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On the Construction of Approximate Solutions for the 1D Pollutant Transport Model

Yacouba ZONGO[1], Brahima ROAMBA[1,2] & Boulaye YIRA[2]

[1] UFR/ST, Université Nazi Boni, 01 BP 1091 Bobo-Dioulasso, Burkina Faso

[2] IUT, Université Nazi Boni, 01 BP 1091 Bobo-Dioulasso, Burkina Faso

Correspondence: Brahima ROAMBA, UFR/ST, Université Nazi Boni, 10 BP 1091 Bobo-Dioulasso, Burkina Faso

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Abstract

The purpose of this paper is to build sequences of suitably smooth approximate solutions to the 1D pollutant transport model that preserve the mathematical structure discovered in (Roamba, Zabsonré, Zongo, 2017). The stability arguments in this paper then apply to such sequences of approximate solutions, which leads to the global existence of weak solutions for this model. We show that when the Reynold number goes to infinity, we have always an existence of global weak solutions result for the corresponding model.

Keywords: shallow water equations, bilayer models, viscosity, friction, capillarity, intermolecular forces, construction of weak solutions

1. Introduction

We consider a bilayer model of immiscible fluids where the upper layer can be represented by a Reynolds lubrifications model and the lower layer by a shallow water model. It can be used to simulate for instance the evolution of a pollutant fluid over water. A similar model was studied in (Fernandez-Nieto, Narbona-Reina & Zabsonré, 2017). The model reads as follows:

$$\partial_t h_1 + \partial_x(h_1 u) = 0, \tag{1}$$

$$\partial_t(h_1 u) + \partial_x(h_1 u^2) + \frac{1}{2}g\partial_x h_1^2 - 4\nu_1\partial_x(h_1\partial_x u) + \frac{u}{\beta} - h_1\partial_x(\sigma\partial_x^2 h_1 - V(h_1))$$

$$+r_1 h_1|u|^2 u + rgh_1\partial_x h_2 + rgh_2\partial_x(h_1 + h_2) = 0, \tag{2}$$

$$\partial_t h_2 + \partial_x(h_2 u) - \varepsilon\partial_x^2 h_2 - \partial_x\Big((a{h_2}^2 + b{h_2}^3)\partial_x p_2\Big) = 0, \tag{3}$$

with

$$\partial_x p_2 = \rho_2 g\partial_x(h_1 + h_2) \quad \text{and} \quad V(h_1) = \frac{1}{h_1^3} - \frac{\alpha}{h_1^4} \quad (\alpha > 0), \tag{4}$$

where $(t, x) \in (0, T)\times]0, 1[$.

These equations represent a system composed of two layers of immiscible fluids.

Where we denote h_1, h_2 respectively, the water and the pollutant heights, u is the water velocity. ν_1 is the kinematic viscosity and p_2 the pressure; g is the constant gravity. The coefficients σ, r_1 and β are respectively the coefficients of the intrefaz tension, quadratic friction and positive slip length parameters ; a and b respectively depend on the friction at the interfaz and coefficient of the viscosity of the pollutant. α, ε are positive constants. r is the ratio of densities given by $r = \dfrac{\rho_2}{\rho_1}$ where ρ_1 and ρ_2 denoted respectively the densities of the water and the pollutant. $V(h_1)$ represents the force of Van Der Waals which is given by $V(h_1) = \dfrac{1}{h_1^3} - \dfrac{\alpha}{h_1^4}$ $(\alpha > 0)$, see (Kitavtsev, Laurençot & Niethammer, 2011; Roamba, Zabsonré & Zongo, 2017; Seemann, Herminghaus & Jacobs, 2001).

We complete the system studied with the initial conditions

$$h_1(0, x) = h_{1_0}(x), \quad h_2(0, x) = h_{2_0}(x), \quad (h_1 u)(0, x) = \mathbf{m}_0(x) \quad \text{in } [0, 1]. \tag{5}$$

$$\begin{aligned} & h_{1_0},\quad h_{2_0} \in L^2(0,1), \quad \partial_x(h_{1_0}) \in L^2(0,1), \\ & \partial_x \mathbf{m}_0 \in L^1(0,1), \quad \mathbf{m}_0 = 0 \quad \text{if} \quad h_{1_0} = 0, \\ & \frac{|\mathbf{m}_0|^2}{h_{1_0}} \in L^1(0,1), \quad \varphi(h_{1_0}) \in L^1(0,1), \end{aligned} \tag{6}$$

where $\varphi(h_1) = 4\nu_1 \log h_1$.

The energy inequality associated to the system (1)-(3) is:

$$\begin{aligned} & \frac{d}{dt}\int_0^1 \left[\frac{1}{2}h_1|u|^2 + U(h_1) + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}\sigma|\partial_x h_1|^2\right] \\ & +4\nu_1\int_0^1 h_1|\partial_x u|^2 + \frac{1}{\beta}\int_0^1 |u|^2 + \frac{1}{2}gr\varepsilon\int_0^1 |\partial_x h_2|^2 \\ & \qquad +r_1\int_0^T\int_0^1 h_1|u|^4 + \rho_2 rg^2\int_0^1 h_2^2|\partial_x(h_1+h_2)|^2(a+bh_2) \le \frac{1}{2}rg\varepsilon\int_0^1 |\partial_x h_1|^2 \end{aligned} \tag{7}$$

where the potential function U is the indefinite integral of V defined by
$U(h_1) = -\dfrac{1}{2h_1^2} + \dfrac{\alpha}{3h_1^3}, \qquad h_1 > 0$. The entropy inequality associated with system (1)-(3) reads as

$$\begin{aligned} & \frac{d}{dt}\int_0^1 \left[\frac{1}{2}h_1|u+\partial_x\varphi(h_1)|^2 - \frac{1}{\beta}\varphi(h_1) + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}\sigma|\partial_x h_1|^2 + U(h_1)\right] \\ & +\frac{1}{\beta}\int_0^1 |u|^2 + 4\nu_1\int_0^1 (g + gr\frac{h_2}{h_1} + V'(h_1))|\partial_x h_1|^2 + rg\int_0^1 (\varepsilon + 4\nu_1\frac{h_2}{h_1})\partial_x h_1 \partial_x h_2 + 4\nu_1\sigma\int_0^1 |\partial_x^2 h_1|^2 \\ & +r_1\int_0^T\int_0^1 h_1|u|^4 + gr\varepsilon\int_0^1 |\partial_x h_2|^2 + rg^2\int_0^1 h_2^2(a+bh_2)\Big(\partial_x(h_1+h_2)\Big)^2 \le \frac{1}{2}rg\varepsilon\int_0^1 |\partial_x h_1|^2 \end{aligned} \tag{8}$$

We say that (h_1, h_2, u) is a weak solution of (1)-(3), with the initial condition verifying the entropy inequality (8) for all smooth test functions $\phi = \phi(t,x)$ with $\phi(T,.) = 0$, we have:

$$h_{0_1}\phi(0,.) - \int_0^T\int_0^1 h_1\partial_t\phi - \int_0^T\int_0^1 h_1 u\partial_x\phi = 0, \tag{9}$$

$$\begin{aligned} & -h_{0_2}\phi(0,.) - \int_0^T\int_0^1 h_2\partial_t\phi - \int_0^T\int_0^1 h_2 u\partial_x\phi + \varepsilon\int_0^T\int_0^1 \partial_x h_2\partial_x\phi \\ & \qquad + \int_0^T\int_0^1 \Big((ah_2{}^2 + bh_2{}^3)\partial_x p_2\Big)\partial_x\phi = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} & h_{01}u_0\phi(0,.) - \int_0^T\int_0^1 h_1 u\partial_t\phi - \int_0^T\int_0^1 h_1 u^2\partial_x\phi + 4\nu_1\int_0^T\int_0^1 h_1\partial_x u\partial_x\phi \\ & +\frac{1}{\beta}\int_0^T\int_0^1 u\phi + \int_0^T\int_0^1 (\sigma\partial_x^2 h_1 - V(h_1))\phi\partial_x h_1 + \int_0^T\int_0^1 (\sigma\partial_x^2 h_1 - V(h_1))h_1\partial_x\phi \\ & -\frac{1}{2}g\int_0^T\int_0^1 h_1^2\partial_x\phi - rg\int_0^T\int_0^1 h_2 h_1\partial_x\phi + r_1\int_0^T\int_0^1 h_1|u|^2 u\phi \\ & -rg\int_0^T\int_0^1 \phi h_2\partial_x h_1 - rg\int_0^T\int_0^1 (h_1+h_2)h_2\partial_x\phi - rg\int_0^T\int_0^1 (h_1+h_2)\partial_x h_2\phi = 0. \end{aligned} \tag{11}$$

This work follows the work done in (Roamba, Zabsonré & Zongo, 2017). In (Roamba, Zabsonré & Zongo, 2017) as in this present work, we use a model of transport of pollutant in 1D formally derived in (Fernandez-Nieto, Narbona-Reina & Zabsonré, 2013). In (Roamba, Zabsonré & Zongo, 2017), the authors showed the existence of global weak solutions of similar model derived in (Fernandez-Nieto, Narbona-Reina & Zabsonré, 2013). To lead well this result, the authors considered the condition according to which $h_2 \le h_1$ (the water layer is more important than the layer of the pollutant). We suppose in this paper the existence of molecular interactions between molecules and this leads us to use the Van Der

Waals force which is given by $V(h_1) = \dfrac{1}{h_1^3} - \dfrac{\alpha}{h_1^4}$ $(\alpha > 0)$, see (Kitavtsev, Laurençot & Niethammer, 2011; Seemann, Herminghaus & Jacobs, 2001). This force of Van Der Waals allows us to lower the height of water which allows us to get around hypothesis made in (Roamba, Zabsonré & Zongo, 2017).

From a theoretical point of view several studies have been carried out on the construction of global weak solutions of shallow-water equations model. The construction of global weak solutions for a shallow water model is done in (Bresch & Desjardins, 2006) for the two-dimensional case. In (Kitavtsev, Laurençot & Niethammer, 2011), the authors, to prove the existence of global weak solutions for one-dimensional lubrification models, have constructed approximate solutions.

In this paper, our contribution is to build sequences of suitably smooth approximate solutions to the 1D pollutant transport model for a similar model studied in (Roamba, Zabsonré & Zongo, 2017). A similar method of construction of weak solutions has been made in (Gamba, j*ü*ngel & Vasseur, 2009). In (Vasseur & Yu, 2016; Roamba & Zabsonré, 2017), an other method of construction of weak solutions is developped to prove the existence of global weak solutions by deriving the Mellet-Vasseur type inequality.

We complete the system $(1) - (3)$ by:

$$u = 0 \quad at \quad x = 0, 1 \tag{12}$$

$$\partial_x h_i = 0, \quad i = 1, 2 \quad at \quad x = 0, 1 \tag{13}$$

Our paper is organized as follows. On the one hand, the Section 2 is devoted to the actual construction of solutions to a perturbed system that preserves the BD entropy discovered in (Bresch & Desjardins, 2002; Bresch & Desjardins, 2006; Bresch, Desjardins & Lin, 2003), we establish a classical energy equality and the "mathematical BD entropy", which entail some regularities on the unknowns. The BD entropy is a mathematical entropy introduced firstly in (Bresch & Desjardins, 2002). Then we give a proposition allowing us to limit inferiorly the height of water which is very fundamental for the continuation since this limit study gives us additional regularities on the data. We also give an existence theorem of global weak solutions. To end, we give the proof of existence Theorem including the limits passage in the section.

2. Construction of Approximate Solutions

This section is devoted to the construction of approximate solutions to the 1D pollutant transport model. A small parameter β is introduced. For given $\eta > 0$, the approximate system is globally well posed and h_1 is bounded and bounded away from 0. The global existence of weak solutions is obtained by taking the limit $\eta \to 0$ and using the stability arguments detailed in (Roamba, Zabsonré & Zongo, 2017) and (Roamba, Zabsonré & Zongo, 2017). Although the pressure term $V(h_1)$ does not need a regularization as in the case of Bresch and Desjardins (Bresch & Desjardins, 2006), one still needs to regularize the function h_1 sufficiently in order to control additional higher order terms arising in the entropy equality. The approximating systems we take are given by

$$\partial_t h_{1_\eta} + \partial_x (h_{1_\eta} u_\eta) = 0, \tag{14}$$

$$\begin{aligned}\partial_t (h_{1_\eta} u_\eta) + \partial_x (h_{1_\eta} {u_\eta}^2) + \frac{1}{2} g \partial_x (h_{1_\eta})^2 - 4\nu_1 \partial_x (h_{1_\eta} \partial_x u_\eta) + \frac{u_\eta}{\beta} - h_{1_\eta} \partial_x (\sigma \partial_x^2 h_{1_\eta} - V(h_{1_\eta})) \\ + rgh_{1_\eta} \partial_x h_{2_\eta} + rgh_{2_\eta} \partial_x (h_{1_\eta} + h_{2_\eta}) - \eta h_{1_\eta} (\partial_x^7 h_{1_\eta} + \partial_x^3 h_{1_\eta}) + \eta^2 \partial_x^4 u_\eta = 0,\end{aligned} \tag{15}$$

$$\partial_t h_{2_\eta} + \partial_x (h_{2_\eta} u_\eta) - \eta (\partial_x^6 h_{2_\eta} + \partial_x^4 h_{2_\eta}) - \varepsilon \partial_x^2 h_{2_\eta} - \partial_x \Big((a{h_{2_\eta}}^2 + b(h_{2_\eta})^3) \partial_x p_{2_\eta} \Big) = 0, \tag{16}$$

with

$$\partial_x p_{2_\eta} = \rho_2 g \partial_x (h_{1_\eta} + h_{2_\eta}) \quad \text{and} \quad V(h_{1_\eta}) = \frac{1}{(h_{1_\eta})^3} - \frac{\alpha}{(h_{1_\eta})^4} \quad (\alpha > 0), \tag{17}$$

where $(t, x) \in (0, T) \times]0, 1[$ and η is a small parameter. Consider $(14) - (16)$ with boundary condions

$$u_\eta = \partial_x^2 u_\eta = \partial_x h_{1_\eta} = \partial_x^3 h_{1_\eta} = \partial_x^5 h_{1_\eta} = \partial_x h_{2_\eta} = \partial_x^5 h_{2_\eta} = 0, \qquad (t, x) \in (0, T) \times \{0, 1\}. \tag{18}$$

and initial data

$$\begin{aligned}&h_{1,\eta}^0, h_{2,\eta}^0 \in H^1(0, 1), \quad u_\eta^0 \in L^2(0, 1), \\ &u_\eta(x, 0) = u_\eta^0(x), \quad h_{1_\eta}(x, 0) = h_{1_\eta}^0(x) > 0 \quad \text{and} \quad h_{2_\eta}(x, 0) = h_{2,\eta}^0(x) > 0, \quad \text{in} \quad (0, 1),\end{aligned} \tag{19}$$

where u_η^0, $h_{1_\eta}^0$ and $h_{2,0}^\eta$ are smooth functions such as

$$\begin{aligned}&u_\eta^0 \to u_{1,0} \quad \text{in} \quad L^2(0, 1), \quad h_{1_\eta}^0 \to h_{1,0}, \quad h_{2,\eta}^0 \to h_{2,0} \quad \text{in} \quad H^1(0, 1) \\ &\text{and} \quad \eta h_{1_\eta}^0 \to 0, \quad \eta h_{2,\eta}^0 \to 0 \quad \text{in} \quad H^3(0, 1) \quad \text{as} \quad \eta \to 0.\end{aligned} \tag{20}$$

We have the following energy inequality

Lemma 1. *For classical solutions of the system* (14)-(16)*, the following inequality holds*

$$\begin{aligned}
E(u_\eta, h_{1_\eta}, h_{2_\eta}) &+ 4\nu_1 \int_0^T \int_0^1 h_{1_\eta} |\partial_x u_\eta|^2 + \frac{1}{\beta}\int_0^T \int_0^1 |u_\eta|^2 + \frac{1}{2} rg\eta \int_0^T \int_0^1 |\partial_x^2 h_{2_\eta}|^2 \\
&+ \frac{1}{2} gr\varepsilon \int_0^T \int_0^1 |\partial_x h_{2_\eta}|^2 + rg^2\rho_2 \int_0^T \int_0^1 (h_{1_\eta})^2 (a + bh_{2_\eta}) \Big(\partial_x (h_{1_\eta} + h_{2_\eta})\Big)^2 + \eta^2 \int_0^T \int_0^1 |\partial_x^2 u_\eta|^2 \\
&+ \frac{1}{2} rg\eta \int_0^T \int_0^1 |\partial_x^3 h_{2_\eta}|^2 \le \frac{1}{2} rg \int_0^T \int_0^1 \Big[\varepsilon |\partial_x h_{1_\eta}|^2 + \eta |\partial_x^2 h_{1_\eta}|^2 + \eta |\partial_x^3 h_{1_\eta}|^2\Big] + E(u_\eta^0, h_{1_\eta}^0, h_{2,0}^\eta),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
E(u_\eta, h_{1_\eta}, h_{2_\eta}) := \int_0^1 \Big[&\frac{1}{2} h_{1_\eta} |u_\eta|^2 + U(h_{1_\eta}) + \frac{1}{2} g(1-r)|h_{1_\eta}|^2 + \frac{1}{2} rg |h_{1_\eta} + h_{2_\eta}|^2 + \frac{1}{2}\sigma |\partial_x h_{1_\eta}|^2 + \eta \frac{1}{2} |\partial_x^2 h_{1_\eta}|^2 \\
&+ \frac{\eta}{2} |\partial_x^3 h_{1_\eta}|^2\Big],
\end{aligned}$$

and

$$U(h_1) = -\frac{1}{2h_1^2} + \frac{\alpha}{3h_1^3}, \qquad h_1 > 0.$$

Remark 1. *Notice that the two terms in the right can be controlled using Gronwall's lemma.*

Remark 2. *Let* $(h_{1_\eta}, h_{2_\eta}, u_\eta)$ *be a solution of model* $(14)-(16)$*. Then, thanks to the energy inequality, we have:*

$$\sqrt{\frac{1}{2} g(1-r)} h_{1_\eta} \quad \textit{is bounded in} \quad L^\infty(0,T; L^2(0,1)),$$

$$\sqrt{\frac{1}{2}\sigma} \partial_x h_{1_\eta} \quad \textit{is bounded in} \quad L^\infty(0,T; L^2(0,1)),$$

$$\sqrt{\frac{1}{2} rg} (h_{1_\eta} + h_{2_\eta}) \quad \textit{is bounded in} \quad L^\infty(0,T; L^2(0,1)),$$

$$\sqrt{\frac{1}{2}} \sqrt{h_{1_\eta}} u_\eta \quad \textit{is bounded in} \quad L^\infty(0,T; L^2(0,1)),$$

$$2\sqrt{\nu_1} \sqrt{h_{1_\eta}} \partial_x u_\eta \quad \textit{is bounded in} \quad L^2(0,T; L^2(0,1)),$$

$$\frac{1}{\sqrt{\beta}} u_\eta \quad \textit{is bounded in} \quad L^2(0,T; L^2(0,1)),$$

$$g\sqrt{r\rho_2} h_{2_\eta} \sqrt{a + bh_{2_\eta}} \Big(\partial_x (h_{1_\eta} + h_{2_\eta})\Big) \quad \textit{is bounded in} \quad L^2(0,T; L^2(0,1)),$$

$$\sqrt{rg\eta} (h_{1_\eta})^{-\frac{3}{2}} \quad \textit{is bounded in} \quad L^\infty(0,T; L^2(0,1)),$$

$$\sqrt{\frac{\eta}{2}} \partial_x^3 h_{1_\eta} \quad \textit{is bounded in} \quad L^\infty(0,T; L^2(0,1)),$$

$$\sqrt{gr\varepsilon} \partial_x h_{2_\eta} \quad \textit{is bounded in} \quad L^2(0,T; L^2(0,1)),$$

$$\sqrt{\frac{\eta}{2}} \partial_x^2 h_{1_\eta} \quad \textit{is bounded in} \quad L^\infty(0,T; L^2(0,1)),$$

$$\eta |\partial_x^2 u_\eta| \quad \textit{is bounded in} \quad L^2(0,T; L^2(0,1)).$$

The following lemma gives us the inequality of entropy necessary to limit $\partial_x \sqrt{h_{1_\eta}}$.

Lemma 2. *For smooth solutions* $(h_{1_\eta}, h_{2_\eta}, u_\eta)$ *of model* $(14)-(16)$ *satisfying the classical energy equality of the lemma 1, we have the following mathematical BD entropy inequality:*

$$S(u_\eta, h_{1_\eta}, h_{2_\eta}) + \frac{1}{\beta}\int_0^T \int_0^1 |u_\eta|^2 + 4\nu_1 \int_0^T \int_0^1 (g + gr\frac{h_{2_\eta}}{h_{1_\eta}} V'(h_{1_\eta})) |\partial_x h_{1_\eta}|^2 + 4rg\nu_1 \int_0^T \int_0^1 (1 + \frac{h_{1_\eta}}{h_{1_\eta}}) \partial_x h_{1_\eta} \partial_x h_{2_\eta}$$

$$+4\nu_1\sigma\int_0^T\int_0^1|\partial_x^2 h_{1_\eta}|^2+rg^2\rho_2\int_0^T\int_0^1(h_{1_\eta})^2(a+bh2_\eta)\Big(\partial_x(h_{1_\eta}+h_{2_\eta})\Big)^2+\frac{1}{2}rg\eta\int_0^T\int_0^1|\partial_x^2 h_{2_\eta}|^2$$
$$+\int_0^T\int_0^1\Big[\eta^2|\partial_x^2 u_\eta|^2+4\nu\eta|\partial_x^4 h_{1_\eta}|^2+4\nu\eta^2\partial_x^2 u_\eta\partial_x^3 logh_{1_\eta}\Big]$$
$$\leq\frac{1}{2}rg\int_0^T\int_0^1\Big[\varepsilon|\partial_x h_{1_\eta}|^2+\eta|\partial_x^2 h_{1_\eta}|^2+\eta|\partial_x^3 h_{1_\eta}|^2\Big]+S(u_\eta^0,h_{1_\eta}^0,h_{2_0}^\eta),\tag{22}$$

where

$$S(u_\eta,h_{1_\eta},h_{2_\eta}):=\int_0^1\Big[\frac{1}{2}h_{1_\eta}|u_\eta+\partial_x\varphi(h_{1_\eta})|^2-\frac{1}{\beta}\varphi(h_{1_\eta})+\frac{1}{2}rg|h_{1_\eta}+h_{2_\eta}|^2+\frac{1}{2}g(1-r)|h_{1_\eta}|^2$$
$$+\frac{1}{2}\sigma|\partial_x h_{1_\eta}|^2+U(h_{1_\eta})+\frac{\eta}{2}|\partial_x^2 h_{1_\eta}|^2+\frac{\eta}{2}|\partial_x^3 h_{1_\eta}|^2\Big].$$

Remark 3.

In the lemma 2 all the terms, excepted

$$\int_0^T\int_0^1(\varepsilon+4\nu_1\frac{h_{2_\eta}}{h_{1_\eta}})\partial_x h_{1_\eta}\partial_x h_{2_\eta}\quad\int_0^1\int_0^T V'(h_{1_\eta})|\partial_x h_{1_\eta}|^2\quad and\quad\int_0^T\int_0^1 4\nu\eta^2\partial_x^2 u_\eta\partial_x^3 logh_{1_\eta}$$

are controlled since they have the good sign. The control of the term $\int_0^T\int_0^1|u_\eta|^2u_\eta\partial_x h_{1_\eta}$ *takes inspiration in (Roamba, Zabsonré & Traor?, 2016). The term* $\int_0^1\int_0^T V'(h_{1_\eta})|\partial_x h_{1_\eta}|^2$ *can be absorbed thanks to the work done in (Kitavtsev, Laurençot & Niethammer, 2011). It remains for us to control the terms* $\int_0^T\int_0^1(\varepsilon+4\nu_1\frac{h_{2_\eta}}{h_{1_\eta}})\partial_x h_{1_\eta}\partial_x h_{2_\eta}$, $\int_0^T\int_0^1 4\nu\eta^2\partial_x^2 u_\eta\partial_x^3 logh_{1_\eta}$, *see (Roamba, Zabsonré & Zongo, 2017) and (Kitavtsev, Laurençot & Niethammer, 2011) for justifications.*

Remark 4. *Given $\eta>0$, the equation (14) is parabolic in u_η. Also, the quations (15) and (16) are parabolic respectively in h_{1_η} and in h_{2_η}. Relying on the works of Bresch and Desjardins in (Bresch & Desjardins, 2006) and those of Kitavtsev, Laurençot and Niethammer in (Kitavtsev, Laurençot & Niethammer, 2011) the system (14)-(16) with (19)-(20) has a unique classical solution at least locally in time. Arguing as in (Kitavtsev, Laurençot & Niethammer, 2011), the **Proposition 1** and the regularities above guarantee the global in time solvability for (14)-(16) with (19)-(20).*

Proposition 1. *If h_{1_η} has the regularities established in **corollary** 2.1, then there exists constants c_1 and c_2 such as* $0<c_1<h_{1_\eta}<c_2$.

Lemma 3. *For classical solutions of the system* (14) − (16) *with a first component h_{1_η}, we have*

$$\frac{1}{4}\int_0^1 h_{1_\eta}|\partial_x\varphi(h_{1_\eta})|^2\leq\frac{1}{2}\int_0^1 h_{1_\eta}(u_\eta+\partial_x\varphi(h_{1_\eta}))^2+2E(h_{1_\eta},h_{2_\eta},u_\eta)+\frac{1}{3\alpha^2}\tag{23}$$

with

$$E(h_{1_\eta},h_{2_\eta},u_\eta)=\int_0^1\Big[\frac{1}{2}h_{1_\eta}|u_\eta|^2+U(h_{1_\eta})+\frac{1}{2}g(1-r)|h_{1_\eta}|^2+\frac{1}{2}rg|h_{1_\eta}+h_{2_\eta}|^2+\frac{1}{2}\sigma|\partial_x h_{1_\eta}|^2+\frac{\eta}{2}|\partial_x^3 h_{1_\eta}|^2\Big].$$

Proof: See (Roamba, Zabsonré & Zongo, 2017).

Corollary 1. *Let $(h_{1_\eta},h_{2_\eta},u_\eta)$ be a solution of model* (14) − (16).

*Then, thanks to **lemma 3** and the **BD entropy equality**, we have:*

$$\sqrt{h_{1_\eta}}\quad\text{is bounded in}\quad L^\infty(0,T;L^2(0,1)),$$
$$\partial_x\sqrt{h_{1_\eta}}\quad\text{is bounded in}\quad L^\infty(0,T;L^2(0,1)),$$
$$\partial_x^2 h_{1_\eta}\quad\text{is bounded in}\quad L^2(0,T;L^2(0,1)),$$
$$\sqrt{\eta}\partial_x^2 h_{2_\eta}\quad\text{is bounded in}\quad L^2(0,T;L^2(0,1)),$$
$$2\sqrt{\eta}\partial_x^4 h_{1_\eta}\quad\text{is bounded in}\quad L^2(0,T;L^2(0,1)).$$

Remark 5. *1. In the **remark** 2, the estimate*

$$\sqrt{h_{1_\eta}}u_\eta \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(0,1))$$

implies,

$$h_{1_\eta}u_\eta \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(0,1))$$

this leads us

$$\partial_t h_{1_\eta} \quad \textit{is bounded in} \quad L^\infty(0,T;W^{-1,2}(0,1)).$$

2. *We have the additional regularities thanks to **Corollary** 1:*
   (a) h_{1_η}, u_η *are bounded in* $L^2(0,T;H^2(0,1))$,
   (b) h_{2_η} *are bounded in* $L^2(0,T;H^2(0,1))$,
   (c) $h_{1_\eta}u_\eta$ *is bounded in* $L^3(0,T;L^3(0,1)) \cap L^\infty(0,T;L^2(0,1))$.

Remark 6. *We have the following additional regularities:*

1. h_{1_η} *and* h_{2_η} *are bounded in* $L^\infty(0,T;L^2(0,1))$.
2. $\sqrt{h_{1_\eta}}$ *is bounded in* $L^2(0,T;H^1(0,1))$.
   Indeed,
   by integrating the mass equation, we obtain directly $\sqrt{h_{1_\eta}}$ *in* $L^\infty(0,T;L^2(\Omega))$. *As **Corollary 1** gives us* $\partial_x\sqrt{h_{1_\eta}}$ *in* $L^\infty(0,T;L^2(\Omega))$, *so* $\sqrt{h_{1_\eta}}$ *is bounded in* $L^\infty(0,T;H^1(\Omega))$.

Remark 7. *Since we have* $0 < c_1 < h_{1_\eta} < c_2$ *uniformly with respect to* ε, *the limit* h_{1_η} *is bounded and bounded away from zero. The limit system can then be divided by* h_{1_η} *and becomes parabolic with respect to the velocity* u_η. *Arguing as in Bresch and Desjardins (Bresch & Desjardins, 2006), Kitavtsev, Laurençot and Niethammer (Kitavtsev, Laurençot & Niethammer, 2011) the initial-boundary value problem system (14)-(16) with (18)-(19) has a unique classical solution at least locally in time.*

Now, we are going to define a weak formulation of the problem (14)-(16) with boundary conditions (18). Consider (h_{1_0}, h_{2_0}, u_0) satisfying (20).

Definition 1. *A triplet* $(h_{1_\eta}, h_{2_\eta}, u_\eta)$ *is a global weak solution to (14)-(16) with boundary conditions (18) and initial conditions* $(h_{1_0}, h_{2_0}, u_{1_0})$ *if* h_{1_0}, h_{2_0} *and* u_0 *enjoy the regularity properties stated above in this section and the following holds*

$$h^0_{1_\eta}\phi(0,.) - \int_0^T\int_0^1 h_{1_\eta}\partial_t\phi - \int_0^T\int_0^1 h_{1_\eta}u_\eta\partial_x\phi = 0, \tag{24}$$

$$\begin{aligned}&-h^\eta_{2_0}\phi(0,.) - \int_0^T\int_0^1 h_{2_\eta}\partial_t\phi - \int_0^T\int_0^1 h_{2_\eta}u_\eta\partial_x\phi + \varepsilon\int_0^T\int_0^1 \partial_x h_{2_\eta}\partial_x\phi \\ &+ \int_0^T\int_0^1 \Big((a{h_{2_\eta}}^2 + b{h_{2_\eta}}^3)\partial_x p_{2_\eta}\Big)\partial_x\phi - \eta\int_0^1\int_0^T \partial_x^6 h_{2_\eta}\phi - \eta\int_0^1\int_0^T \partial_x^4 h_{2_\eta}\phi = 0,\end{aligned} \tag{25}$$

$$\begin{aligned}&h^\eta_{0_1}u^0_\eta\phi(0,.) - \int_0^T\int_0^1 h_{1_\eta}u_\eta\partial_t\phi - \int_0^T\int_0^1 h_{1_\eta}{u_\eta}^2\partial_x\phi + 4\nu_1\int_0^T\int_0^1 h_{1_\eta}\partial_x u_\eta\partial_x\phi + \frac{1}{\beta}\int_0^T\int_0^1 u_\eta\phi \\ &+ \int_0^T\int_0^1(\sigma\partial_x^2 h_{1_\eta} - V(h_{1_\eta}))\phi\partial_x h_{1_\eta} + \int_0^T\int_0^1(\sigma\partial_x^2 h_{1_\eta} - V(h_{1_\eta}))h_{1_\eta}\partial_x\phi - \frac{1}{2}g\int_0^T\int_0^1 (h_{1_\eta})^2\partial_x\phi \\ &-rg\int_0^T\int_0^1 h_{2_\eta}h_{1_\eta}\partial_x\phi + r_1\int_0^T\int_0^1 h_{1_\eta}|u_\eta|^2u_\eta\phi - rg\int_0^T\int_0^1 \phi h_{2_\eta}\partial_x h_{1_\eta} - \eta\int_0^1\int_0^T \partial_x^2 u_\eta\partial_x^2\phi \\ &-rg\int_0^T\int_0^1(h_{1_\eta} + h_{2_\eta})h_{2_\eta}\partial_x\phi - rg\int_0^T\int_0^1(h_{1_\eta} + h_{2_\eta})\partial_x h_{2_\eta}\phi + \eta\int_0^1\int_0^T h_{1_\eta}\partial_x^7 h_{1_\eta}\phi - \eta\int_0^1\int_0^T \partial_x^3 h_{1_\eta}\phi = 0,\end{aligned} \tag{26}$$

for all $\phi \in C_0^\infty([0,\infty)\times[0,1])$ *such that* $\phi(T,.) = 0$.

We now show that solutions to the system (14)-(16) with boundary and initial conditions (18)-(19) converge to a solution of (9)-(11) as $\eta \longrightarrow 0$.

Theorem 1. *For any positive* σ, β *and initial data* (h_{1_0}, h_{2_0}, u_0) *satisfaying (20), there exists a global weak solution to the system (14)-(16) with boundary conditions (18) and initial conditions (19) in the sense of (24)-(26).*

3. Case $\beta = \infty$

We follow the ideas proposed in (Kitavtsev, Laurençot & Niethammer, 2011).

Let us first consider a sequence of positive real numbers (β_n), $\beta_n \to \infty$, and denote the corresponding solutions to (24)-(26) with $\beta = \beta_n$ by $(h_{1_{\beta_n}}, h_{2_{\beta_n}}, u_{\beta_n})$. The corresponding system reads as:

$$\partial_t h_{1_{\beta_n}} + \partial_x(h_{1_{\beta_n}} u_{\beta_n}) = 0, \tag{27}$$

$$\partial_t(h_{1_{\beta_n}} u_{\beta_n}) + \partial_x(h_{1_{\beta_n}} {u_{\beta_n}}^2) + \frac{1}{2} g\partial_x(h_{1_{\beta_n}})^2 - 4\nu_1\partial_x(1_{\beta_n}\partial_x u_{\beta_n}) + \frac{u_{\beta_n}}{\beta_n} - h_{1_\eta}\partial_x(\sigma\partial_x^2 h_{1_{\beta_n}} - V(h_{1_{\beta_n}}))$$
$$+rgh_{1_{\beta_n}}\partial_x h_{2_{\beta_n}} + rgh_{2_{\beta_n}}\partial_x(h_{1_{\beta_n}} + h_{2_{\beta_n}}) - \eta_n h_{1_{\beta_n}}(\partial_x^7 h_{1_{\beta_n}} + \partial_x^3 h_{1_{\beta_n}}) + \eta_n^2\partial_x^4 u_{\beta_n} = 0, \tag{28}$$

$$\partial_t h_{2_{\beta_n}} + \partial_x(h_{2_{\beta_n}} u_{\beta_n}) - \eta_n(\partial_x^6 h_{2_{\beta_n}} + \partial_x^4 h_{2_{\beta_n}}) - \varepsilon\partial_x^2 h_{2_{\beta_n}} - \partial_x\Big((a{h_{2_{\beta_n}}}^2 + b(h_{2_{\beta_n}})^3)\partial_x p_{2_{\beta_n}}\Big) = 0, \tag{29}$$

with

$$\partial_x p_{1_{\beta_n}} = \rho_2 g\partial_x(h_{1_{\beta_n}} + h_{2_{\beta_n}}) \quad \text{and} \quad V(h_{1_{\beta_n}}) = \frac{1}{(h_{1_{\beta_n}})^3} - \frac{\alpha}{(h_{1_{\beta_n}})^4} \quad (\alpha > 0), \tag{30}$$

where $(t, x) \in (0, T)\times]0, 1[$ and η_n is a small parameter.

For this system (27) – (29), the statement of **Remark** 2, **Corollary** 1 and the **Lemma** 1 are true for the weak solutions to (24)-(26). We may then investigate the behaviour of these solutions as either $\beta \to \infty$. Though the estimate on (u_{β_n}/β_n) is useless in that case, one still recovers the estimate of (u_{β_n}) in $L^2(0, T; H_0^1(0, 1))$ as a consequence of **Remark** 2, **Corollary** 1 and the Poincaré inequality. Arguing as in the proof of **Theorem** 1, we conclude that, after possibly extracting a subsequence, $(h_{1_{\beta_n}}, h_{2_{\beta_n}}, u_{\beta_n})$ converges towards a weak solution to the model

$$\partial_t h_1 + \partial_x(h_1 u) = 0, \tag{31}$$

$$\partial_t(h_1 u) + \partial_x(h_1 u^2) + \frac{1}{2} g\partial_x h_1^2 - 4\nu_1\partial_x(h_1\partial_x u) - h_1\partial_x(\sigma\partial_x^2 h_1 - V(h_1))$$
$$+r_1 h_1|u|^2 u + rgh_1\partial_x h_2 + rgh_2\partial_x(h_1 + h_2) = 0, \tag{32}$$

$$\partial_t h_2 + \partial_x(h_2 u) - \varepsilon\partial_x^2 h_2 - \partial_x\Big((a{h_2}^2 + b{h_2}^3)\partial_x p_2\Big) = 0. \tag{33}$$

4. Conclusion

This article was the subject of the construction of global weak solutions of a model of pollutant transport in dimension 1. Furthermore, we have shown that the existence of global weak solutions of the model is preserved when the Reynolds number tends to infinity. For our future works, we will show the existence of global weak solutions of the model studied in this paper when $\sigma \longrightarrow 0$.

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Appendix

Tha aim of this appendix is to prove technical lemmas that will yield crucial estimates for passing to the limits in the approximate (14)-(16).

***Proof of Lemma** 1*

First, we multiply the momentum equation by u_η and we integrate from 0 to 1. We use the mass conservation equation of the first layer for simplification. Then, we obtain

$$\int_0^1 \frac{1}{2}\partial_t(h_{1_\eta}(u_\eta)^2) + \frac{1}{2}\int_0^1 g\partial_x(h_{1_\eta})^2 u_\eta - 4\int_0^1 \partial_x(\nu_1 h_{1_\eta}\partial_x u_\eta)u_\eta - \int_0^1 h_{1_\eta}u_\eta\partial_x(\sigma\partial_x^2 h_{1_\eta} - V(h_{1_\eta}))$$
$$+ \int_0^1 \frac{(u_\eta)^2}{\beta} + \eta\int_0^1 h_{1_\eta}\partial_x^7 h_{1_\eta}u_\eta - \eta^2\int_0^1 \partial_x^4 u_\eta u_\eta + rg\int_0^1 h_{1_\eta}\partial_x h_{2_\eta}u_\eta$$
$$+rg\int_0^1 h_{2_\eta}\partial_x(h_{1_\eta} + h_{2_\eta})u_\eta = 0. \tag{34}$$

Now, we simplify each term as follows:

- $-4\int_0^1 \partial_x(\nu_1 h_{1_\eta}\partial_x u_\eta)u_\eta = 4\nu_1\int_0^1 h_{1_\eta}(\partial_x u_\eta)^2,$
- $$\begin{aligned} -\int_0^1 h_{1_\eta}u_\eta\partial_x(\sigma\partial_x^2 h_{1_\eta} - V(h_{1_\eta})) &= \int_0^1 \partial_x(h_{1_\eta}u_\eta)(\sigma\partial_x^2 h_{1_\eta} - V(h_{1_\eta}))\\ &= -\int_0^1 \partial_t h_{1_\eta}(\sigma\partial_x^2 h_{1_\eta} - V(h_{1_\eta}))\\ &= \int_0^1 \sigma\partial_{xt}h_{1_\eta}\partial_x h_{1_\eta} + \int_0^1 \partial_t(U(h_{1_\eta}))\\ &= \int_0^1 \partial_t\Big(\frac{1}{2}\sigma|\partial_x h_{1_\eta}|^2 + U(h_{1_\eta})\Big), \end{aligned}$$

- $rg\int_0^1 h_{1_\eta}\partial_x h_{2_\eta}u_\eta = -rg\int_0^1 h_{2_\eta}\partial_x(h_{1_\eta}u_\eta) = rg\int_0^1 h_{2_\eta}\partial_t h_{1_\eta}$,
- $\frac{1}{2}g\int_0^1 \partial_x(h_{1_\eta})^2 u_\eta = \frac{1}{2}g\frac{d}{dt}\int_0^1 |h_{1_\eta}|^2$,

$$\bullet\eta\int_0^1 h_{1_\eta}\partial_x^7 h_{1_\eta}u_\eta = -\eta\int_0^1 \partial_x(h_{1_\eta}u_\eta)\partial_x^6 h_{1_\eta}$$
$$= \eta\int_0^1 \partial_t h_{1_\eta}\partial_x^6 h_{1_\eta}$$
$$= -\eta\int_0^1 \partial_{xt} h_{1_\eta}\partial_x^5 h_{1_\eta}$$
$$= \eta\int_0^1 \partial_t\partial_x^2 h_{1_\eta}\partial_x^4 h_{1_\eta}$$
$$= -\eta\int_0^1 \partial_t\partial_x^3 h_{1_\eta}\partial_x^3 h_{1_\eta}$$
$$= \frac{-\eta}{2}\frac{d}{dt}\int_0^1 |\partial_x^3 h_{1_\eta}|^2,$$

$$\bullet\eta\int_0^1 h_{1_\eta}\partial_x^3 h_{1_\eta}u_\eta = -\eta\int_0^1 \partial_x(h_{1_\eta}u_\eta)\partial_x^2 h_{1_\eta}$$
$$= \eta\int_0^1 \partial_t h_{1_\eta}\partial_x^2 h_{1_\eta}$$
$$= -\eta\int_0^1 \partial_{xt} h_{1_\eta}\partial_x h_{1_\eta}$$
$$= -\eta\frac{1}{2}\frac{d}{dt}\int_0^1 |\partial_x h_{1_\eta}|^2,$$

$$\bullet\eta^2\int_0^1 u_\eta\partial_x^4 u_\eta = -\eta^2\int_0^1 \partial_x u_\eta\partial_x^3 u_\eta$$
$$= \eta^2\int_0^1 |\partial_x^2 u_\eta|^2,$$

- $rg\int_0^1 h_{2_\eta}\partial_x(h_{1_\eta} + h_{1_\eta})u_\eta = -rg\int_0^1 (h_{1_\eta} + h_{2_\eta})\partial_x(h_{2_\eta}u_\eta)$.

The pollutant transport equation gives us:

$\partial_x(h_{2_\eta}u_\eta) = -\partial_t h_{2_\eta} + \varepsilon\partial_x^2 h_{2_\eta} + \partial_x\Big((a{h_{1_\eta}}^2 + b{h_{2_\eta}}^3)\partial_x p_{2_\eta}\Big) + \eta(\partial_x^6 h_{2_\eta} + \partial_x^4 h_{2_\eta})$ and we have:

- $$rg\int_0^1 h_{2_\eta}\partial_x(h_{1_\eta} + h_{1_\eta})u_\eta = rg\varepsilon\int_0^1 \partial_x h_{1_\eta}\partial_x h_{2_\eta} + rg\varepsilon\int_0^1 |\partial_x h_{2_\eta}|^2 + \frac{1}{2}rg\frac{d}{dt}\int_0^1 |h_{2_\eta}|^2$$
$$+\rho_2 rg^2\int_0^1 (h_{1_\eta})^2|\partial_x(h_{1_\eta} + h_{1_\eta})|^2(a + bh_{2_\eta}) + rg\int_0^1 h_{1_\eta}\partial_t h_{2_\eta} + rg\eta\int_0^1 \partial_x^3 h_{1_\eta}\partial_x^3 h_{2_\eta}$$
$$+rg\eta\int_0^1 |\partial_x^3 h_{2_\eta}|^2 + rg\eta\int_0^1 \partial_x^2 h_{1_\eta}\partial_x^2 h_{2_\eta} + rg\eta\int_0^1 |\partial_x^2 h_{2_\eta}|^2.$$

Substituting all these terms in (34), we get (21) by integrating under 0 to T.

***Proof of Lemma** 2*

Let us multiply the equation (15) by $\partial_x\varphi(h_{1_{\eta_k}})$, integrate with respect to x and use an integration by parts, and using (1), we have:

$$4\nu_1\int_0^1 (\partial_t u_\eta + u_\eta\partial_x u_\eta)\partial_x h_{1_\eta} + 4\nu_1 g\int_0^1 |\partial_x h_{1_\eta}|^2 + 16\nu_1^2\int_0^1 h_{1_\eta}\partial_x u_\eta\partial_x\Big(\frac{\partial_x h_{1_\eta}}{h_{1_\eta}}\Big)$$

$$+4\nu_1\int_0^1 \frac{u_\eta\partial_x h_{1_\eta}}{\beta h_{1_\eta}} + 4\nu_1\sigma\int_0^1 |\partial_x^2 h_{1_\eta}|^2 + 4\nu_1\int_0^1 V'(h_{1_\eta})|\partial_x h_{1_\eta}|^2 + 4\nu_1 rg\int_0^1 \partial_x h_{1_\eta}\partial_x h_{1_\eta}$$

$$+4\nu_1 rg\int_0^1 \frac{h_{2_\eta}}{h_{1_\eta}}|\partial_x h_{1_\eta}|^2 + 4\nu_1 rg\int_0^1 \frac{h_{2_\eta}}{h_{1_\eta}}\partial_x h_{2_\eta}\partial_x h_{1_\eta} + 4\nu_1\eta\int_0^1 |\partial_x^4 h_{1_\eta}|^2 + \eta^2\int_0^1 \partial_x^2 u_\eta\partial_x^3\varphi(h_{1_\eta}) = 0. \tag{35}$$

On the one hand, a further integration by parts of the first integral of (35), equation (1), and the energy inequality (21) give

$$\begin{aligned}
&4\nu_1\int_0^1 (\partial_t u_\eta + u_\eta\partial_x u_\eta)\partial_x h_{1_\eta}\\
&= 4\nu_1\Big(\frac{d}{dt}\int_0^1 u_\eta\partial_x h_{1_\eta} - \int_0^1 u_\eta\partial_{xt}^2 h_{1_\eta} + \int_0^1 u_\eta\partial_x u_\eta\partial_x h_{1_\eta}\Big)\\
&= 4\nu_1\Big(\frac{d}{dt}\int_0^1 u_\eta\partial_x h_{1_\eta} - \int_0^1 \partial_x u_\eta\partial_x(h_{1_\eta}u_\eta) + \int_0^1 u_\eta\partial_x u_\eta\partial_x h_{1_\eta}\Big)\\
&= 4\nu_1\Big(\frac{d}{dt}\int_0^1 u_\eta\partial_x h_{1_\eta} - \int_0^1 h_{1_\eta}(\partial_x u_\eta)^2\Big)\\
&= \frac{d}{dt}\int_0^1 \Big[4\nu_1 u_\eta\partial_x h_{1_\eta} + \frac{1}{2}h_{1_\eta}|u_\eta|^2 + U(h_{1_\eta}) + \frac{1}{2}g(1-r)|h_{1_\eta}|^2 + \frac{1}{2}rg|h_{1_\eta}+h_{2_\eta}|^2 + \frac{1}{2}\sigma|\partial_x h_{1_\eta}|^2\\
&+\frac{\eta}{2}|\partial_x^2 h_{1_\eta}|^2 + \frac{\eta}{2}|\partial_x^3 h_{1_\eta}|^2\Big] + \frac{1}{\beta}\int_0^1 |u_\eta|^2 + rg\varepsilon\int_0^1 \partial_x h_{1_\eta}\partial_x h_{2_\eta} + rg\eta\int_0^1 \partial_x^3 h_{1_\eta}\partial_x^3 h_{2_\eta} + rg\varepsilon\int_0^1 |\partial_x h_{2_\eta}|^2\\
&+rg^2\int_0^1 (h_{2_\eta})^2(a+bh_{1_\eta})\Big(\partial_x(h_{1_\eta}+h_{1_\eta})\Big)^2 + 4\nu_1\eta\int_0^1 |\partial_x^4 h_{1_\eta}|^2 + \eta^2\int_0^1 \partial_x^2 u_\eta\partial_x^3\varphi(h_{1_\eta})\\
&\qquad +rg\eta\int_0^1 \partial_x^2 h_{1_\eta}\partial_x^2 h_{2_\eta} + rg\eta\int_0^1 |\partial_x^2 h_{2_\eta}|^2.
\end{aligned} \tag{36}$$

On the other hand,we can write the third and the fourth integrals of (35) as

- $$16\nu_1^2\int_0^1 \partial_x\Big(\frac{\partial_x h_{1_\eta}}{h_{1_\eta}}\Big)\partial_x u_\eta h_{1_\eta} = \frac{1}{2}\frac{d}{dt}\int_0^1 h_{1_\eta}|\varphi(h_{1_\eta})|^2,$$
- $$4\nu_1\int_0^1 \frac{u_\eta\partial_x h_{1_\eta}}{\beta h_{1_\eta}} = -4\nu_1\int_0^1 \frac{\partial_x(u_\eta h_{1_\eta})}{\beta h_{1_\eta}} + 4\nu_1\int_0^1 \frac{\partial_x u_\eta}{\beta} = -\frac{1}{\beta}\frac{d}{dt}\int_0^1 \varphi(h_{1_\eta}).$$

Substituting finally the last three identities into (35), we obtain (22).

***Proof of Proposition** 1*

We follow the lines performed in (Kitavtsev, Laurençot & Niethammer, 2011). Using the bound on $\partial_x h_{1_\eta}$ we obtain:

$$h_{1_\eta}(x,t) - h_{1_\eta}(y,t) \le \Big|\int_x^y \partial_x h_{1_\eta}(z,t)dz\Big| \le \|x-y\|^{1/2}\|\partial_x h_{1_\eta}(t)\|_2 \le \frac{c_1}{\sqrt{\sigma}}|x-y|^{1/2}$$

for all $(x,y)\in(0,1)\times(0,1)$ and $t\in(0,T)$. Next we integrate the above inequality with respect to $y\in(0,1)$, readily give the upper bound. To establish the lower bound for h_{1_η} , we combine the $L^\infty(0,T;L^2(0,1))$-estimates on $(h_{1_\eta})^{-3/2}$ and $\partial_x h_{1_\eta}$ just established to obtain a bound on the norm of $1/\sqrt{h_{1_\eta}}$ in $L^\infty(0,T;W^{1,1}(0,1))$ since

$$\int_0^1 \big|\partial_x(h_1^{-1/2})\big| = \frac{1}{2}\int_0^1 \frac{|\partial_x h_{1_\eta}|}{(h_{1_\eta})^{3/2}} \le \frac{1}{2\sqrt{\sigma}}\|\sqrt{\sigma}\partial_x h_{1_\eta}\|_2\|(h_{1_\eta})^{-3/2}\|_2.$$

Due to the continuous embedding of $W^{1,1}(0,1)$ in $L^\infty(0,1)$, we get the positive lower bound.

***Proof of Theorem** 1*

In this section, we give a proof of the **Theorem** 1. Let be $(h_{1_{\eta_k}}, h_{2_{\eta_k}}, u_{1_{\eta_k}})$ a sequence of weak solutions with initial data

$$h_{1_{\eta_k}|t=0} = h^0_{1_{\eta_k}},\quad h_{2_{\eta_k}|t=0} = h^0_{2_{\eta_k}},\quad (h_{1_{\eta_k}}u_{\eta_k})_{|t=0} = m^0_{\eta_k}$$

such as

$$h^0_{1_{\eta_k}} \longrightarrow h_{1_0} \text{ in } H^1(\Omega), \quad h^0_{2_{\eta_k}} \longrightarrow h_{2_0} \text{ in } H^1(\Omega), \quad m^0_{\eta_k} \longrightarrow m_0 \text{ in } (L^1(\Omega))^2,$$

and satisfying the following inequality:

$$-\frac{1}{\beta}\int_0^1 \varphi(h^0_{1_{\eta_k}}) + \int_0^1 \Big[h^0_{1_{\eta_k}}\big|u^0_{1_{\eta_k}}\big|^2 + 64\nu_1^2\big|\partial_x\sqrt{h^0_{1_{\eta_k}}}\big|^2 + \frac{1}{2}g(1-r)\big|h^0_{1_{\eta_k}}\big|^2 + \frac{1}{2}rg\big|h^0_{1_{\eta_k}} + h^0_{2_{\eta_k}}\big|^2 + \frac{1}{2}\sigma\big|\partial_x h^0_{1_{\eta_k}}\big|^2 + \frac{\eta}{2}|\partial_x^3 h^0_{1_{\eta_k}}|^2\Big] \le C.$$

Take a sequence $\{\eta_k\}_{k\ge1} \to 0$ and, for each $k \ge 1$, denote the corresponding solution to the approximate system (14)-(16)-(18)-(19) with $\eta = \eta_k$ by $(h_{1_{\eta_k}}, h_{2_{\eta_k}}, u_{\eta_k})$.

Convergence of $\sqrt{h_{1_{\eta_k}}}$, $h_{1_{\eta_k}}$ and $h_{2_{\eta_k}}$

From the **remark 6**:

$$\sqrt{h_{1_{\eta_k}}} \quad \text{is bounded in } L^\infty(0,T;H^1(\Omega)). \tag{37}$$

Moreover, using the mass equation, we obtain the following equality:

$$\partial_t\sqrt{h_{1_{\eta_k}}} = \frac{1}{2}\sqrt{h_{1_{\eta_k}}}\partial_x u_{\eta_k} - \partial_x(\sqrt{h_{1_{\eta_k}}}u_{\eta_k}),$$

which gives that $\partial_t\sqrt{h_{1_{\eta_k}}}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.

Applying Aubin-Simon lemma (see (Lions, 1989; Simon, 1987), we can extract a subsequence, still denoted $(h_{1_{\eta_k}})_{1\le k}$, such as

$$\sqrt{h_{1_{\eta_k}}} \quad \text{converges strongly to} \quad \sqrt{h_1} \text{ in } C^0(0,T;L^2(0,1)).$$

According to the **Proposition** 1, we show that

$$\Big|h_{1_{\eta_k}} - h_1\Big| \le \sqrt{c_2}\Big|\sqrt{h_{1_{\eta_k}}} - \sqrt{h_1}\Big| \Rightarrow \Big|h_{1_{\eta_k}} - h_1\Big|^2 \le c_2\Big|\sqrt{h_{1_{\eta_k}}} - \sqrt{h_1}\Big|^2.$$

This ensures

$$h_{1_{\eta_k}} \quad \text{converges strongly to} \quad h_1 \text{ in } L^2(0,T;L^2(0,1)).$$

We have $h_{2_{\eta_k}}$ bounded in $L^2(0,T;H^1(0,1))$. Moreover, we have

$$\partial_t h_{2_{\eta_k}} = -\partial_x(h_{2_{\eta_k}}u_{\eta_k}) + \varepsilon\partial_x^2 h_{2_{\eta_k}} + \eta_k(\partial_x^6 h_{2_{\eta_k}} + \partial_x^4 h_{2_{\eta_k}}) + \partial_x((ah_{2_{\eta_k}} + b(h_{2_{\eta_k}})^3)\partial_x p_{2_{\eta_k}}).$$

Let us study each term separately

•Since $h_{2_{\eta_k}}$ is in $L^\infty(0,T;L^2(0,1))$ and u_{η_k} is in $L^2(0,T;L^2(0,1))$, we show that the first term is in $L^2(0,T;W^{-1,1}(0,1))$.

•For the second term, since $\partial_x h_2^{\eta_k}$ is in $L^2(0,T;L^2(0,1))$, we have $\partial_x^2 h_2^{\eta_k}$ in $L^2(0,T;W^{-1,1}(0,1))$.

• For the third one, for any $\psi \in C_0^\infty((0,1)\times(0,T))$, using integration by parts and regularities in the previous section,

$$\begin{aligned}\Big|\int_0^T\int_0^1 \psi\partial_x^6 h_{2_{\eta_k}}\Big| &\le \|\partial_x^2\psi\|_{L^2(0,T;L^2(0,1))}\|\partial_x^4 h_{2_{\eta_k}}\|_{L^2(0,T;W^{-1,1}(0,1))}\\ &\le C\|\partial_x^4 h_{2_{\eta_k}}\|_{L^2(0,T;W^{-1,1}(0,1))}.\end{aligned}$$

$$\begin{aligned}\Big|\int_0^T\int_0^1 \psi\partial_x^4 h_{2_{\eta_k}}\Big| &\le \|\partial_x^2\psi\|_{L^2(0,T;L^2(0,1))}\|\partial_x^3 h_{2_{\eta_k}}\|_{L^2(0,T;W^{-1,1}(0,1))}\\ &\le C\|\partial_x^3 h_{2_{\eta_k}}\|_{L^2(0,T;W^{-1,1}(0,1))}.\end{aligned}$$

•For the last term, as $h_2^{\eta_k}\sqrt{a + bh_2^{\eta_k}}\Big(\partial_x(h_{1_{\eta_k}} + h_2^{\eta_k})\Big)$ is in $L^2(0,T;L^2(0,1))$, we have $\partial_x\Big(h_2^{\eta_k}\sqrt{a + bh_2^{\eta_k}}\Big(\partial_x(h_{1_{\eta_k}} + h_2^{\eta_k})\Big)\Big)$ in $L^2(0,T;W^{-1,1}(0,1))$.

So, the third term is in $L^2(0,T;W^{-1,1}(0,1))$ and therefore, $\partial_t h_{2_{\eta_k}}$ is in $L^2(0,T;W^{-1,1}(0,1))$.

Convergence of $h_{1_{\eta_k}} u_{\eta_k}$

According to the remark 5, $u_{\eta_k} \in L^2(0,T;H^1(0,1))$. This fact with the lemma 1 allows us to get

$$(h_{1_{\eta_k}} u_{\eta_k}) \quad in \quad L^2(0,T;H^1(0,1)) \tag{38}$$

Moreover, the momentum equation (2) enables us to write the time derivation of the water discharge:

$$\partial_t(h_{1_{\eta_k}} u_{\eta_k}) = -\partial_x(h_{1_{\eta_k}} {u_{\eta_k}}^2) - \frac{1}{2} g\partial_x {h_{1_{\eta_k}}}^2 + 4\nu_1 \partial_x(h_{1_{\eta_k}} \partial_x u_{\eta_k}) - \frac{u_{\eta_k}}{\beta} + h_{1_{\eta_k}} \partial_x(\sigma\partial_x^2 h_{1_{\eta_k}} - V(h_{1_{\eta_k}}))$$
$$-rgh_{1_{\eta_k}} \partial_x h_{2_{\eta_k}} - rgh_{2_{\eta_k}} \partial_x(h_{1_{\eta_k}} + h_{2_{\eta_k}}) + \eta_k h_{1_{\eta_k}} (\partial_x^7 h_{1_{\eta_k}} + \partial_x^3 h_{1_{\eta_k}}) - \eta_k^2 \partial_x^4 u_{\eta_k}. \tag{39}$$

We then study each term:

- $\partial_x(h_{1_{\eta_k}} (u_{\eta_k})^2) = \partial_x((h_{1_{\eta_k}} u_{\eta_k})u_{\eta_k})$ which is in $L^2(0,T;W^{-1,1}(0,1))$.
- As $h_{1_{\eta_k}}$ is in $L^\infty(0,T;L^2(0,1))$, we have:

$$\partial_x[(h_{1_{\eta_k}})^2] \text{ is in } L^\infty(0,T;W^{-1,1}(0,1)).$$

- $\partial_x(h_{1_{\eta_k}} \partial_x u_{\eta_k}) = \partial_x(\sqrt{h_{1_{\eta_k}}}\sqrt{h_{1_{\eta_k}}} \partial_x u_{\eta_k})$ is bounded in $L^2(0,T;W^{-1,1}(0,1))$.
- $rgh_{1_{\eta_k}} \partial_x h_{2_{\eta_k}}$ is bounded in $L^2(0,T;W^{-1,1}(0,1))$.
- $h_{1_{\eta_k}} \partial_x \partial_x^2 h_{1_{\eta_k}}$ is bounded in $L^\infty(0,T;W^{-1,1}(0,1))$.
- $rgh_{2_{\eta_k}} \partial_x(h_{1_{\eta_k}} + h_{2_{\eta_k}})$ is bounded in $L^2(0,T;W^{-1,1}(0,1))$.
- For any $\psi \in C_0^\infty((0,T)\times(0,1))$, we obtain, using integration by parts and the regularities in the previous section,

$$\left|\int_0^T \int_0^1 \psi h_{1_{\eta_k}} \partial_x^7 h_{1_{\eta_k}}\right| = \left|\int_0^T \int_0^1 \partial_x^4 h_{1_{\eta_k}} [\psi\partial_x^3 h_{1_{\eta_k}} + 3\partial_x\psi\partial_x^2 h_{1_{\eta_k}} + 3\partial_x^2\psi\partial_x h_{1_{\eta_k}} + h_{1_{\eta_k}} \partial_x^3\psi]\right|$$
$$\le \int_0^T ||\partial_x^4 h_{1_{\eta_k}}||_{L^2(0,1)}[||\psi||_{L^\infty(0,1)}||\partial_x^3 h_{1_{\eta_k}}||_{L^2(0,1)} + ||h_{1_{\eta_k}}||_{L^\infty(0,1)}||\partial_x^3\psi||_{L^2(0,1)}$$
$$+3||\partial_x\psi||_{L^\infty(0,1)}||\partial_x^2 h_{1_{\eta_k}}||_{L^2(0,1)} + 3||\partial_x^2\psi||_{L^\infty}||\partial_x h_{1_{\eta_k}}||_{L^2(0,1)}]$$
$$\le C||\partial_x^4 h_{1_{\eta_k}}||^2_{L^2(0,1)}||\psi||_{H^3(0,1)} \le ||\psi||_{L^2(0,T;H^3(0,1))},$$

$$\left|\int_0^T \int_0^1 \psi h_{1_{\eta_k}} \partial_x V(h_{1_{\eta_k}})\right| = \left|\int_0^T \int_0^1 \partial_x\psi V_1(h_{1_{\eta_k}})\right|$$
$$\le ||V_1(h_{1_{\eta_k}})||_{L^\infty((0,1)\times(0,T))}\Big(\int_0^T ||\psi||_{H^1(0,1)}\Big)^{\frac{1}{2}}$$

where $V_1(h_{1_{\eta_k}}) := -\int_h^\infty \tau V_1^{'}(\tau)d\tau$,

and

$$\left|\int_0^T \int_0^1 \psi h_{1_{\eta_k}} \partial_x^3 h_{1_{\eta_k}} dxdt\right| = \left|\int_0^T \int_0^1 \partial_x^2 h_{1_{\eta_k}} [h_{1_{\eta_k}} \partial_x\psi + \psi\partial h_{1_{\eta_k}}]\right|$$

$$\le \int_0^T ||\partial_x^2 h_{1_{\eta_k}}||_{L^2(0,1)}[||h_{1_{\eta_k}}||_{L^\infty(0,1)}||\partial_x\psi||_{L^2(0,1)} + ||\psi||_{L^\infty(0,1)}||\partial_x h_{1_{\eta_k}}||_{L^2(0,1)}]$$

$$\le C\left(\int_0^T ||\psi||^2_{H^1(0,1)}\right)^{\frac{1}{2}}.$$

Finally (u_{η_k}) and $(\eta_k\partial_x^4 u_{\eta_k})$ are bounded in $L^2(0,1;H^1(0,T))$ and $L^2(0,T;H^{-2}(0,1))$ respectively. Collecting the above information completes the proof of the boundness of the right-hand side of (39), whence

$$\partial_t(h_{1_{\eta_k}} u_{\eta_k}) \quad \text{is bounded in } L^2(0,T;H^{-3}(0,1)).$$

Combining this with (38) and corollary 4 in (Simon, 1987) ensures that $(h_{1_{\eta_k}} u_{\eta_k})$ is compact in $L^2((0,T);L^2(0,1))$. So, there exists $\mathbf{m} \in L^2((0,T);L^2(0,1))$ such that

$$h_{1_{\eta_k}} u_{\eta_k} \quad \text{converges to } \mathbf{m} \text{ in} \quad L^2((0,T);L^2(0,1)). \tag{40}$$

Convergences of $(h_{1_{\eta_k}})^{-1}$, u_{η_k} and $\sqrt{h_{1_{\eta_k}}} u_{\eta_k}$

• As $(h_{1_{\eta_k}})_k$ converges strongly to h_1 in $L^2(0,T;W^{1,p}(0,1)) \cap C([0,T]\times(0,1)$ for $p \in [1,\infty)$ and we have $0 < c_1 \leq h_{1_{\eta_k}} \leq c_2$, we deduce that

$$(h_{1_{\eta_k}})^{-1} \quad \text{converges strongly to} \quad h_1^{-1} \quad in \quad C([0,T]\times(0,1)). \tag{41}$$

• Considering (40) and (41), there exists $u_1 \in L^2(0,T;H^1(0,1))$ such that

$$u_{\eta_k} \text{converges strongly to} \quad u \quad in \quad L^2(0,T;L^2(0,1)). \tag{42}$$

• Since $\sqrt{h_{1_{\eta_k}}}$ converges strongly to $\sqrt{h_1}$ in $C^0(0,T;L^2(0,1))$, by using (42), $\sqrt{h_{1_{\eta_k}}} u_{\eta_k}$ converges strongly to $\sqrt{h_1} u_1$ in $L^2(0,T;L^1(0,1))$.

Convergences of $\partial_x h_{1_{\eta_k}}$, $h_{2_{\eta_k}} \partial_x h_{1_{\eta_k}}$, $\partial_x^2 h_{1_{\eta_k}}$, $h_{1_{\eta_k}} \partial_x^2 h_{1_\eta}$ and $\partial_x h_{1_{\eta_k}} \partial_x^2 h_{1_{\eta_k}}$

• We have $\partial_x h_{1_{\eta_k}}$ bounded in $L^2(0,T;H^1(0,1))$ and $\partial_t \partial_x h_{1_{\eta_k}}$ is bounded in $L^\infty(0,T;H^{-1}(0,1))$ since $\partial_t h_{1_{\eta_k}}$ is bounded in $L^\infty(0,T;H^{-1}(0,1))$. Thanks to compact injection of $H^1(0,1)$ in $L^2(0,1)$ in one dimension, we have:

$$\partial_x h_{1_{\eta_k}} \quad \text{converges strongly to} \quad \partial_x h_1 \quad \text{in } L^2(0,T;L^2(0,1)).$$

• The bound of $\partial_x^2 h_{1_{\eta_k}}$ in $L^2(0,T;L^2(0,1))$ and $\partial_x h_{2_{\eta_k}}$ in $L^2(0,T;L^2(0,1))$ gives us:

$$\partial_x^2 h_{1_{\eta_k}} \quad \text{converges strongly to} \quad \partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$\partial_x h_{2_{\eta_k}} \quad \text{converges strongly to} \quad \partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)).$$

• Thanks to the strong convergence of $h_{1_{\eta_k}}$, $h_{2_{\eta_k}}$, $\partial_x h_{1_{\eta_k}}$ and the weak convergence of $\partial_x^2 h_{1_{\eta_k}}$, we have:

$$h_{2_{\eta_k}} \partial_x h_{1_{\eta_k}} \quad \text{converges strongly to} \quad h_2 \partial_x h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$h_{1_{\eta_k}} \partial_x^2 h_{1_{\eta_k}} \quad \text{converges strongly to} \quad h_1 \partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$\partial_x h_{1_{\eta_k}} \partial_x^2 h_{1_{\eta_k}} \quad \text{converges wealkly to} \quad \partial_x h_1 \partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$h_{1_{\eta_k}} \partial_x h_{2_{\eta_k}} \quad \text{converges strongly to} \quad h_1 \partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$h_{2_{\eta_k}} \partial_x h_{2_{\eta_k}} \quad \text{converges strongly to} \quad h_2 \partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_{1_{\eta_k}})^2 \quad \text{converges strongly to} \quad {h_1}^2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_{2_{\eta_k}})^2 \quad \text{converges strongly to} \quad {h_2}^2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$h_{1_{\eta_k}} h_{2_{\eta_k}} \quad \text{converges strongly to} \quad h_1 h_2 \quad \text{in } L^1(0,T;L^1(0,1)).$$

Convergences of $h_{1_{\eta_k}} \partial_x u_{\eta_k}$, u_{η_k}

As u_{η_k} is bounded in $L^2(0,T;L^2(0,1))$, then $\partial_x u_{\eta_k}$ is bounded in $L^2(0,T;W^{-1,2}(0,1))$.

Then,

$$u_{\eta_k} \quad \text{converges strongly to} \quad u \quad \text{in } L^1(0,T;L^1(0,1)).$$

At last, the function $(h_{1_{\eta_k}}, \partial_x u_{\eta_k}) \longmapsto h_{1_{\eta_k}} \partial_x u_{\eta_k}$ is a continuous in $L^\infty(0,T;H^1(0,1)) \times L^2(0,T;W^{-1,2}(0,1))$ to $L^2(0,T;W^{-1,2}(0,1))$.

So,

$$h_{1_{\eta_k}} \partial_x u_{\eta_k} \quad \text{converges weakly to} \quad h_1 \partial_x u \quad \text{in } L^2(0,T;H^{-1}(0,1)).$$

Convergences of $h_{2_{\eta_k}} u_{\eta_k}$ and $\partial_x^2 h_{2_{\eta_k}}$

We know that $\partial_x h_{2_{\eta_k}}$ is bounded in $L^2(0,T;L^2(0,1))$ this implies $\partial_x^2 h_2^k$ is in $L^1(0,T;W^{-1,2}(0,1))$.
So,

$$\partial_x^2 h_{2_{\eta_k}} \quad \text{converges weakly to} \quad \partial_x^2 h_2 \in L^1(0,T;W^{-1,2}(0,1)).$$

To conclude, we have u_{η_k} converges weakly to u in $L^2(0,T;L^2(0,1))$ and the strong convergence of h_2^k to h_2, gives us:

$$h_{2_{\eta_k}} u_{\eta_k} \quad \text{converges weakly to} \quad h_2 u \quad \text{in } L^1(0,T;L^1(0,1)).$$

Convergence of $\Big(a(h_{2_{\eta_k}})^2 + b(h_{2_{\eta_k}})^3)\Big)\partial_x(h_{1_{\eta_k}} + h_{2_{\eta_k}})$

We know that $\partial_x(h_{1_{\eta_k}} + h_{2_{\eta_k}})$ converges weakly to $\partial_x(h_1 + h_2)$ in $L^2(0,T;L^2(0,1))$ and $(a(h_{2_{\eta_k}})^2 + b(h_{2_{\eta_k}})^3)$converges strongly to $ah_2^2 + bh_2^3$ in $L^1(0,T;L^1(0,1))$.
So,

$$\Big(a(h_{2_{\eta_k}})^2 + b(h_{2_{\eta_k}})^3)\Big)\partial_x(h_{1_{\eta_k}} + h_{1_{\eta_k}}) \quad \text{converges weakly to} \quad (ah_2^2 + bh_2^3)\partial_x(h_1 + h_2) \quad \text{in } L^1(0,T;L^1(0,1))$$

Convergences of $h_{1_\eta} V(h_{1_{\eta_k}})$ and $V(h_{1_{\eta_k}})\partial_x h_{1_{\eta_k}}$

We will begin by studying the convergence of the term $h_1 V(h_{1_{\eta_k}})$. We have $h_1 V(h_{1_{\eta_k}}) = \dfrac{1}{(h_{1_{\eta_k}})^2} - \dfrac{\alpha}{(h_{1_{\eta_k}})^3}$ and

$$\left|\frac{1}{(h_{1_{\eta_k}})^2} - \frac{\alpha}{(h_{1_{\eta_k}})^3} - (\frac{1}{h_{1_\eta}^2} - \frac{\alpha}{h_{1_\eta}^3})\right| \le \left|\frac{1}{(h_{1_{\eta_k}})^2} - \frac{1}{h_1^2}\right| + \left|\frac{1}{(h_{1_{\eta_k}})^3} - \frac{1}{h_1^3}\right|$$

$$\left|\frac{1}{(h_{1_{\eta_k}})^2} - \frac{\alpha}{(h_{1_{\eta_k}})^3} - (\frac{1}{h_1^2} - \frac{\alpha}{h_1^3})\right| \le \frac{|h_{1_{\eta_k}} - h_1||h_{1_{\eta_k}} + h_1|}{(h_{1_{\eta_k}})^2 h_1^2} + \frac{|h_{1_{\eta_k}} - h_1||(h_{1_{\eta_k}})^2 + h_{1_{\eta_k}} h_1 + h_1^2|}{(h_{1_{\eta_k}})^3 h_1^3}.$$

We use the **Proposition** 1 to find two constants δ_1 and δ_2 such as

$$\left|\frac{1}{(h_{1_{\eta_k}})^2} - \frac{\alpha}{(h_{1_{\eta_k}})^3} - (\frac{1}{h_1^2} - \frac{\alpha}{h_1^3})\right| \le \delta_1|h_{1_{\eta_k}} - h_1| + \delta_2|h_{1_{\eta_k}} - h_1|.$$

So

$$\left|\frac{1}{(h_{1_{\eta_k}})^2} - \frac{\alpha}{(h_{1_{\eta_k}})^3} - (\frac{1}{h_1^2} - \frac{\alpha}{h_1^3})\right|^2 \le \delta_3^2|h_1^k - h_1|^2 \to 0, \quad \text{with } \delta_3 = 2\max(\delta_1,\delta_2).$$

We have

$$\frac{1}{(h_{1_{\eta_k}})^2} - \frac{\alpha}{(h_{1_{\eta_k}})^3} \quad \text{converges strongly to} \quad \frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \text{ in } L^2(0,T;L^2(0,1)).$$

A similar reasoning ensures the strong convergence of $\dfrac{1}{(h_{1_{\eta_k}})^3} - \dfrac{\alpha}{(h_{1_{\eta_k}})^4}$ to $\dfrac{1}{h_1^3} - \dfrac{\alpha}{h_1^4}$ in $L^2(0,T;L^2(0,1))$.
The strong convergence of $\partial_x h_{1_{\eta_k}}$ in $L^2(0,T;L^2(0,1))$ gives us

$$V(h_{1_{\eta_k}})\partial_x h_{1_{\eta_k}} \quad \text{converges weakly to} \quad V(h_1)\partial_x h_1 \quad \text{in } L^1(0,T;L^1(0,1)).$$

Convergence of $\partial_x^2 u_{\eta_k}$, $h_{1_{\eta_k}}\partial_x^7 h_{1_{\eta_k}}$, $\partial_x^6 h_{2_{\eta_k}}$

• The bound of $\partial_x^2 u_{\eta_k}$ in $L^2(0,T;L^2(0,1))$ gives us:

$$\partial_x^2 u_{\eta_k} \quad \text{converges strongly to} \quad \partial_x^4 u \quad \text{in } L^1(0,T;L^1(0,1)).$$

•, we have:

$$\int_0^T\int_0^1 h_{1_{\eta_k}}\partial_x^7 h_{1_{\eta_k}}\phi = -3\int_0^T\int_0^1 \partial_x^2 h_{1_{\eta_k}}\partial_x\phi\partial_x^4 h_{1_{\eta_k}} - 3\int_0^T\int_0^1 \partial_x h_{1_{\eta_k}}\partial_x^2\phi\partial_x^4 h_{1_{\eta_k}} - \int_0^T\int_0^1 \phi\partial_x^3 h_{1_{\eta_k}}\partial_x^4 h_{1_{\eta_k}}$$
$$-\int_0^T\int_0^1 h_{1_{\eta_k}}\partial_x^3\phi\partial_x^4 h_{1_{\eta_k}}$$

The function $(\partial_x^2 h_{1_{\eta_k}}, \partial_x^4 h_{1_{\eta_k}}) \longmapsto \partial_x^2 h_{1_{\eta_k}} \partial_x^4 h_{1_{\eta_k}}$ is a continuous in $L^\infty(0,T;H^1(0,1)) \times L^2(0,T;W^{-1,2}(0,1))$ to $L^2(0,T;W^{-1,2}(0,1))$, so $(\partial_x^2 h_{1_{\eta_k}} \partial_x^4 h_{1_{\eta_k}})_k$ converges weakly to $\partial_x^2 h_{1_\eta} \partial_x^4 h_{1_\eta}$ in $L^2(0,T;W^{-1,2}(0,1))$. Next, the function $(\partial_x^3 h_{1_{\eta_k}}, \partial_x^4 h_{1_{\eta_k}}) \longmapsto \partial_x^3 h_{1_{\eta_k}} \partial_x^4 h_{1_{\eta_k}}$ is a continuous in $L^\infty(0,T;H^1(0,1)) \times L^2(0,T;W^{-1,2}(0,1))$ to $L^2(0,T;W^{-1,2}(0,1))$, so $(\partial_x^3 h_{1_{\eta_k}} \partial_x^4 h_{1_{\eta_k}})_k$ converges weakly to $\partial_x^3 h_1 \partial_x^4 h_1$ in $L^2(0,T;W^{-1,2}(0,1))$.

$\partial_x h_{1_{\eta_k}} \partial_x^4 h_{1_{\eta_k}}$ converges weakly to $\partial_x h_1 \partial_x^4 h_1$ in $L^2(0,T;L^1(0,1))$. Finally, $h_{1_{\eta_k}} \partial_x^4 h_{1_{\eta_k}}$ converges strongly to $h_1 \partial_x^4 h_1$ in $L^1(0,T;L^1(0,1))$.

• We have:

$$\int_0^T \int_0^1 \partial_x^6 h_{2_{\eta_k}} \phi = \int_0^T \int_0^1 h_{2_{\eta_k}} \partial_x^6 \phi$$

The bound of $h_{2_{\eta_k}}$ in $L^\infty(0,T;L^2(0,1))$, give us:

the strong convergence of $h_{2_{\eta_k}}$ in $L^2(0,T;W^{-1,1}(0,1))$.

Convergence of $h_{1_{\eta_k}} \partial_x^3 h_{1_{\eta_k}}$

The function $(h_{1_{\eta_k}}, \partial_x^3 h_{1_{\eta_k}}) \longmapsto h_{1_{\eta_k}} \partial_x^3 h_{1_{\eta_k}}$ is a continuous in $L^\infty(0,T;H^1(0,1)) \times L^2(0,T;W^{-1,2}(0,1))$ to $L^2(0,T;W^{-1,2}(0,1))$, so $h_{1_{\eta_k}} \partial_x^3 h_{1_{\eta_k}}$ converges weakly to $h_1 \partial_x^3 h_1$ in $L^2(0,T;W^{-1,2}(0,1))$.

These above convergences then allow us to pass to the limit as $n \to \infty$ in the weak formulation of the approximating systems (14) − (16) − (18) − (19) in order to get that (h_1, h_2, u_1) satisfies (24) − (25).

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