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Anti-Symplectic Involutions on Symplectic Manifolds $S^2 \times S^2$

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Abstract

Let X be the symplectic manifold $S^2 \times S^2$. In this paper, we discuss anti-symplectic involutions on X. We construct one equivalent class of anti-symplectic involutions on X from the equivalent class of anti-symplectic involutions on S^2.

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1. Introduction

To study group actions on 4-dim manifolds, fixed point set is helpful. For example, (Atiyah & Bott, 1968) study the relation between fixed point set of group actions and the induced group actions on homology of manifolds. (Atiyah, 1982) find for compact symplectic manifold and the n-torus hamiltonian actions on it, they can recover the cohomology of manifold by the cohomology of fixed point set. Fixed point set can also be used to study the classification of group actions (Li & Liu, 2008), (Liu & Nakamura, 2007 & 2008). By using fixed point set, (Karsho, 1995) study the classification of periodic hamiltonian flows on compact symplectic 4-manifold.

In recent years, there are some studies about anti-symplectic involutions on symplectic manifolds. Such as the fixed point set of anti-symplectic involutions and the quotient space of the involutions (Biss, Guillemin & Holm, 2004) and (Cho & Joe, 2002). Besides, there are also many studies on lagrangian spheres in rational manifolds. Such as (Hind, 2004) prove there is only one isotopy class of lagrangian spheres in $S^2 \times S^2$. Since lagrangian spheres could be the fixed point set of some involutions, the classification of anti-symplectic involutions on symplectic manifolds is naturally be a research focus.

In this paper, we try to study the classification of anti-symplectic involutions on rational manifolds $S^2 \times S^2$. And we give one equivalent class of anti-symplectic involutions on $S^2 \times S^2$. Concretely, we get the following results.

Theorem 1. *For (S^2, ω) , where $\omega = d\theta \wedge dh$, (θ, h) is the cylindrical coordinate of S^2. T is the circle action on it. Suppose σ_1, σ_2 are two anti-symplectic involutions induced by reflections on S^2 and compatible with T. Then σ_1, σ_2 are smooth equivalent.*

Theorem 2. *Let T be the circle action on S^2, σ_1, σ_2 be two anti-symplectic involutions on S^2 induced by two reflections compatible with T, then the anti-symplectic involutions $\tau_{\sigma_1}, \tau_{\sigma_2}$ on $(S^2 \times S^2, \omega \oplus \omega)$ must be smooth equivalent, where $\tau_{\sigma_1}, \tau_{\sigma_2}$ are defined as below*

$$\tau_{\sigma_i} : S^2 \times S^2 \to S^2 \times S^2$$

$$(\theta_1, h_1, \theta_2, h_2) \mapsto (\sigma_i^{-1}(\theta_2, h_2), \sigma_i(\theta_1, h_1)),$$

where $i = 1, 2$.

Theorem 3. *Let T be the circle action on S^2, σ_1 be the anti-symplectic involution on S^2 induced by reflection compatible with T, σ_2 be the anti-symplectic involutions on S^2 induced by antipodal map, then the anti-symplectic involutions $\tau_{\sigma_1}, \tau_{\sigma_2}$ on $(S^2 \times S^2, \omega \oplus \omega)$ must be smooth equivalent and compatible with the hamiltonian diagonal torus action on X.*

We organize the remainder of this paper as follows. In section 2, we review some preliminaries used in study. In section 3, we study the anti-symplectic involutions on S^2 and prove Theorem 1, Furthermore, we study the anti-symplectic involutions on $S^2 \times S^2$ and prove Theorem 2 and Theorem 3.

2. Preliminary

Let X be a symplectic manifold, that means X is a smooth manifold and there is a closed nondegenerate 2-form on X. A symplectic manifold is necessarily of even dimension $2n$ and the n-fold wedge product $\omega \wedge \ldots \wedge \omega$ never vanishes, thus

M is orientable. For details about symplectic manifolds, refer to (Mcduff & Salamo, 1995).

Let $(X;\omega)$ be a closed, symplectic manifold with a symplectic structure ω. A smooth map $\sigma : X \to X$ is an anti-symplectic involution if and only if $\sigma^*\omega = \omega$ and $\sigma^2 = -\mathrm{Id}$. Denote X^σ the fixed point set of σ. We call it the real locus of X. If the real locus X^σ is nonempty, then it is a Lagrangian submanifold of X (Sjamaar, 2007). Moreover, $\dim X^\sigma = \frac{1}{2}\dim X$.

By the definition of symplectic manifold, sphere S^2 is a 2-dim symplectic manifold. Under the cylindrical coordinate, we can take $\omega = d\theta \wedge dh$ as the symplectic form. Let T be the S^1 action on X. If for any $\tau \in T$, we have $\tau \circ \sigma \circ \tau = \sigma$, then we call the anti-symplectic involution σ and τ are compatible. By (Sjamaar, 2007), if σ and τ are compatible, then

$$\dim H^*(X; Z_2) = \dim H^*(X^T; Z_2).$$

$$\dim H^*(X^\sigma; Z_2) = \dim H^*(X^\sigma \cap X^T; Z_2).$$

Thus X^σ is empty set or S^1 that passing through the south and north poles of sphere under the sense of homeomorphism. And by the property of involution, if fixed point set is S^1, then the intersection of S^1 and horizontal plane must be a pair of antipodal points. To be simple, we call this kind of S^1 the symmetrical S^1. In fact, only the reflection and antipodal maps can induce anti-symplectic involution on X, in which the fixed point set of the anti-symplectic involution that compatible with T is symmetrical S^1. While the fixed point set of antipodal map is empty set.

Definition 3. *For the smooth actions g_1, g_2, if there is a diffeomorphism φ, such that $\varphi \circ g_1 = g_2 \circ \varphi$ for any point on X, then we say g_1 and g_2 are smooth equivalent.*

By Definition 3, it is obvious that the anti-symplectic involutions induced by the reflections and antipodal maps on X are not equivalent.

3. Main Result

In fact, we can look every symmetrical S^1 as the fixed point of some anti-symplectic involution induced by reflection. Concretely,

$$\sigma : S^2 \to S^2, \sigma(\theta, h) = (2\varphi - \theta, h),$$

where (θ, h) is the cylindrical coordinate of the points on the symmetrical S^1. Obviously, the above σ is a reflection about symmetrical S^1, so it can induce an anti-symplectic involution on sphere and its fixed point set is the given symmetrical S^1. We can prove all these anti-symplectic involutions on sphere are smooth equivalent.

Theorem 1. *Let (X, ω) be a standard sphere, $\omega = d\theta \wedge dh$ be the symplectic form. Let T be S^1 action on X. Suppose σ_1, σ_2 are two anti-symplectic involutions on X induced by reflections on sphere X and compatible with T action, then σ_1, σ_2 are smooth equivalent.*

Proof. Suppose

$$\phi : S^2 \to S^2, \phi(\theta, h) = (\theta + \varphi_2 - \varphi_1, h).$$

Obviously, it is a smooth diffeomorphism, where (φ_1, h), (φ_2, h) are cylindrical coordinates of the fixed point set of σ_1 and σ_2. Thus for any points (θ, h) on sphere, we have

$$(\phi \circ \sigma_1)(\theta, h) = \phi(2\varphi_1 - \theta, h) = (\varphi_1 + \varphi_2 - \theta, h),$$

$$(\sigma_2 \circ \phi)(\theta, h) = \sigma_2(\theta + \varphi_2 - \varphi_1, h) = (\varphi_1 + \varphi_2 - \theta, h).$$

Thus $\phi \circ \sigma_1 = \sigma_2 \circ \phi$ is always true on X, that is anti-symplectic involutions σ_1 and σ_2 are smooth equivalent.

By the anti-symplectic involution σ on sphere, we can construct anti-symplectic involution τ_σ on 4-dim manifold $(S^2 \times S^2, \omega \oplus \omega)$ as follows,

$$\tau_\sigma : S^2 \times S^2 \longrightarrow S^2 \times S^2, (x, y) \longmapsto (\sigma(y), \sigma(x)).$$

Obviously, $\tau_\sigma^2 = \mathrm{Id}$ and $\tau_\sigma^*(\omega \oplus \omega) = -(\omega \oplus \omega)$. We can prove easily the fixed point set of τ_σ is exactly the graph of σ. That is $(S^2 \times S^2)^{\tau_\sigma} = \{(x, y) | y = \sigma(x)\} \cong S^2$. For any anti-symplectic involutions induced by reflections and antipodal maps on sphere, we can construct anti-symplectic involutions on $(S^2 \times S^2, \omega \oplus \omega)$ like this. And we can prove these anti-symplectic involutions are smooth equivalent. Concretely, we can prove the following result.

Theorem 2. *Let X be symplectic manifold $(S^2 \times S^2, \omega \oplus \omega)$, T be S^1 action on X, σ_1, σ_2 are two anti-symplectic involutions induced by reflections and compatible with T, then τ_{σ_1}, τ_{σ_2} are smooth equivalent.*

Proof. By assumption and Theorem 1, σ_1, σ_2 are smooth equivalent. Thus there exist a smooth diffeomorphism ϕ, such that $\phi \circ \sigma_1 = \sigma_2 \circ \phi$. Thus, we can construct a smooth diffeomorphism on $(S^2 \times S^2, \omega \oplus \omega)$ as below,

$$\Phi = \phi \times \phi : S^2 \times S^2 \longrightarrow S^2 \times S^2$$

$$\Phi(x, y) = (\phi(x), \phi(y)),$$

where $(x, y) \in S^2 \times S^2$. And we have

$$(\Phi \circ \tau_{\sigma_1})(x, y) = \Phi(\sigma_1(y), \sigma_1(x)) = (\phi(\sigma_1(y)), \phi(\sigma_1(x))) = ((\sigma_2 \circ \phi)(y), (\sigma_2 \circ \phi)(x)),$$

$$(\tau_{\sigma_2} \circ \Phi)(x, y) = \tau_{\sigma_2}(\phi(x), \phi(y)) = ((\sigma_2 \circ \phi)(y), (\sigma_2 \circ \phi)(x)).$$

That is $\Phi \circ \tau_{\sigma_1} = \tau_{\sigma_2} \circ \Phi$ for any points on $S^2 \times S^2$. Hence the anti-symplectc involutions τ_{σ_1}, τ_{σ_2} are smooth equivalent.

Theorem 3. *Let X be symplectic manifold $(S^2 \times S^2, \omega \oplus \omega)$, T be S^1 action on X. σ_1 is an anti-symplectic involution on sphere induced by reflection and compatible with T, σ_2 is the anti-symplectic involution on sphere induced by antipodal map. Then the anti-symplectic involutions τ_{σ_1}, τ_{σ_2} are smooth equivalent.*

Proof. Suppose φ is a reflection on X about some given symmetric S^1, such that $\varphi(\theta, h) = (2\alpha - \theta, h)$, where (α, h) is the cylindrical coordinate of the symmetric S^1. Obviously $\varphi^2 = \mathbf{Id}_X$. Set $f = \sigma_2 \circ \varphi \circ \sigma_2$, then

$$f^2 = \sigma_2 \circ \varphi \circ \sigma_2(\theta, h) = \sigma_2 \circ \varphi(\theta + \pi, -h) = \sigma_2(2\alpha - \theta - \pi, -h) = (2\alpha - \theta, h).$$

Hence f is a reflection on X with the above symmetric S^1 as fixed-point set, and compatible with the T action on X. Thus by Theorem 1, f and σ_1 are smooth equivalent. Furthermore, by the proof of Theorem 2, τ_f and τ_{σ_1} are smooth equivalent. Hence to prove τ_{σ_1}, τ_{σ_2} are smooth equivalent, we only need to prove τ_f and τ_{σ_2} are smooth equivalent.

Set

$$\Phi = (\varphi \circ \sigma_2) \times \mathbf{Id_X} : S^2 \times S^2 \longrightarrow S^2 \times S^2$$

$$\Phi(x, y) = (\varphi \circ \sigma_2(x), y), (x, y) \in S^2 \times S^2.$$

Obviously, Φ is a diffeomorphism on $(S^2 \times S^2, \omega \oplus \omega)$, and satisfy

$$(\Phi \circ \tau_f)(x, y) = \Phi(f(y), f(x)) = \Phi(\sigma_2 \circ \varphi \circ \sigma_2(y), \sigma_2 \circ \varphi \circ \sigma_2(x)) = (\sigma_2(y), \sigma_2 \circ \varphi \circ \sigma_2(x)),$$

$$(\tau_{\sigma_2} \circ \Phi)(x, y) = \tau_{\sigma_2}(\varphi \circ \sigma_2(x), y) = (\sigma_2(y), \sigma_2 \circ \varphi \circ \sigma_2(x)).$$

So for any $(x, y) \in S^2 \times S^2$, we have $\Phi \circ \tau_f = \tau_{\sigma_2} \circ \Phi$. Thus τ_f, τ_{σ_2} are smooth equivalent.

By Theorem 3, we construct infinitely many anti-symplectic involutions on standard 4-dim manifold $S^2 \times S^2$ and prove these anti-symplectic involutions are smooth equivalent.

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