of the surface in the Cartesian coordinates in a three-dimensional case has been proposed, and its generalization in the *n*- dimension case of measuring Euclidean space has been given.

Keywords: Gaussian curvature, corner element, Jacobian

1. Introduction

The issue that is touched upon in this work is directly related to differential geometry, and it is related to the common properties of smooth surfaces. It must be said that in this direction there are a lot of original publications and monographs (see, for example, Mac Connell (1957), Rashevsky (1967), Fihtenholtz (1969), Deepmala (2015), Kisi, et all (2016), Sezgin Buyukkutuk, et. all (2016), Laurian-Ioan, et. all (2017), Vandana, et. all (2016), Laurian-Ioan, et. all (2017)) in which all the basic properties of surfaces are presented in great detail and with strict mathematical evidence, including detailed calculations of their curvature radius R.

However, the approach that will now be demonstrated is very different from the methodical approach outlined in the classical monograph Rashevsky (1967), and it is based on a qualitatively different assumption.

The essence of it is this. As is customary in differential geometry, all calculations are reduced to an analysis of dependency not by the curvature radius R, but by the curvature k associated with R a simple inversion $k = \frac{1}{R}$ conversion, in the case of a flat task. In a three-dimensional case, the curvature is usually introduced in a slightly different way, namely, in the form of $K = \frac{1}{R'^2}$.

In this work, we will approach this problem, based on the idea of a large-scale theory, which allows using a simple algorithm to offer a very compact and strict conclusion for the curvature of the surface K, allowing to bring its generalization to case $n-$ of measuring Euclidean space.

This task will be part of the first part of this work. In the second part of the article, we will give a simple summary of the proposed approach for the case of arbitrary curved coordinates.

1. Euclidean $n-$ Is a Dimensional Space

Recall at first some commonly used formulas from the theory of differential geometry. As the main calculation formula for solving various tasks from mechanics and physics related to the theory of two-dimensional curved motion, as a rule, the expression is used

$$k = \frac{1}{R} = \frac{y''}{\left(1 + y'^2\right)^{\frac{3}{2}}}, \tag{1}$$

where $y = y(x)-$ is the trajectory of the body in the coordinates $x-y$ and strokes traditionally indicates the appropriate derivatives.

The formula (1) is easily derived from a simple large-scale identity

$$k = \frac{1}{R} = \frac{d\alpha}{dl}, \tag{2}$$

where $dl = dx\sqrt{1 + y'^2} -$ is a curve arc length element, and the angle α is a sharp angle between the tangent drawn to

this element of the arc dl and the absciss axis x.

Thus, it becomes quite clear that if we are talking about a three-dimensional case, instead of equality (2) we should write a ratio $R^2 dO = dS$, where $dO = \sin\alpha d\alpha d\beta -$ is the element of the body angle, and $dS -$ is the surface element corresponding to this element of the body angle, with the radius of curvature R.

This means that the curvature in three-dimensional space we can enter as a fraction

$$K_2 = \frac{1}{R^2} = \frac{dO_2}{dS_2}. \tag{3}$$

It is quite clear that by following the algorithm of composing expressions (2) and (3) they can be easily summarized in case of space of arbitrary dimension n, and write down the general expression for curvature in the form of

$$K_{n-1} = \frac{1}{R^{n-1}} = \frac{d\omega_{n-1}}{d\Omega_{n-1}}, \tag{4}$$

where $d\omega_{n-1} -$ is body angle element covering the surface element $d\Omega_{n-1}$ of the $n-$ dimensional space.

An element $d\omega_{n-1}$ calculates perfectly trivial, as well as the hypersurface element $d\Omega_{n-1}$, which we will now demonstrate on a simple example of three-dimensional Euclidean space.

With this goal let's use a simple geometric interpretation. Indeed, let the surface be set by a clear equation $z = z(x,y)$.

Then from the decomposition the radius - vector on a single orthogonal basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the accordance to the expression $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z(x,y)\mathbf{k}$, follows that the slanted base on the surface $z = z(x,y)$ is

$$\begin{cases} \mathbf{e}_1 = \dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{i} + z_x \mathbf{k}, \\[2mm] \mathbf{e}_2 = \dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + z_y \mathbf{k}, \end{cases} \tag{5}$$

where the cuts are introduced $z_x = \dfrac{\partial z}{\partial x}, \ z_y = \dfrac{\partial z}{\partial y}$ and $\mathbf{e}_1 \cdot \mathbf{e}_2 \neq 0$.

Therefore, the surface element will be

$$dS = \left| \mathbf{e}_1 \times \mathbf{e}_2 \right| dxdy = \sqrt{1 + z_x^2 + z_y^2}\, dxdy = qdxdy. \tag{6}$$

As to the element of the bodily angle dO_2 we have

$$dO_2 = \sin\alpha d\alpha d\beta. \tag{7}$$

Calculating the expression (7) is quite simple. Really, the projections of the curvature radius on the corresponding axis coordinates are

$$\begin{cases} R_x = R\sin\alpha\cos\beta, \\ R_y = R\sin\alpha\sin\beta, \\ R_z = R\cos\alpha, \end{cases} \tag{8}$$

where the angles α, β are respectively azimuth and polar angles and, as usual, they are enclosed in segments $0 \leq \alpha \leq \pi, \ 0 \leq \beta \leq 2\pi$

On the other side, the projection of the single normal vector to the surface dS is defined as

$$\begin{cases} n_x = \dfrac{R_x}{R} = \sin\alpha\cos\beta = -\dfrac{z_x}{q}, \\[2mm] n_y = \dfrac{R_y}{R} = \sin\alpha\sin\beta = -\dfrac{z_y}{q}, \\[2mm] n_z = \dfrac{R_z}{R} = \cos\alpha = \dfrac{1}{q}. \end{cases} \tag{9}$$

Where does it come

$$\begin{cases} tg\,\beta = \dfrac{z_y}{z_x}, \\[2mm] \cos\alpha = \dfrac{1}{\sqrt{1+z_x^2+z_y^2}}. \end{cases} \tag{10}$$

This means that Jacobians moving from variables β, α to variables x, y can be easily calculated by following the formula

$$I = \begin{vmatrix} \dfrac{\partial\beta}{\partial x} & \dfrac{\partial\beta}{\partial y} \\[3mm] \dfrac{\partial\alpha}{\partial x} & \dfrac{\partial\alpha}{\partial y} \end{vmatrix}. \tag{11}$$

With the help expr. (10), performing a simple differentiation, we find that

$$I = \frac{\left(z_x^2+z_y^2\right)\left(z_{xx}z_{yy}-z_{xy}^2\right)\cos^2\beta}{q^3 z_x^2 \sin\alpha}. \tag{12}$$

Substituting (12) in expr. (7) and expressing $\cos\beta$ through $tg\,\beta$ according to (10), for the element of the bodily angle come to the next expression

$$d\mathrm{O}_2 = \sin\alpha\, d\alpha\, d\beta = \frac{\left(z_{xx}z_{yy}-z_{xy}^2\right)}{q^3} dx\,dy = \frac{dx\,dy}{\left(1+z_x^2+z_y^2\right)^{\frac{3}{2}}} \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix}. \tag{13}$$

This means that the desired curvature of the surface according to (3) taking into account (6) and (13) will be such as

$$K_2 = \frac{1}{R^2} = \frac{1}{\left(1+z_x^2+z_y^2\right)^2} \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix}. \tag{14}$$

Note that the formula (14) has exactly the same look as Rashevsky (1967). That is the approach we have outlined is quite correct.

It is quite clear that in a four-dimensional case, if the three-dimensional surface is set as a clear dependence $u = u(x, y, z)$, the element of the bodily angle will be

$$d\omega_3 = \sin^2\alpha\sin\beta\, d\alpha\, d\beta\, d\gamma = \frac{dx\,dy\,dz}{\left(1+u_x^2+u_y^2+u_z^2\right)^2} \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix}. \tag{15}$$

Conversions are set in the form of

$$\begin{cases} R_x = R\sin\alpha\sin\beta\cos\gamma, \\ R_y = R\sin\alpha\sin\beta\sin\gamma, \\ R_z = R\sin\alpha\cos\beta, \\ R_u = R\cos\alpha. \end{cases} \tag{16}$$

By putting here $\alpha = \dfrac{\pi}{2}$, we come to transformations (8). Jacobian crossing here, obviously is $I = R^2\sin^2\alpha\sin\beta$. That's

why the body angle element

$$d\omega_3 = \sin^2\alpha\sin\beta\,d\alpha\,d\beta\,d\gamma . \tag{17}$$

This is reflected in the formula (15). The three – dimensional hypersurface element according to (6) in the accordance with the designations (4) is

$$d\Omega_3 = \left|\mathbf{e}_1 \times [\mathbf{e}_2 \times \mathbf{e}_3]\right| dxdydz = \sqrt{1 + u_x^2 + u_y^2 + u_z^2}\,dxdydy = q_3 dxdydz , \tag{18}$$

Hence due to the determination (4) curvature will be determined privately from expressions (15) and (18). That is

$$K_3 = \frac{1}{R^3} = \frac{d\omega_3}{d\Omega_3} = \frac{1}{\left(1 + u_x^2 + u_y^2 + u_z^2\right)^{\frac{5}{2}}} \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix}. \tag{19}$$

Thus, the curvature computation algorithm is generally absolutely clear, and we can summarize the formulas (13), (15) and (19) in case of $n-$ arbitrary dimensional Euclidean space.

Indeed, if we are enter designations for independent variables in the form of a vector $\mathbf{x}$ with coordinates

$$\mathbf{x} = \left(x_1, x_2, x_3, \dots x_{n-1}\right), \tag{20}$$

and set the surface with a clear equation

$$u = u(\mathbf{x}), \tag{21}$$

that instead of (14) and (19) we will have

$$K_{n-1} = \frac{1}{R^{n-1}} = \frac{1}{\left(1 + u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2 + \dots u_{x_{n-1}}^2\right)^{\frac{n+1}{2}}} \begin{vmatrix} u_{x_1 x_1} & u_{x_1 x_2} & u_{x_1 x_3} & \dots & u_{x_1 x_{n-1}} \\ u_{x_2 x_1} & u_{x_2 x_2} & u_{x_2 x_3} & \dots & u_{x_2 x_{n-1}} \\ u_{x_3 x_1} & u_{x_3 x_2} & u_{x_3 x_3} & \dots & u_{x_3 x_{n-1}} \\ \dots & \dots & \dots & \dots & \dots \\ u_{x_{n-1} x_1} & u_{x_{n-1} x_2} & u_{x_{n-1} x_3} & \dots & u_{x_{n-1} x_{n-1}} \end{vmatrix}. \tag{22}$$

Since we are interested in the curvature that can be written in the form $K = \dfrac{1}{R^2}$, according to (22) it can be imagined as

$$K = \frac{1}{R^2} = \frac{1}{\left(1 + u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2 + \dots u_{x_{n-1}}^2\right)^{\frac{n+1}{n-1}}} \begin{vmatrix} u_{x_1 x_1} & u_{x_1 x_2} & u_{x_1 x_3} & \dots & u_{x_1 x_{n-1}} \\ u_{x_2 x_1} & u_{x_2 x_2} & u_{x_2 x_3} & \dots & u_{x_2 x_{n-1}} \\ u_{x_3 x_1} & u_{x_3 x_2} & u_{x_3 x_3} & \dots & u_{x_3 x_{n-1}} \\ \dots & \dots & \dots & \dots & \dots \\ u_{x_{n-1} x_1} & u_{x_{n-1} x_2} & u_{x_{n-1} x_3} & \dots & u_{x_{n-1} x_{n-1}} \end{vmatrix}^{\frac{2}{n-1}} \tag{23}$$

Formula (23) adequately responds to the question of the curvature radius of any $n-$ dimensional Euclidean surface set by an explicit equation (21) set at the beginning of the article.

As an example, consider the three-dimensional of the sphere of radius b, the implicit equation of which will be set in the form of

$$x^2 + y^2 + z^2 = b^2. \tag{24}$$

Because

$$z_x = -\frac{x}{z}, \ \ z_y = -\frac{y}{z}, \ \ z_{xx} = \frac{y^2 - b^2}{z^3}, \ \ z_{yy} = \frac{x^2 - b^2}{z^3}, \ \ z_{xy} = -\frac{xy}{z^3},$$

then, substituting them in the formula (14), we are immediately come to the desired answer $R = b$. As it should be. Let's move on to the second part of the work now.

3. Curved Coordinates

In this section, we'll elaborate on calculating the curvature of the surface in a three-dimensional case, if the coordinate conversions are set in a parametric form, just as in the Rashevsky (1967), Fichtenholtz (1969)

$$x = x(u, v), \ \ y = y(u, v), \ \ z = z(u, v). \tag{25}$$

As a result, the base vectors on the surface are

$$\begin{cases} \mathbf{e}_1 = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k}, \\[2mm] \mathbf{e}_2 = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k} \end{cases} \tag{26}$$

and the surface element

$$dS_2 = |\mathbf{e}_1 \times \mathbf{e}_2| \, d\,udv = Dd\,udv, \tag{27}$$

where

$$D = \sqrt{I_1^2 + I_2^2 + I_3^2}, \tag{28}$$

where abbreviated designations are introduced as

$$I_1 = n_x = \frac{\partial(y, z)}{\partial(u, v)}, \ \ I_2 = n_y = -\frac{\partial(x, z)}{\partial(u, v)}, \ \ I_3 = n_z = \frac{\partial(x, y)}{\partial(u, v)}. \tag{29}$$

And the expressions of the type $\dfrac{\partial(x, y)}{\partial(u, v)}$ are nothing more than Jacobian transitions, that is, $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$.

Next. Because a single normal vector to the surface is defined as

$$\mathbf{n} = \frac{\mathbf{N}}{N} = \frac{1}{D}(I_1\mathbf{i} + I_2\mathbf{j} + I_3\mathbf{k}), \tag{30}$$

the projections of the normal vector will be

$$\begin{cases} n_x = \dfrac{R_x}{R} = \sin\alpha\cos\beta = \dfrac{I_1}{D}, \\[2mm] n_y = \dfrac{R_y}{R} = \sin\alpha\sin\beta = \dfrac{I_2}{D}, \\[2mm] n_z = \dfrac{R_z}{R} = \cos\alpha = \dfrac{I_3}{D}. \end{cases} \tag{31}$$

Therefore, for angular variables, we have

$$\beta = arctg \frac{I_2}{I_1},$$

$$\alpha = \arccos \frac{I_3}{D}. \tag{32}$$

Thus, the Jacobians transition from angular variables α, β to new variables u, v we carry out according to the formula

$$I = \frac{\partial(\beta, \alpha)}{\partial(u, v)} = \frac{\partial \beta}{\partial u}\frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial v}\frac{\partial \alpha}{\partial u} =$$

$$= \frac{1}{D\left(I_1^2 + I_2^2\right)^{\frac{3}{2}}}\left[\left(I_{2u}I_1 - I_{1u}I_2\right)\left(I_{3v}D - I_3 D_v\right) - \left(I_{2v}I_1 - I_{1v}I_2\right)\left(I_{3u}D - I_3 D_u\right)\right]. \tag{33}$$

where lower u, v indices mean appropriate according private derivatives.

Therefore, the element of the bodily angle will be

$$dO_2 = \sin\alpha d\alpha d\beta = \sin\alpha I d u\, d v =$$

$$= \frac{1}{D^2\left(I_1^2 + I_2^2\right)}\left[\left(I_{2u}I_1 - I_{1u}I_2\right)\left(I_{3v}D - I_3 D_v\right) - \left(I_{2v}I_1 - I_{1v}I_2\right)\left(I_{3u}D - I_3 D_u\right)\right] d u\, d v. \tag{34}$$

Therefore, taking into account the surface element (27) curvature there is

$$K_2 = \frac{1}{R^2} = \frac{dO_2}{dS_2} =$$

$$= \frac{1}{D^3\left(I_1^2 + I_2^2\right)}\left[\left(I_{2u}I_1 - I_{1u}I_2\right)\left(I_{3v}D - I_3 D_v\right) - \left(I_{2v}I_1 - I_{1v}I_2\right)\left(I_{3u}D - I_3 D_u\right)\right]. \tag{35}$$

(cf. with a similar formula given in the Rashevsky (1967) see also the works Gladkov (2019)).

The formula (34) answers the question of the element of the bodily angle, the expression for which we have not found in the literature we know, as well as for expressions (13), (17), (19) and (22).

4. Conclusion

Ending this short message, once again it is worth noting a few main points.

1. An alternative calculation of gauss curvature by calculating the element of the bodily angle is proposed;
2. Shows the complete correspondence of the results obtained with its help with the classic;
3. Summary of formula for curvature of an arbitrary surface for an $n-$ dimensional case.

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