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Feedback Systems on Extended Hilbert Space-Normality and Linearization

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Abstract

The study of the normality of a feedback system on an extended Hilbert space has been made. The results of approximation of the solutions of such a nonlinear system by another linear are also established. This study represents an extension of the work of (*Vaclav Dolezal*, 1979), on a Hilbert space.

Keywords: feedback system, maximum monotone operator, extended Hilbert space, normality and linearization

AMS classification: 2000, 65J15; 65J10; 93A05; 47H07; 93B18

1. Introduction

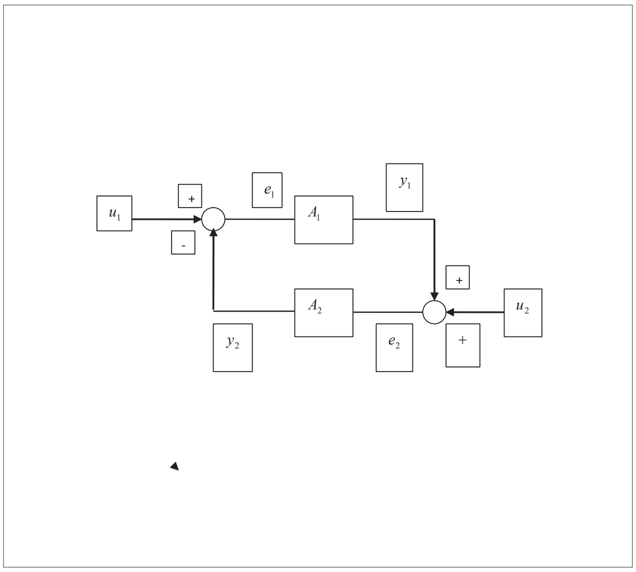
In recent decades, special attention was devoted, to the study and the development of systems analysis, more precisely: electrical engineering, telecommunications and economic systems around the world. The fundamental publication of (*G.Zames*, 1963), has shown the important role of functional analysis in the study of nonlinear systems. (*Vaclav Dolezal*, 1979) introdced the feedback systems described by certain special types of operators, defined on appropriate spaces. He has established, a series of existence and uniqueness results, of the solutions of this system on a Hilbert space H. He obtained among others conditions of causality, stability and Lipschitz continuity. In addition, (*Vaclav Dolezal*, 1980; 1990) demonstrated, how these results are applicable, in several domains such that: control theory, network theory, solving the Hammerstein equation...etc. The techniques used by the author are based, on the surjectivity theorem, of the monotonic and coercive maximal operators of (*R.T.Rockafellar*, 1970). Since, the resolution of some special cases of feedback systems, on normed spaces, is often a difficult task, (*Vaclav Dolezal*, 1979) introduced, the notion of extended Hilbert space He, and obtained, a normality result for a feedback system, on this space. Morover, (*Vaclav Dolezal*, 1991), showed how to use such a space, in the study of stability robustness, and the sensitivity of this system. In the present work, we propose to formalize and generalize in He, the results obtained in H. One of our fundamental results is that, the behavior of $[A_1, A_2]$ is completely determined, by the inverse of some application $M_a = I + A_2(a + A_1)$ (see (2)). Note that, in the case where the operators A_1 and A_2 are not linear, and if $(u_1, u_2) \mapsto (e_1, e_2)$, then $(e_1, e_2) = (M_{u_2}^{-1} u_1, u_2 + A_1 M_{u_2}^{-1} u_1)$. If one of the two operators is linear, the writing of the solution (e_1, e_2), can take forms, that do not necessarily depend, on the inverse of the operator M_{u_2}, (section 4, (4)&(5)). These forms, play an important role in the study of the sensitivity (*Vaclav Dolezal*, 1990), and give suitable estimates of the solutions in the sense of section 3.2. For more details, on the study of the inverse of such an operator, which is non linear, one consult (*Vaclav Dolezal*, 1998; 1999; 2003). It is then natural, to proceed to the approximation method. Therefore, to find an approximate solution of $[A_1, A_2]$, supposed nonlinear, by one linearizes, in the neighborhood of zero. We then consider, a linear $[A_1^0, A_2^0]$ on He, and prove that, if $(u_1, u_2) \mapsto (e_1, e_2) \in H_e^2$ and $(u_1, u_2) \mapsto (e_1^0, e_2^0) \in H_e^2$, where $(u_1, u_2) \in H_P^2$, with $\|u_1\| \leq r$, $\|u_2\| \leq r$ $(r > 0)$ and (e_1, e_2), (e_1^0, e_2^0) the respective solutions of $[A_1, A_2]$ and $\left[A_1^0, A_2^0\right]$. There exists, $k_{11}, k_{12}, k_{21}, k_{22}$, positive real constants such that

$$\left\|e_1 - e_1^0\right\| \leq k_{11} \|u_1\| + k_{12} \|u_2\|,$$

and

$$\left\|e_2 - e_2^0\right\| \leq k_{21} \|u_1\| + k_{22} \|u_2\|.$$

Our work is organized as follows: in section 2, we recall some definitions concerning, the existence and uniqueness of solutions of feedback systems on a vector space, the definitions of an extended Hilbert space, the spaces $\mathcal{M}$ and Lip. Section 3 is reserved for our results of, normality and linearization of nonlinear feedback systems on H_e. Section 4 contains the reminders of the results used in this paper.



2. Definitions and Notations

Let H be a real vector space, 2^H the set of parts of H, A an application of H into 2^H and $D(A) = \{x \in H;\ Ax \neq \emptyset\}$, the domain of A. We say that, A is an operator, if $D(A) = H$ and Ax is a singleton for all x in H.

Definition 2.1.

(a) We call feedback system on H, and we write FS, any pair $[A_1, A_2]$ of applications of H in 2^H.

(b) We say that, an element (e_1, e_2) (error) of H^2 is a solution of $[A_1, A_2]$, corresponding to the given (u_1, u_2) (input) of H^2 and we write $(u_1, u_2) \mapsto (e_1, e_2)$, if there exists (y_1, y_2) (output) in $A_1e_1 \times A_2e_2$ such that:

$$\begin{cases} e_1 = u_1 - y_2; \\ e_2 = u_2 + y_1. \end{cases} \tag{1}$$

The meaning of the preceding notions, can be understood for exemple, from a physical point of view, by looking at the above representative schema

Definition 2.2. We say that the FS $[A_1, A_2]$ on H is:

(i) Resoluble, if for all $(u_1, u_2) \in H^2$, there exists a solution $(e_1, e_2) \in H^2$, corresponding to (u_1, u_2).

(ii) Unambiguous, if each solution is unique.

(iii) Normal, if it is resoluble and unambiguous.

The existence and uniqueness results of the solutions of $[A_1, A_2]$ over H, are based on the mapping $M_a : H \mapsto 2^H$ defined for all $(a, x) \in H^2$, by

$$M_a x = x + A_2(a + A_1 x). \tag{2}$$

Let H be a Hilbert space, $<,>$ the scalar product over H, $\|.\|$ the norm induced by the scalar product, H_e a vector space containing H, and $\mathcal{P} = \{P_\alpha;\ \alpha \in I\}$ a non-empty family of linear operators on H_e. For all $P \in \mathcal{P}$, $x^{(P)}$ will denote an element of $H_P := PH$.

Definition 2.3. We say that, H_e is an extended Hilbert, or an extension of H, if the following axioms are verified:

(i) $P^2 = P$, $\forall P \in \mathcal{P}$.

(ii) ($P_1P_2 = P_2P_1$ and $P_1P_2 \in \mathcal{P}$), $\forall P_1, P_2 \in \mathcal{P}$.

(iii) If $x \in H_e$, then $Px \in H$, $\forall P \in \mathcal{P}$.

(iv) If $\forall P \in \mathcal{P}$, the element $x^{(P)}$ is in H_P, and $P_0 x^{(P_1)} = P_0 x^{(P_2)}$ $\forall P_1, P_2 \in \mathcal{P}$, where $P_0 = P_1 P_2$. Then, there exists x in H_e, such that $x^{(P)} = Px$, $\forall P \in \mathcal{P}$.

(v) If $x \in H$, then $\|Px\| \leq \|x\|$, $\forall P \in \mathcal{P}$.

(vi) If $x \in H_e$ and $\|Px\| \leq a$, $\forall P \in \mathcal{P}$, where $a \geq 0$. Then $x \in H$ and $\|x\| \leq a$.

It is straightforward to cheque that:

(a) $\forall P_1, P_2 \in \mathcal{P}$, $P_0 P_1 = P_0 P_2 = P_0$, where $P_0 = P_1 P_2$.

(b) $\forall P \in \mathcal{P}$, $PH_e = H_P$, and H_P is a closed subspace of H.

(c) If $x \in H_e$ and $Px = 0$ for all $P \in \mathcal{P}$, then $x = 0$.

(d) The element $x \in H$, in axiom (iv) is unique.

(e) $\langle Px, y \rangle = \langle x, Py \rangle$, for every $x, y \in H$ and every $P \in \mathcal{P}$.

Exemple2.1. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$, n times the product of $\mathbb{R}$, and $\overline{C}(\mathbb{R}_+; \mathbb{R}^n)$ the vector space of the continuous functions $x : \mathbb{R}_+ \to \mathbb{R}^n$. $C(\mathbb{R}_+; \mathbb{R}^n)$ denotes the subspace of the functions of $\overline{C}(\mathbb{R}_+; \mathbb{R}^n)$, which are bounded, for the norm defined by: for all $x \in C(\mathbb{R}_+; \mathbb{R}^n)$, $\|x\| = \sup\{|x(t)| ; t \in \mathbb{R}_+\}$ where $|.|$ is a norm of $\mathbb{R}^n$.

For the family $\mathcal{P}=\left\{P_\alpha : \overline{C}(\mathbb{R}_+; \mathbb{R}^n) \to C(\mathbb{R}_+; \mathbb{R}^n) ;\ \alpha \in \mathbb{R}_+\right\}$ where, for all $\alpha \in \mathbb{R}_+$

$$(P_\alpha x)(t) = \{.x(t) \text{ if } t \in [0, \alpha[\ ; x(\alpha) \text{ if } t \in [\alpha, +\infty[\ ,$$

$\overline{C}(\mathbb{R}_+; \mathbb{R}^n)$ is an extended space of $C(\mathbb{R}_+; \mathbb{R}^n)$.

Exemple2.2. Let $\overline{L_2}(\mathbb{R}_+)$ be the space of the functions $t \in \mathbb{R}_+ \longmapsto x(t) \in \mathbb{R}$, which are locally square integrable on $\mathbb{R}_+$, and $L_2(\mathbb{R}_+)$ the subspace of the functions x, which are square integrable on $\mathbb{R}_+$.

We define the family $\mathcal{P} = \left\{P_\alpha : \overline{L_2}(\mathbb{R}_+) \to L_2(\mathbb{R}_+) ;\ \alpha \in \mathbb{R}_+\right\}$ by:

$$(P_\alpha x)(t) = \{.x(t) \text{ if } t \in [0, \alpha] \ ; 0 \text{ if } t \in]\alpha, +\infty[\ .$$

for all $\alpha \in \mathbb{R}_+$. $\overline{L_2}(\mathbb{R}_+)$ is an extended space of $L_2(\mathbb{R}_+)$.

Definition 2.4. An operator $A : H_e \to H_e$ is called causal if, $\forall P \in \mathcal{P}$, PA=PAP.

Definition 2.5. An normal FS $[A_1, A_2]$ on H_e, is called causal if, for $(u_1, u_2) \mapsto (e_1, e_2)$ and $(u_1^{'}, u_2^{'}) \mapsto (e_1^{'}, e_2^{'})$, such that $\forall P \in \mathcal{P}$, $Pu_1 = Pu_1^{'}$ and $Pu_2 = Pu_2^{'}$, then $Pe_1 = Pe_1^{'}$ and $Pe_2 = Pe_2^{'}$.

Definition 2.6. We say that, an operator $A : H \to H$ is hemicontinuous in $x_0 \in H$, if for all $\omega \in H$ and for any real sequence $t_n \to 0$; the sequence $A(x_0 + t_n\omega)$ converges weakly to $A(x_0)$ in H. A is hemicontinuous on H, if it is hemicontinuous in any point of H.

Before stating the results of normalities, we introduce the two following spaces:

$$\mathcal{M}=\left\{ N : H \to H \text{ such that } \mu_N := \inf_{\substack{x_1, x_2 \in H \\ x_1 \neq x_2}} \frac{\langle Nx_1 - Nx_2, x_1 - x_2 \rangle}{\|x_1 - x_2\|^2} > -\infty \right\}$$

and

$$Lip=\left\{ N : H \to H \text{ such that } \|N\|^* := \sup_{\substack{x_1, x_2 \in H \\ x_1 \neq x_2}} \frac{\|Nx_1 - Nx_2\|}{\|x_1 - x_2\|} < +\infty \right\}$$

It is clear that $Lipis \subset \mathcal{M}$ and $\forall M, N \in \mathcal{M}$, $\forall \alpha \geq 0$

(i) $M + N, \alpha N \in \mathcal{M}$, $\mu_{M+N} \geq \mu_M + \mu_N$ and $\mu_{\alpha N} = \alpha \mu_N$.

(ii) N is monotone (respectively strictly monotone) iff $\mu_N \geq 0$ (respectively $\mu_N > 0$).

On the other hand, $\forall M, N \in Lip$, $\forall \alpha, \beta \in \mathbb{R}$

(iii) $\|N\|^* \geq |\mu_N|$, $\|N\|^* \geq 0$ and $\|N\|^* = 0$ iff N is constant.

(iv) $\alpha N + \beta M$, $NM \in Lip$, $\|\alpha N\|^* = |\alpha| \|N\|^*$, $\|N + M\|^* \leq \|N\|^* + \|M\|^*$ and $\|NM\|^* \leq \|N\|^* \|M\|^*$.

(v) If N is linear, then N is bounded iff $N \in Lip$, in this case $\|N\|^* = \|N\|$.

3. Fundamental Results

This section is divided into two subsections. In the first one, we give and prove two normality results. In the second, we have formalize and obtained linearization results.

3.1 Normality of the Feedback System on H_e

The first result of normality in this work is:

Theorem 3.1. Let $A_1, A_2 : H_e \to H_e$ be two causal operators, whose A_2 is linear, and for all $P \in \mathcal{P}$, A_{1P} and A_{2P} the respective restrictions of PA_1 and PA_2 to H_P. We assumed that for any $P \in \mathcal{P}$

(i) $A_{1P} \in \mathcal{M}$, A_{1P} is hemicontinuous and $\mu_{A_{1P}} \leq 0$.

(ii) $A_{2P} \in \mathcal{M}$, *and* $\mu_{A_{2P}} > 0$.

(iii) $\mu_{A_{2P}} + \mu_{A_{1P}} \|A_{2P}\|^2 > 0$.

Then, the FS $[A_1, A_2]$ on H_e is normal and causal.

If $\|A_{2P}\| \leq k$, $\left(\mu_{A_{2P}} + \mu_{A_{1P}} \|A_{2P}\|^2\right)^{-1} \leq \lambda$; $(u_1, u_2) \mapsto (e_1, e_2) \in H_e^2$ and $(u_1^{'}, u_2^{'}) \mapsto (e_1^{'}, e_2^{'}) \in H_e^2$, with $\left(u_1 - u_1^{'},\ u_2 - u_2^{'}\right) \in H^2$ then $\left(e_1 - e_1^{'}, e_2 - e_2^{'}\right) \in H^2$ and

$$\left\|e_1 - e_1^{'}\right\| \leq \lambda k \left\|u_1 - u_1^{'}\right\| + \lambda k^2 \left\|u_2 - u_2^{'}\right\|.$$

If in addition, $A_{1P} \in Lip$ and $\|A_{1P}\|^* \leq k^*$, then

$$\left\|e_2 - e_2^{'}\right\| \leq \lambda k k^* \left\|u_1 - u_1^{'}\right\| + \left(1 + \lambda k^* k^2\right) \left\|u_2 - u_2^{'}\right\|,$$

where λ, k and k^* are positive real constants.

Proof. Let for all $P \in \mathcal{P}$, $N_P = I + A_{2P}A_{1P}$. The operator N_P satisfies the conditions of lemma 4.3, so it is invertible, the inverse $N_P^{-1} \in Lip$ and $\left\|N_P^{-1}\right\|^* \leq \|A_{2P}\| \left(\mu_{A_{2P}} + \mu_{A_{1P}} \|A_{2P}\|^2\right)^{-1}$. Moreover N_P is the restriction to H_P of the operator PN, with $N = I + A_2A_1$ and it is causal. Indeed, since for all $P \in \mathcal{P}$, $P\left[P(I + A_2A_1) - (I + PA_2A_1)\right] = 0$, then $PN = P(I + A_2A_1) = I + PA_2A_1$ (see (c) in section 2). So, for all $P \in \mathcal{P}$, $PN = I + PA_2PA_1 = I + A_{2P}A_{1P} = N_P$ and

$$\begin{aligned} PN_PP &= P\left(I + A_{2P}A_{1P}\right)P = P\left(P + A_{2P}A_{1P}P\right) = P\left(P + PA_2PA_1P\right) \\ &= P\left(P + P^2A_2PA_1\right) = P^2\left(I + PA_2PA_1\right) = P\left(I + A_{2P}A_{1p}\right) = PN_P. \end{aligned}$$

We deduce (cf lemma 4.3, lemma 4.6), that the operators N, N_P^{-1} are invertible and causal. According to corollary 4.2 and (4) the FS $[A_1, A_2]$ is normal,

$$(e_1, e_2) = \left((I + A_2A_1)^{-1}(u_1 - A_2u_2), u_2 + A_1(I + A_2A_1)^{-1}(u_1 - A_2u_2)\right)$$

therefore

$$\left(Pe_1, Pe_1^{'}\right) = \left(N_P^{-1}\left(Pu_1 - A_{2P}u_2\right), N_P^{-1}\left(Pu_1^{'} - A_{2P}u_2^{'}\right)\right)$$

and

$$\begin{aligned} \left\|P(e_1 - e_1^{'})\right\| &= \left\|N_P^{-1}\left(Pu_1 - A_{2P}u_2\right) - N_P^{-1}\left(Pu_1^{'} - A_{2P}u_2^{'}\right)\right\| \\ &\leq \left\|N_P^{-1}\right\|^* \left\|\left(Pu_1 - A_{2P}u_2\right) - \left(Pu_1^{'} - A_{2P}u_2^{'}\right)\right\| \\ &\leq k\lambda\left(\left\|P\left(u_1 - u_1^{'}\right)\right\| + \left\|A_{2P}\left(u_2 - u_2^{'}\right)\right\|\right) \\ &\leq k\lambda\left(\left\|u_1 - u_1^{'}\right\| + \|A_{2P}\| \left\|u_2 - u_2^{'}\right\|\right) \\ &\leq k\lambda \left\|u_1 - u_1^{'}\right\| + \lambda k^2 \left\|u_2 - u_2^{'}\right\| \end{aligned}$$

where the axiom (v) was used.

According to the axiom (vi), $e_1 - e_1^{'} \in H$ and

$$\left\| e_1 - e_1^{'} \right\| \leq \lambda k \left\| u_1 - u_1^{'} \right\| + \lambda k^2 \left\| u_2 - u_2^{'} \right\|.$$

If now $Pu_1 = Pu_1^{'}$ and $Pu_2 = Pu_2^{'}$, the causality of A_2 and the first inequality above leads to $Pe_1 = Pe_1^{'}$, . According to (1) $e_2 = A_1 e_1 + u_2$ and $e_2^{'} = A_1 e_1^{'} + u_2^{'}$, then $Pe_2 = PA_1 e_1 + Pu_2 = PA_1 Pe_1 + Pu_2$ and $Pe_2^{'} = PA_1 e_1^{'} + Pu_2^{'} = PA_1 Pe_1^{'} + Pu_2^{'}$. So $Pe_2 = Pe_2^{'}$ and the $FS\ [A_1, A_2]$ is causal. On the other hand,

$$\begin{aligned} Pe_2 &= Pu_2 + PA_1 (I + A_2 A_1)^{-1} (u_1 - A_2) \\ &= Pu_2 + PA_1 P (I + A_2 A_1)^{-1} (u_1 - A_2 u_2) \\ &= Pu_2 + PA_1 P (I + A_2 A_1)^{-1} P (u_1 - A_2 u_2) \\ &= Pu_2 + A_{1P} N_P^{-1} (Pu_1 - A_{2P} u_2) \end{aligned}$$

likewise

$$Pe_2^{'} = Pu_2^{'} + A_{1P} N_P^{-1} \left(Pu_1^{'} - A_{2P} u_2^{'} \right),$$

therfore

$$\begin{aligned} \left\| P(e_2 - e_2^{'}) \right\| &\leq \left\| u_2 - u_2^{'} \right\| + \\ &\quad + \left\| A_{1P} N_P^{-1} (Pu_1 - A_{2P} u_2) - A_{1P} N_P^{-1} \left(Pu_1^{'} - A_{2P} u_2^{'} \right) \right\| \\ &\leq \left\| u_2 - u_2^{'} \right\| + \| A_{1P} \|^* \left\| N_P^{-1} \right\|^* \left\| P \left(u_1 - u_1^{'} \right) + A_{2P} \left(u_2^{'} - u_2 \right) \right\| \\ &\leq \left\| u_2 - u_2^{'} \right\| + \| A_{1P} \|^* \left\| N_P^{-1} \right\|^* \left(\left\| u_1 - u_1^{'} \right\| + \| A_{2P} \| \left\| u_2^{'} - u_2 \right\| \right) \\ &\leq \left\| u_2 - u_2^{'} \right\| + \lambda k k^* \left(\left\| u_1 - u_1^{'} \right\| + k \left\| u_2^{'} - u_2 \right\| \right) \\ &\leq \lambda k k^* \left\| u_1 - u_1^{'} \right\| + \left(1 + \lambda k^2 k^* \right) \left\| u_2 - u_2^{'} \right\|, \end{aligned}$$

hence $e_2 - e_2^{'} \in H$ and

$$\left\| e_2 - e_2^{'} \right\| \leq \lambda k k^* \left\| u_1 - u_1^{'} \right\| + \left(1 + \lambda k^2 k^* \right) \left\| u_2 - u_2^{'} \right\|.$$

The second result of normality in this work is:

Theorem 3.2. Let $A_1, A_2 : H_e \rightarrow H_e$ be two causal operators whose A_1 is linear, and for all $P \in \mathcal{P}$, A_{1P} and A_{2P}, the respective restrictions of PA_1 and PA_2 to H_P. It is assumed that, for any $P \in \mathcal{P}$

(i) $A_{1P} \in Lip$ and $\mu_{A_{1P}} > 0$.

(ii) $A_{2P} \in \mathcal{M}$, A_{2P} is hemicontinuous and $\mu_{A_{2P}} \leq 0$.

(iii) $\mu_{A_{1P}} + \mu_{A_{2P}} \| A_{1P} \|^2 > 0$.

Then the $FS\ [A_1, A_2]$ on H_e is normal and causal.

Moreover, if $\| A_{1P} \| \leq k, \left(\mu_{A_{1P}} + \mu_{A_{2P}} \| A_{1P} \|^2 \right)^{-1} \leq \lambda$; $(u_1, u_2) \mapsto (e_1, e_2) \in H_e^2$ and $(u_1^{'}, u_2^{'}) \mapsto (e_1^{'}, e_2^{'}) \in H_e^2$ with $\left(u_1 - u_1^{'},\ u_2 - u_2^{'} \right) \in H^2$, then, $\left(e_1 - e_1^{'}, e_2 - e_2^{'} \right) \in H^2$,

$$\left\| e_1 - e_1^{'} \right\| \leq \lambda k \left\| u_1 - u_1^{'} \right\| + (1 + \lambda k) \mu_{A_{1P}}^{-1} \left\| u_2 - u_2^{'} \right\|,$$

and

$$\left\| e_2 - e_2^{'} \right\| \leq \lambda k^2 \left\| u_1 - u_1^{'} \right\| + \left(1 + \left(1 + \lambda k^2 \right) \mu_{A_{1P}}^{-1} \right) \left\| u_2 - u_2^{'} \right\|,$$

where λ and k are two positive real constants.

Proof. Let $z \in H_e$ and be the two operators $M_z, B_z : H_e \rightarrow H_e$ defined respectively by: $M_z x = x + A_2(z + A_1 x)$ and $B_z x = z + A_1 x$ for all $x \in H_e$. It is clear that $M_z = I + A_2 B_z$ and that, for all $P \in \mathcal{P}$, $PB_z = P(z + A_1) = Pz + PA_1 P = P(z + A_1 P) = PB_z P$ and

$$\begin{aligned} PM_z P &= P (I + A_2 B_z) P = P (P + A_2 B_z P) = P^2 + PA_2 B_z P \\ &= P^2 + PA_2 PB_z P = P^2 + PA_2 PB_z = P + PA_2 B_z \\ &= P (I + A_2 B_z) = PM_z, \end{aligned}$$

therfore B_z and M_z are causal. With the same argumentation used in the proof of theorem 3.1, we have on H_P, for all $P \in \mathcal{P}$, $PM_z = P(I + A_2B_z) = I + PA_2B_z$ and therfore $PM_z = I + PA_2PB_z = I + A_{2P}B_{Pz} = N_P$, where $B_{Pz} = z + A_{1P}$. Then the operator N_P is the restriction to H_P of the oprerator PM_z and N_P satisfy the conditions of lemma 4.4, so it is causal, invertible, the inverse $N_P^{-1} \in Lip$, and $\left\|N_P^{-1}\right\|^* \leq \|A_{1P}\| \left(\mu_{A_{1P}} + \mu_{A_{2P}} \|A_{1P}\|^2\right)^{-1} \leq \lambda k$ ($Lip \subset \mathcal{M}$, $\mu_{A_{1P}} = \mu_{B_{Pz}}$ and $\|A_{1P}\| = \|B_{Pz}\|^*$). It is deduced (cf, lemma 4.1) that the operator M_z is invertible. According to corollary 4.2, the FS $[A_1, A_2]$ is normal and the solution is given (see (3)) by

$$(e_1, e_2) = (M_{u_2}^{-1}u_1, u_2 + A_1M_{u_2}^{-1}u_1).$$

Using (5), on get

$$\left(e_1, e_1'\right) = \left(N^{-1}(u_1 + A_1^{-1}u_2) - A_1^{-1}u_2, N^{-1}(u_1' + A_1^{-1}u_2') - A_1^{-1}u_2'\right),$$

where $N = I + A_2A_1$, so

$$\begin{aligned}
\left\|P(e_1 - e_1')\right\| &\leq \left\|PN^{-1}(Pu_1 + PA_1^{-1}u_2) - PN^{-1}(Pu_1' + PA_1^{-1}u_2')\right\| \\
&\quad + \left\|PA_1^{-1}u_2' - PA_1^{-1}u_2\right\| \\
&\leq \left\|N_P^{-1}\right\|^* \left\|u_1 - u_1'\right\| + \left(\left\|N_P^{-1}\right\|^* + 1\right)\left\|A_{1P}^{-1}\right\| \left\|u_2 - u_2'\right\| \\
&\leq \lambda k \left\|u_1 - u_1'\right\| + (1 + \lambda k)\,\mu_{A_{1P}}^{-1} \left\|u_2 - u_2'\right\|,
\end{aligned}$$

where lemma 4.1 was used. From where

$$\left\|e_1 - e_1'\right\| \leq \lambda k \left\|u_1 - u_1'\right\| + (1 + \lambda k)\,\mu_{A_{1P}}^{-1} \left\|u_2 - u_2'\right\|.$$

Using the first inequality above, we deduce the causality of the FS $[A_1, A_2]$ as in the proof of theorem3.1. On the other hand

$$\begin{aligned}
\left(e_2, e_2'\right) = \Big(&u_2 + A_1N^{-1}(u_1 + A_1^{-1}u_2) - A_1^{-1}u_2, \\
&u_2' + A_1N^{-1}(u_1' + A_1^{-1}u_2') - A_1^{-1}u_2'\Big)
\end{aligned}$$

therefore

$$\begin{aligned}
P\left(e_2 - e_2'\right) &= P\left(u_2 - u_2'\right) + PA_1PN^{-1}(u_1 + A_1^{-1}u_2) \\
&\quad -PA_1N^{-1}(u_1' + A_1^{-1}u_2') + PA_1^{-1}\left(u_2' - u_2\right) \\
&= P\left(u_2 - u_2'\right) + A_{1P}N_P^{-1}P(u_1 + A_1^{-1}u_2) \\
&\quad -A_{1P}N_P^{-1}P(u_1' + A_1^{-1}u_2') + A_{1P}^{-1}P\left(u_2' - u_2\right) \\
&= P\left(u_2 - u_2'\right) + A_{1P}N_P^{-1}(Pu_1 + PA_1^{-1}u_2) \\
&\quad -A_{1P}N_P^{-1}(Pu_1' + PA_1^{-1}u_2') + A_{1P}^{-1}P\left(u_2' - u_2\right) \\
&= P\left(u_2 - u_2'\right) + A_{1P}N_P^{-1}(Pu_1 + A_{1P}^{-1}Pu_2) \\
&\quad -A_{1P}N_P^{-1}(Pu_1' + A_{1P}^{-1}Pu_2') + A_{1P}^{-1}P\left(u_2' - u_2\right),
\end{aligned}$$

and

$$\begin{aligned}
\left\|P(e_2 - e_2')\right\| &\leq \left\|u_2 - u_2'\right\|\left(1 + \left\|A_{1P}^{-1}\right\|\right) \\
&\quad + \left\|A_{1P}N_P^{-1}(Pu_1 + A_{1P}^{-1}Pu_2) - A_{1P}N_P^{-1}(Pu_1' + A_{1P}^{-1}Pu_2')\right\| \\
&\leq \left(1 + \left\|A_{1P}^{-1}\right\|\right)\left\|u_2 - u_2'\right\| + \|A_{1P}\| \left\|N_P^{-1}\right\|^* \left\|u_1 - u_1'\right\| \\
&\quad + \|A_{1P}\| \left\|N_P^{-1}\right\|^* \left\|A_{1P}^{-1}\right\| \left\|u_2 - u_2'\right\| \\
&\leq \lambda k^2 \left\|u_1 - u_1'\right\| + \left(1 + \left(1 + \lambda k^2\right)\mu_{A_{1P}}^{-1}\right)\left\|u_2 - u_2'\right\|.
\end{aligned}$$

From where

$$\left\|e_2 - e_2'\right\| \leq \lambda k^2 \left\|u_1 - u_1'\right\| + \left(1 + \left(1 + \lambda k^2\right)\mu_{A_{1P}}^{-1}\right)\left\|u_2 - u_2'\right\|.$$

3.2 Linearization of $FS\,[A_1, A_2]$ on H_e

Let a non linair $FS\ [A_1, A_2]$ on H_e. The main idea in this subsection is to linearize $[A_1, A_2]$ in the neighbourhood of the zero. We then consider a linear $FS\ \left[A_1^0, A_2^0\right]$ on H_e and prove that, if $(u_1,u_2) \mapsto (e_1,e_2) \in H_e^2$ and $(u_1,u_2) \mapsto (e_1^0,e_2^0) \in H_e^2$ where $(u_1,u_2) \in H_P^2$ with $\|u_1\|, \|u_2\| \leq r\ (r > 0)$ and (e_1,e_2), (e_1^0,e_2^0) the respective solutions of $[A_1, A_2]$ and $\left[A_1^0, A_2^0\right]$. There existes k_{11}, k_{12}, k_{21} and k_{22} positive real constants such that

$$\left\|e_1 - e_1^0\right\| \leq k_{11}\|u_1\| + k_{12}\|u_2\|,$$

and

$$\left\|e_2 - e_2^0\right\| \leq k_{21}\|u_1\| + k_{22}\|u_2\|.$$

The inequalities above are given by theorem 3.3. To have suitable estimates, in the sense that the solutions of the two systems become sufficiently close. It is assumed that, one of the two operators of $[A_1, A_2]$ is linear, this is the subject of theorems 3.4 and 3.5. Before establishing the first linearization result of this part, we need the following two notions:

Definition 3.1.

(i) We say that a normal $FS\ [A_1, A_2]$ on H_e is Lipschitz continuous for the first inputs, if there are positive numbers λ_{11} and λ_{21} such that $\left\|e_1 - e_1'\right\| \leq \lambda_{11}\left\|u_1 - u_1'\right\|$ and $\left\|e_2 - e_2'\right\| \leq \lambda_{12}\left\|u_1 - u_1'\right\|$ where $(u_1,u^*) \mapsto (e_1,e_2)$, $(u_1',u^*) \mapsto (e_1',e_2')$ and $u_1 - u_1' \in H$.

(ii) We say that a normal $FS\ [A_1, A_2]$ on H_e is Lipschitz continuous for both inputs, if there are positive numbers $\lambda_{11,}$ $\lambda_{12,}$ λ_{21} and λ_{22} such that:

$$\left\|e_1 - e_1'\right\| \leq \lambda_{11}\left\|u_1 - u_1'\right\| + \lambda_{12}\left\|u_2 - u_2'\right\|$$

and

$$\left\|e_2 - e_2'\right\| \leq \lambda_{21}\left\|u_1 - u_1'\right\| + \lambda_{22}\left\|u_2 - u_2'\right\|$$

where $(u_1,u_2) \mapsto (e_1,e_2)$, $(u_1',u_2') \mapsto (e_1',e_2')$ and $\left(u_1 - u_1', u_2 - u_2'\right) \in H^2$.

Let in the Hilbert space H_P be the closed ball B_r, centered in zero with radius $r > 0$. Then we have:

Theorem 3.3. Let $A_1, A_2 : H_e \mapsto H_e$ be the causal operators. For all $P \in P$, A_{1P}, A_{2P}, the respective restrictions of PA_1, PA_2 to H_P. Assume that:

(a) $A_{1P}, A_{2P} \in Lip$ for any $P \in \mathcal{P}$.

(b) There exist a linear and causal operator $A_1^0 : H_e \mapsto H_e$ such that: for all $x \in B_{\nu r}$

$$\left\|\left(A_{1P} - A_{1P}^0\right)x\right\| \leq a_1\|x\|$$

where for all $P \in \mathcal{P}$, A_{1P}^0 is the restriction of PA_1^0 to H_P, $0 \leq a_1 \leq \mu_{A_{1P}}$ and $\nu = \mu_{A_{1P}}^{-1}\left(\mu_{A_{2P}} + \mu_{A_{1P}}\|A_{1P}\|^{*-2}\right)^{-1}$.

(c) There exist a linear and causal operator $A_2^0 : H_e \mapsto H_e$ such that: for all $x \in B_{(1+\nu\|A_{1P}\|^*)r}$

$$\left\|\left(A_{2P} - A_{2P}^0\right)x\right\| \leq a_2\|x\|$$

where for all $P \in \mathcal{P}$, A_{2P}^0 the restriction of PA_2^0 to H_P and $a_2 > 0$.

(d) $(\mu_{A_{2P}} - a_2) + (\mu_{A_{1P}} - a_1)(a_{1+}\|A_{1P}\|^*)^{-2} > 0$. Then:

(i) The $FS's$ $[A_1, A_2]$ and $\left[A_1^0, A_2^0\right]$ on H_e are normal and Lipschitz continuous for the first inputs.

(ii) If $(u_1,u_2) \mapsto (e_1,e_2) \in H_e^2$ and $(u_1,u_2) \mapsto (e_1^0,e_2^0) \in H_e^2$ where $(u_1,u_2) \in H_P^2$ with $\|u_1\|, \|u_2\| \leq r$ and (e_1,e_2), (e_1^0,e_2^0) the respective solutions of $[A_1, A_2]$ and $\left[A_1^0, A_2^0\right]$.We have:

$$\begin{aligned}\left\|e_1 - e_1^0\right\| &\leq k_{11}\|u_1\| + k_{12}\|u_2\|;\\ \left\|e_2 - e_2^0\right\| &\leq k_{21}\|u_1\| + k_{22}\|u_2\|,\end{aligned}$$

where

$$k_{11} = k\nu\left(a_2 + \|A_{1P}\|^* + a_1\left\|A^0_{2P}\right\|\right), k_{12} = ka_2$$
$$k_{21} = a_1\nu + \left\|A^0_{1P}\right\|k_{11}, k_{22} = k\left\|A^0_{1P}\right\|a_2$$

and

$$k = (\mu_{A_{1P}} - a_1)$$

Proof. Let $x \in B_r$, since $\left\|\left(A_{1P} - A^0_{1P}\right)x\right\| \leq a_1\|x\|$ so $A_{1P}0 = 0$, moreover for $x \neq 0$

$$\left\|A^0_{1P}x\right\| \leq \left\|\left(A_{1P} - A^0_{1P}\right)x\right\| + \|A_{1P}x\| \leq (a_1 + \|A_{1P}\|^*)\|x\|,$$

then $\left\|A^0_{1P}\right\| \leq a_1 + \|A_{1P}\|^*$, hence it is bounded on H_p
On the other hand, $\left\langle A^0_{1P}x, x\right\rangle = \langle A_{1P}x, x\rangle + \left\langle A^0_{1P}x - A_{1P}x, x\right\rangle$ therefore

$$\left\langle\left(A_{1P} - A^0_{1P}\right)x, x\right\rangle = \langle A_{1P}x, x\rangle - \left\langle A^0_{1P}x, x\right\rangle$$

From the cauchy-schwarz inequality,

$$\left\|\langle A_{1P}x, x\rangle - \left\langle A^0_{1P}x, x\right\rangle\right\| \leq \left\|\left(A_{1P} - A^0_{1P}\right)x\right\|\|x\| \leq a_1\|x\|^2.$$

By definition of $\mu_{A_{1P}}$, we have for all $x \in B_{\nu r}$ $(x \neq 0)$, $\langle A_{1P}x, x\rangle \geq \mu_{A_{1P}}\|x\|^2$ therefore, for all $x \in H_P$, $\left\langle A^0_{1P}x, x\right\rangle \geq (\mu_{A_{1P}} - a_1)\|x\|^2$, from where $\dfrac{\left\langle A^0_{1P}x, x\right\rangle}{\|x\|^2} \geq \mu_{A_{1P}} - a_1$, and thus $\mu_{A^0_{1P}} \geq \mu_{A_{1P}} - a_1 > 0$. The operator A^0_{1P} is linear and bounded so it is hemicontinuous, according to lemma 4.1, A^0_{1P} is invertible. It is similarly shown that $\left\|A^0_{2P}\right\| \leq a_2 + \|A_{2P}\|^*$ and $\mu_{A^0_{2P}} \geq \mu_{A_{2P}} - a_2$. Let now, for all x and z in H_e, $M_zx = x + A_2(z + A_1x)$, $M^0_zx = x + A^0_2(z + A^0_1x)$, $M_{Pz}x = x + A_{2P}(z + A_{1P}x)$, and $M^0_{Pz}x = x + A^0_{2P}(z + A^0_{1P}x)$ or $M_{Pz} = I + A_{2P}B_{Pz}$, and $M^0_{Pz} = I + A^0_{2P}B^0_{Pz}$, (with $B_{Pz} = z + A_{1P}$, $B^0_{Pz} = z + A^0_{1P}$). Using, lemma 4.5, M_{Pz} and M^0_{Pz} are invertible and therefore according to corollary 4.2, the FS $[A_1, A_2]$ is normal, moreover (cf, lemma 4.2) $M^{-1}_{Pz} \in Lip$,

$$\begin{aligned}\left\|M^{-1}_{Pz}\right\|^* &\leq \mu^{-1}_{B_{Pz}}\left(\mu_{A_{2P}} + \mu_{B_{Pz}}\|B_{Pz}\|^{*-2}\right)^{-1}\\ &= \mu^{-1}_{A_{1P}}\left(\mu_{A_{2P}} + \mu_{A_{1P}}\|A_{1P}\|^{*-2}\right)^{-1}\\ &= \nu.\end{aligned}$$

where $\mu_{B_{Pz}} = \mu_{A_{1P}}$ and $\|B_{Pz}\|^* = \|A_{1P}\|^*$.

Now, we demonstrate that $[A_1, A_2]$ is Lipschitz continuous for the first input. We know (cf lemma 4.5) that for all $P \in \mathcal{P}$, $PM^{-1}_z = N^{-1}_P P$ and M^{-1}_z is causal, so

$$\begin{aligned}Pe_1 &= PM^{-1}_{u_2}u_1 = PM^{-1}_{u_2}Pu_1 = N^{-1}_P Pu_1;\\ Pe'_1 &= PM^{-1}_{u_2}u'_1 = PM^{-1}_{u_2}Pu'_1 = N^{-1}_P Pu'_1.\end{aligned}$$

Since $N^{-1}_P \in Lip$ (cf lemma 4.2) with $N_P = I + A_{2P}B_{zP}$, we have

$$\begin{aligned}\left\|P(e_1 - e'_1)\right\| &\leq \left\|N^{-1}_P Pu_1 - N^{-1}_P Pu'_1\right\|\\ &\leq \left\|N^{-1}_P\right\|^*\left\|P(u_1 - u'_1)\right\|\\ &\leq \lambda_{11}\left\|u_1 - u'_1\right\|.\end{aligned}$$

hence, $e_1 - e'_1 \in H$ and $\left\|e_1 - e'_1\right\| \leq \lambda_{11}\left\|u_1 - u'_1\right\|$, where $\lambda_{11} = \nu$.

On the other hand, $e_2 = A_1e_1 + u_2$ and $e_2' = A_1e_1' + u_2$ so

$$\begin{aligned}\left\|P\left(e_2 - e_2'\right)\right\| &\leq \|A_{1P}\|^* \left\|e_1 - e_1'\right\| \\ &\leq \lambda_{11}\|A_{1P}\|^* \left\|u_1 - u_1'\right\|,\end{aligned}$$

we deduce that $\left\|e_2 - e_2'\right\| \leq \lambda_{21}\left\|u_1 - u_1'\right\|$, where $\lambda_{21} = \|A_{1P}\|^* \lambda_{11}$.

To demonstrate that $\left[A_1^0, A_2^0\right]$ is normal, it is sufficient to check that $\mu_{A_{2P}^0} + \mu_{A_{1P}^0}\left\|A_{1P}^0\right\|^{*-2} > 0$. We have $\mu_{A_{2P}^0} \geq \mu_{A_{2P}} - a_2$; $\mu_{A_{1P}^0} \geq \mu_{A_{1P}} - a_1$ and $\left\|A_{1P}^0\right\| \leq a_1 + \|A_{1P}\|^*$, so

$$\begin{aligned}\mu_{A_{2P}^0} + \mu_{A_{1P}^0}\left\|A_{1P}^0\right\|^{*-2} &\geq \mu_{A_{2P}} - a_2 + \mu_{A_{1P}^0}\left\|A_{1P}^0\right\|^{*-2} \\ &\geq \mu_{A_{2P}} - a_2 + \left(\mu_{A_{1P}} - a_1\right)\left\|A_{1P}^0\right\| \\ &\geq \mu_{A_{2P}} - a_2 + \left(\mu_{A_{1P}} - a_1\right)\left(a_1 + \|A_{1P}\|^*\right)^{-1} > 0.\end{aligned}$$

Since M_{Pz}^0 is invertible, $\left(M_{Pz}^0\right)^{-1} \in Lip$ and

$$\left\|\left(M_{Pz}^0\right)^{-1}\right\|^* \leq \mu_{B_{Pz}^0}^{-1}\left(\mu_{A_{2P}^0} + \mu_{B_{Pz}^0}\left\|B_{Pz}^0\right\|^{*-2}\right)^{-1},$$

therefore

$$\left\|\left(M_{Pz}^0\right)^{-1}\right\|^* \leq \left(\mu_{A_{1P}} - a_1\right)^{-1}\left(\mu_{A_{2P}^0} - a_2 + \left(\mu_{A_{1P}} - a_1\right)\left\|A_{1P}^0\right\|^{*-2}\right)^{-1}.$$

The two operators $A_{1P}^0, A_{2P}^0 \in Lip$ and they are hemicontinuous with $\mu_{A_{1P}^0} > 0$ and $\mu_{A_{2P}^0} + \mu_{A_{1P}^0}\left\|A_{1P}^0\right\|^{*-2} > 0$. According to lemma 4.2, $\left[A_1^0, A_2^0\right]$ is normal. The Lipschitzian continuity of $\left[A_1^0, A_2^0\right]$ is demonstrated in the same way as that of $[A_1, A_2]$. On the other hand, the operator $N_P^0 = I + A_{2P}^0A_{1P}^0$ is such that: $A_{2P}^0 \in Lip$ and it is hemicontinuous, $A_{1P}^0 \in Lip$, $\mu_{A_{1P}^0} > 0$ and $\mu_{A_{2P}^0} + \mu_{A_{1P}^0}\left\|A_{1P}^0\right\|^{*-2} > 0$. By using lemma 4.2, N_P^0 is invertible, $N_P^{0-1} \in Lip$ and $\left\|N_P^{0-1}\right\|^* \leq \mu_{A_{1P}^0}^{-1}\left(\mu_{A_{2P}^0} + \mu_{A_{1P}^0}\left\|A_{1P}^0\right\|^{*-2}\right)^{-1} \leq k$. Then we can write from (5)

$$\begin{aligned}M_{Pz}^{-1}x - M_{Pz}^{0-1}x &= M_{Pz}^{-1}x - N_P^{0-1}\left(x + A_{1P}^{0-1}z\right) + A_{1P}^{0-1}z \\ &= M_{Pz}^{-1}x - N_P^{0-1}x - N_P^{0-1}A_{1P}^{0-1}z + A_{1P}^{0-1}z \\ &= M_{Pz}^{-1}x - N_P^{0-1}x + \left(I - N_P^{0-1}\right)A_{1P}^{0-1}z \\ &= -N_P^{0-1}\left(M_{Pz} - N_P^0\right)M_{Pz}^{-1}x + N_P^{0-1}\left(N_P^0 - I\right)A_{1P}^{0-1}z \\ &= -N_P^{0-1}\left(M_{Pz}\omega - N_P^0\omega\right) + N_P^{0-1}A_{2P}^0A_{1P}^0A_{1P}^{0-1}z \\ &= -N_P^{0-1}\left(\omega + A_{2P}\left(z + A_{1P}\omega\right) - \omega - A_{2P}^0A_{1P}^0\omega\right) + N_P^{0-1}A_{2P}^0z \\ &= -N_P^{0-1}\left(A_{2P}\left(z + A_{1P}\omega\right) - A_{2P}^0A_{1P}^0\omega\right) + N_P^{0-1}A_{2P}^0z \\ &= -N_P^{0-1}[A_{2P}\left(z + A_{1P}\omega\right) - A_{2P}^0\left(z + A_{1P}\omega\right) \\ &\quad + A_{2P}^0\left(z + A_{1P}\omega\right) - A_{2P}^0A_{1P}^0\omega] + N_P^{0-1}A_{2P}^0z \\ &= -N_P^{0-1}(A_{2P} - A_{2P}^0)\left(z + A_{1P}\omega\right) - N_P^{0-1}(A_{2P}^0(z + A_{1P}\omega) \\ &\quad - A_{2P}^0A_{1P}^0\omega)) + N_P^{0-1}A_{2P}^0z \\ &= -N_P^{0-1}(A_{2P} - A_{2P}^0)\left(z + A_{1P}\omega\right) \\ &\quad - N_P^{0-1}(A_{2P}^0z + A_{2P}^0A_{1P}\omega - A_{2P}^0A_{1P}^0\omega) + N_P^{0-1}A_{2P}^0z \\ &= -N_P^{0-1}(A_{2P} - A_{2P}^0)\left(z + A_{1P}\omega\right) - N_P^{0-1}A_{2P}^0\left(A_{1P} - A_{1P}^0\right)\omega\end{aligned}$$

where $\omega = M_{Pz}^{-1}x$. Since $\|\omega\| = \left\|M_{Pz}^{-1}x\right\| \leq \left\|M_{Pz}^{-1}\right\|^* \|x\| \leq \nu r, \forall x \in B_r$ then $\omega \in B_{\nu r}$ and $A_{1P}0 = 0$, so

$$\begin{aligned}\|z + A_{1P}\omega\| &\leq \|z\| + \|A_{1P}\omega\| \leq \|z\| + \|A_{1P}\|^* \|\omega\| \\ &\leq \|z\| + \nu\|A_{1P}\|^* \|x\|\end{aligned}$$

It is deduced that for all $x, z \in B_r$, $z + A_{1P}\omega \in B_{(\|z\|+\nu\|A_{1P}\|^*)r}$ and

$$\begin{aligned}
\left\|M_{P_z}^{-1}x - M_{P_z}^{0-1}x\right\| &\leq \left\|N_P^{0-1}\right\| \left\|(A_{2P} - A_{2P}^0)(z + A_{1P}\omega)\right\| \\
&\quad + \left\|N_P^{0-1}\right\| \left\|A_{2P}^0\right\| \left\|(A_{1P} - A_{1P}^0)\omega\right\| \\
&\leq \left\|N_P^{0-1}\right\| \left[a_2 \|z + A_{1P}\omega\| + \left\|A_{2P}^0\right\| a_1 \|\omega\|\right] \\
&\leq k(a_2 \|z\| + a_2 \|A_{1P}\|^* \|\omega\|) + ka_1 \left\|A_{2P}^0\right\| \|\omega\| \\
&\leq ka_2 \|z\| + k \|\omega\| (a_2 \|A_{1P}\|^* + a_1 \left\|A_{2P}^0\right\|) \\
&\leq ka_2 \|z\| + \nu k(a_2 \|A_{1P}\|^* + a_1 \left\|A_{2P}^0\right\|) \|x\| \\
&\leq k_{11} \|x\| + k_{12} \|z\|,
\end{aligned}$$

hence

$$\left\|e_1 - e_1^0\right\| \leq k_{11} \|u_1\| + k_{12} \|u_2\|.$$

On the other hand,

$$\begin{aligned}
\left\|e_2 - e_2^0\right\| &\leq \left\|A_{1P}M_{u_2}^{-1}u_1 - A_{1P}^0 M_{u_2}^{0-1}u_1\right\| \\
&= \left\|A_{1P}M_{u_2}^{-1}u_1 - A_{1P}^0 M_{u_2}^{-1}u_1 + A_{1P}^0 M_{u_2}^{-1}u_1 - A_{1P}^0 M_{u_2}^{0-1}u_1\right\| \\
&= \left\|\left(A_{1P} - A_{1P}^0\right) M_{u_2}^{-1}u_1 + A_{1P}^0 \left(M_{u_2}^{-1}u_1 - M_{u_2}^{0-1}u_1\right)\right\| \\
&\leq \left\|A_{1P} - A_{1P}^0\right\| \left\|M_{u_2}^{-1}u_1\right\| + \left\|A_{1P}^0\right\| \left\|M_{u_2}^{-1}u_1 - M_{u_2}^{0-1}u_1\right\| \\
&\leq a_1\nu \|u_1\| + \left\|A_{1P}^0\right\| (k_{11} \|u_1\| + k_{12} \|u_2\|) \\
&\leq \left(a_1\nu + \left\|A_{1P}^0\right\| k_{11}\right) \|u_1\| + k_{12} \left\|A_{1P}^0\right\| \|u_2\| \\
&\leq k_{21} \|u_1\| + k_{22} \|u_2\|,
\end{aligned}$$

where

$$k_{21} = a_1\nu + \left\|A_{1P}^0\right\| k_{11} \text{ and } k_{22} = \left\|A_{1P}^0\right\| k_{12}.$$

The second linearization result is:

Theorem 3.4. Let $A_1, A_2 : H_e \mapsto H_e$ be, the causal operators whose A_2 is linear, and let for all $P \in P$, A_{1P} and A_{2P} be the respective restrictions of PA_1, PA_2, to H_P. we suppose that:

a) $A_{1P} \in Lip$ and $\mu_{A_{1P}} \leq 0$ for any $P \in \mathcal{P}$.

(*b*) $A_{2P} \in \mathcal{M}$ and $\mu_{A_{2P}} > 0$ for any $P \in \mathcal{P}$.

(*c*) There exists a linear and causal operator, $A_1^0 : H_e \mapsto H_e$ with $\mu_{A_{1P}^0} \leq 0$, and $a_1 \geq 0$ such that:

for all $x \in B_{\varpi(1+\|A_{2P}\|)r}$

$$\left\|\left(A_{1P} - A_{1P}^0\right) x\right\| \leq a_1 \|x\|$$

where for all $P \in \mathcal{P}$, A_{1P}^0 is the restriction of PA_1^0 to H_P and

$$\varpi = \|A_{2P}\| \left(\mu_{A_{2P}} + \mu_{A_{1P}} \|A_{2P}\|^2\right)^{-1}.$$

(*d*) $\mu_{A_{2P}} + (\mu_{A_{1P}} - a_1) \|A_{2P}\|^2 > 0$. Then:

(*i*) The $FS's$, $[A_1, A_2]$ and $\left[A_1^0, A_2\right]$ on H_e, are normal and Lipschitz continuous for both inputs.

(*ii*) If $(u_1, u_2) \mapsto (e_1, e_2) \in H_e^2$; $(u_1, u_2) \mapsto (e_1^0, e_2^0) \in H_e^2$ where $(u_1, u_2) \in H_P^2$ with $\|u_1\|, \|u_2\| \leq r$ and (e_1, e_2); (e_1^0, e_2^0) the respective solutions of $[A_1, A_2]$ and $\left[A_1^0, A_2\right]$. Then

$$\left\|e_1 - e_1^0\right\| \leq \lambda \|u_1\| + \lambda \|A_{2P}\| \|u_2\|$$

and

$$\left\|e_2 - e_2^0\right\| \leq \left(a_1\varpi + \lambda \left\|A_{1P}^0\right\|\right) (\|u_1\| + \|A_{2P}\| \|u_2\|)$$

where

$$\lambda = a_1 \|A_{2P}\|^3 \left(\mu_{A_{2P}} + \mu_{A_{1P}} \left\|A_{2p}\right\|^2\right)^{-1} \left(\mu_{A_{2P}} + \mu_{A^0_{1P}} \|A_{2P}\|^2\right)^{-1}.$$

Proof. Let for all x and z in H_e, $M_{pz}x = x + A_{2P}(z + A_{1P}x)$; $M^0_{Pz}x = x + A^0_{2P}(z + A^0_{1P}x)$; $B_{Pz} = z + A_{1P}$ and $B^0_{pz} = z + A^0_{1P}$. The operators M_{Pz} and M^0_{Pz} are causal; M_{Pz} and M^0_{Pz} are invertible. Indeed, A_{1P} is hemicontinuous, so M_{Pz} and M^0_{Pz} satisfy the conditions of lemma 4.3. Then M_z and M^0_z are invertible (see lemma 4.4).

Demonstrate that $[A_1, A_2]$ is Lipschitz continuous for both inputs. We know from (4) that:

$$\begin{aligned} e_1 &= (I + A_2A_1)^{-1}(u_1 - A_2u_2) = M^{-1}_{Pz}(u_1 - A_{2P}u_2); \\ e_2 &= u_2 + A_1(I + A_2A_1)^{-1}(u_1 - A_2u_2) \end{aligned}$$

therefore (see lemma 4.6)

$$\begin{aligned} Pe_1 &= PM^{-1}_{Pz}(u_1 - A_{2P}u_2) = N^{-1}_P P(u_1 - A_{2P}u_2) \\ &= N^{-1}_P (Pu_1 - A_{2P}Pu_2); \\ Pe'_1 &= N^{-1}_P \left(Pu'_1 - A_{2P}Pu'_2\right), \end{aligned}$$

and

$$\begin{aligned} \left\|P\left(e_1 - e'_1\right)\right\| &= \left\|N^{-1}_P (Pu_1 - A_{2P}Pu_2) - N^{-1}_P \left(Pu'_1 - A_{2P}Pu'_2\right)\right\| \\ &\leq \left\|N^{-1}_P\right\|^* \left\|P\left(u_1 - u'_1\right) - A_{2P}P\left(u_2 - u'_2\right)\right\| \\ &\leq \left\|N^{-1}_P\right\|^* \left(\left\|u_1 - u'_1\right\| + \|A_{2P}\| \left\|u_2 - u'_2\right\|\right) \\ &\leq \varpi \left(\left\|u_1 - u'_1\right\| + \|A_{2P}\| \left\|u_2 - u'_2\right\|\right), \end{aligned}$$

from where

$$\left\|e_1 - e'_1\right\| \leq \lambda_{11} \left\|u_1 - u'_1\right\| + \lambda_{12} \|A_{2P}\| \left\|u_2 - u'_2\right\|,$$

where $\lambda_{11} = \varpi$ and $\lambda_{12} = \varpi \|A_{2P}\|$.

Always from (4)

$$\begin{aligned} e_2 &= u_2 + A_1 \left((I + A_2A_1)^{-1}(u_1 - A_2u_2)\right) \\ &= u_2 + A_{1P}M^{-1}_{Pz}(u_1 - A_{2P}u_2); \\ e'_2 &= u'_2 + A_1 \left((I + A_2A_1)^{-1}(u'_1 - A_2u'_2)\right) \\ &= u'_2 + A_{1P}M^{-1}_{Pz}(u'_1 - A_{2P}u'_2), \end{aligned}$$

therefore

$$\begin{aligned} Pe_2 &= Pu_2 + PA_{1P}M^{-1}_{Pz}(u_1 - A_{2P}u_2) \\ &= Pu_2 + A_{1P}N^{-1}_P(Pu_1 - A_{2P}Pu_2); \\ Pe'_2 &= Pu'_2 + PA_{1P}M^{-1}_{Pz}(u'_1 - A_{2P}u'_2) \\ &= Pu'_2 + A_{1P}N^{-1}_P(Pu'_1 - A_{2P}Pu'_2) \end{aligned}$$

then

$$\begin{aligned} P\left(e_2 - e'_2\right) &= P\left(u_2 - u'_2\right) + A_{1P}N^{-1}_P \left((Pu_1 - A_{2P}Pu_2) - (Pu'_1 - A_{2P}Pu'_2)\right) \\ &= P\left(u_2 - u'_2\right) + A_{1P}N^{-1}_P \left(P(u_1 - u'_1) + A_{2P}P(u_2 - u'_2)\right) \end{aligned}$$

and

$$\begin{aligned}
\left\|P\left(e_2-e_2^{'}\right)\right\| &\leq \left\|P\left(u_2-u_2^{'}\right)\right\| + \left\|A_{1P}N_P^{-1}\left(P(u_1-u_1^{'})+A_{2P}P(u_2-u_2^{'})\right)\right\| \\
&\leq \left\|P\left(u_2-u_2^{'}\right)\right\| + \left\|A_{1P}N_P^{-1}\left(P(u_1-u_1^{'})+A_{2P}P(u_2-u_2^{'})\right)\right\| \\
&\leq \left\|u_2-u_2^{'}\right\| + \|A_{1P}\|\left\|N_P^{-1}\right\|^*\left(\left\|u_1-u_1^{'}\right\|+\|A_{2P}\|\left\|u_2-u_2^{'}\right\|\right) \\
&\leq \left\|u_2-u_2^{'}\right\|(1+\varpi\|A_{1P}\|\|A_{2P}\|)+\varpi\|A_{1P}\|\left\|u_1-u_1^{'}\right\| \\
&\leq \lambda_{21}\left\|u_1-u_1^{'}\right\|+\lambda_{22}\left\|u_2-u_2^{'}\right\|,
\end{aligned}$$

where $\lambda_{21}=\varpi\|A_{1P}\|$ and $\lambda_{22}=1+\lambda_{21}\|A_{2P}\|$.

Then

$$\left\|e_2-e_2^{'}\right\| \leq \lambda_{21}\left\|u_1-u_1^{'}\right\|+\lambda_{22}\left\|u_2-u_2^{'}\right\|.$$

To estimate the solutions of $[A_1, A_2]$ and $\left[A_1^0, A_2^0\right]$, notice first that $\forall x \in B_{\varpi(1+\|A_{2P}\|)r,}$ $\left\|N_P^{-1}x\right\| \leq \left\|N_P^{-1}\right\|^*\|x\| \leq \varpi\|x\| \leq \varpi(1+\|A_{2P}\|)r$, so $N_P^{-1}x \in B_{\varpi(1+\|A_{2P}\|)r}$. Since

$$\begin{aligned}
\left\|N_P^{-1}x-N_P^{0-1}x\right\| &= \left\|N_P^{0-1}\left(N_P^0-N_P\right)N_P^{-1}x\right\| \\
&= \left\|N_P^{0-1}A_{2P}\left(A_{1P}^0-A_{1P}\right)N_P^{-1}x\right\| \\
&\leq \left\|N_P^{0-1}\right\|\|A_{2P}\|\left\|\left(A_{1P}^0-A_{1P}\right)N_P^{-1}x\right\| \\
&\leq a_1\left\|N_P^{0-1}\right\|\|A_{2P}\|\left\|N_P^{-1}x\right\| \leq ka_1\|A_{2P}\|\left\|N_P^{-1}x\right\| \\
&\leq ka_1\|A_{2P}\|\left\|N_P^{-1}\right\|^*\|x\| \leq ka_1\varpi\|A_{2P}\|\|x\| \leq \lambda\|x\|
\end{aligned}$$

where $\lambda = ka_1\ \varpi\|A_{2P}\|$ and $k=\|A_{2P}\|\left(\mu_{A_{2P}}+\mu_{A_{1P}^0}\|A_{2P}\|^2\right)^{-1}$. Let $w=u_1-A_{2P}u_2$, then, for u_1 and u_2 in B_r, we have $\|w\|=\|u_1-A_{2P}u_2\| \leq \|u_1\|+\|A_{2P}\|\|u_2\| \leq (1+\|A_{2P}\|)r$, so $w \in B_{(1+\|A_{2P}\|)r,}$ then

$$\begin{aligned}
\left\|e_1-e_1^0\right\| &= \left\|\left(N_P^{-1}-N_P^{0-1}\right)w\right\| \\
&\leq \lambda\|w\| \leq \lambda\|u_1\|+\lambda\|A_{2P}\|\|u_2\|
\end{aligned}$$

and

$$\begin{aligned}
e_2-e_2^0 &= A_{1P}N_P^{-1}w-A_{1P}^0N_P^{0-1}w \\
&= A_{1P}N^{-1}w-A_{1P}^0N^{-1}w+A_{1P}^0N^{-1}w-A_{1P}^0N^{0-1}w
\end{aligned}$$

from where

$$\begin{aligned}
\left\|e_2-e_2^0\right\| &\leq \left\|\left(A_{1P}-A_{1P}^0\right)N^{-1}w\right\|+\left\|A_{1P}^0\left(N^{-1}w-N^{0-1}w\right)\right\| \\
&\leq a_1\left\|N^{-1}w\right\|+\left\|A_{1P}^0\right\|\left\|N^{-1}w-N^{0-1}w\right\| \\
&\leq a_1\varpi\|w\|+\lambda\left\|A_{1P}^0\right\|\|w\| \\
&\leq \left(a_1\varpi+\lambda A_{1P}^0\right)(\|u_1\|+\|A_{2P}\|\|u_2\|).
\end{aligned}$$

The third linearization result is:

Theorem 3.5. Let $A_1, A_2, A_2^0 : H_e \mapsto H_e$ be the causal operators, where $A_1$is linear, and be for all $P \in P$, A_{1P}, A_{2P}, the respective restrictions of PA_1, PA_2, to H_P. We assume that:

(a) $A_{1P} \in \mathcal{M}$, with $\mu_{A_{1P}} > 0$ for any $P \in \mathcal{P}$, .

(b) $A_{2P} \in Lip$, with $\mu_{A_{2P}} \leq 0$ for any $P \in \mathcal{P}$.

(c) There exist a causal and linear operator, $A_2^0 : H_e \mapsto H_e$, A_{2P}^0 and, there exists $a_1 \geq 0$ such that, for all $x \in B_{\rho\|A_{1P}\|(1+\mu_{A_{1P}}^{-1})r}$

$$\left\|\left(A_{2P}-A_{2P}^0\right)x\right\| \leq a_2\|x\|,$$

where for all $P \in \mathcal{P}$, A^0_{2P} is the restriction of PA^0_2 to H_P, $\mu_{A^0_{2P}} \leq 0$ and

$$\rho = \|A_{1P}\| \left(\mu_{A_{1P}} + \mu_{A_{2P}} \|A_{1P}\|^2\right)^{-1}.$$

(d) $\mu_{A_{1P}} + (\mu_{A_{2P}} - a_2) \|A_{1P}\|^2 > 0$. So

(i) The $FS's$ $[A_1, A_2]$ and $\left[A_1, A^0_2\right]$ on H_e, are normal and Lipschitz continuous, for both inputs.

(ii) If $(u_1,u_2) \mapsto (e_1,e_2) \in H^2_e$; $(u_1,u_2) \mapsto (e^0_1, e^0_2) \in H^2_e$ where, $(u_1,u_2) \in H^2_P$ with $\|u_1\|, \|u_2\| \leq r$ and (e_1,e_2), (e^0_1, e^0_2) the respective solutions of $[A_1, A_2]$ and $\left[A_1, A^0_2\right]$. Then

$$\left\|e_1 - e^0_1\right\| \leq \lambda \|u_1\| + \lambda \mu^{-1}_{A_{1P}} \|u_2\|,$$

and

$$\left\|e_2 - e^0_2\right\| \leq \lambda \|A_{1P}\| \|u_1\| + \lambda \mu^{-1}_{A_{1P}} \|A_{1P}\| \|u_2\|,$$

where

$$\lambda = a_2 \|A_{1P}\|^3 \left(\mu_{A_{1P}} + \mu_{A_{2P}} \|A_{1P}\|^2\right)^{-1} \left(\mu_{A_{1P}} + \mu_{A^0_{2P}} \|A_{1P}\|^2\right)^{-1}.$$

Proof

(i) Let for all x and z in H_e, $M_z x = x + A_2(z + A_1 x)$; $M^0_z x = x + A^0_2(z + A^0_1 x)$; $B_{Pz} = z + A_{1P}$ and $B^0_{Pz} = z + A^0_{1P}$,.The operators M_z and M^0_z are causal; $M_{Pz} = I + A_{2P} B_{Pz}$ and $M^0_{Pz} = I + A^0_{2P} B_{Pz}$ are invertible (see lemme 4.3), threfore M_z and M^0_z are invertible (see lemma 4.4).

Since the operator A^0_{2P} is hemicontinuous, the FS $[A_1, A_2]$ and $\left[A_1, A^0_2\right]$ are normal.

Let's show that, $[A_1,\ A_2]$ is Lipschitz continuous for both inputs. We know that

$$\begin{aligned}(e_1, e_2) &= (M^{-1}_{u_2} u_1, u_2 + A_1 M^{-1}_{u_2} u_1);\\ (e'_1, e'_2) &= (M^{-1}_{u_2} u'_1, u'_2 + A_1 M^{-1}_{u'_2} u'_1).\end{aligned}$$

Since, from lemma 4.4

$$M^{-1}_{u_2} u_1 = N^{-1}(u_1 + A^{-1}_2 u_2) - A^{-1}_2 u_2$$

we have

$$\begin{aligned}Pe_1 &= PN^{-1}(u_1 + A^{-1}_{1P} u_2) - PA^{-1}_{1P} u_2;\\ Pe'_1 &= PN^{-1}(u'_1 + A^{-1}_{1P} u'_2) - PA^{-1}_{1P} u'_2,\end{aligned}$$

and (see lemma 4.6)

$$\begin{aligned}\left\|P\left(e_1 - e'_1\right)\right\| &\leq \left\|PN^{-1}_P(u_1 + A^{-1}_{1P} u^2{}_2) - PN^{-1}_P(u'_1 + A^{-1}_{1P} u'_2)\right\|\\ &\quad + \left\|A^{-1}_{1P} P u'_2 - A^{-1}_{1P} P u_2\right\|\\ &\leq \left\|N^{-1}_P\right\|^* (\left\|u_1 - u'_1\right\| + \left\|A^{-1}_{1P}\right\|^* \left\|u_2 - u'_2\right\|)\\ &\quad + \left\|A^{-1}_{1P}\right\|^* \left\|u_2 - u'_2\right\|\\ &\leq \rho \left\|u_1 - u'_1\right\| + (1 + \rho) \left\|A^{-1}_{1P}\right\|^* \left\|u_2 - u'_2\right\|\end{aligned}$$

then,

$$\begin{aligned}\left\|e_1 - e'_1\right\| &\leq \rho \left\|u_1 - u'_1\right\| + (1 + \rho) \left\|A^{-1}_{1P}\right\|^* \left\|u_2 - u'_2\right\|\\ &\leq \lambda_{11} \left\|u_1 - u'_1\right\| + \lambda_{12} \left\|u_2 - u'_2\right\|\end{aligned}$$

with $\lambda_{11} = \rho$ and $\lambda_{12} = (1 + \rho) \left\|A^{-1}_{1P}\right\|^*$.

On the other hand

$$\begin{aligned} e_2 &= u_2 + A_{1P} M_{u_2}^{-1} u_1 \\ &= u_2 + A_{1P}\left(N^{-1}(u_1 + A_{1P}^{-1}u_2) - A_{1P}^{-1}u_2\right); \\ e_2^{'} &= u_2^{'} + A_{1P}\left(N^{-1}(u_1^{'} + A_{1P}^{-1}u_2^{'}) - A_{1P}^{-1}u_2^{'}\right), \end{aligned}$$

therefore

$$\begin{aligned} Pe_2 &= Pu_2 + PA_{1P}\left(N^{-1}(u_1 + A_{1P}^{-1}u_2) - A_{1P}^{-1}u_2\right); \\ Pe_2^{'} &= Pu_2^{'} + PA_{1P}\left(N^{-1}(u_1^{'} + A_{1P}^{-1}u_2^{'}) - A_{1P}^{-1}u_2^{'}\right), \end{aligned}$$

hence

$$\begin{aligned} &\left\|P\left(e_2 - e_2^{'}\right)\right\| = \\ &\left\|\left(Pu_2 - Pu_2^{'}\right) + A_{1P}\left(PN^{-1}(u_1 + A_{1P}^{-1}u_2) - A_{1P}^{-1}u_2\right) - A_{1P}\left(PN^{-1}(u_1^{'} + A_{1P}^{-1}u_2^{'}) - A_{1P}^{-1}u_2^{'}\right)\right\| \\ &\leq \left\|u_2 - u_2^{'}\right\| + \|A_{1P}\|^* \left\|\left(N_P^{-1}(u_1 + A_{1P}^{-1}u_2) - A_{1P}^{-1}u_2\right) - \left(N_P^{-1}(u_1^{'} + A_{1P}^{-1}u_2^{'}) - A_{1P}^{-1}u_2^{'}\right)\right\| \\ &\leq \left\|u_2 - u_2^{'}\right\| + \|A_{1P}\|^* \left(\left\|N_P^{-1}(u_1 + A_{1P}^{-1}u_2) - N_P^{-1}(u_1^{'} + A_{1P}^{-1}u_2^{'}\right\| + \left\|A_{1P}^{-1}u_2 - A_{1P}^{-1}u_2^{'}\right\|\right) \\ &\leq \left\|u_2 - u_2^{'}\right\| + \|A_{1P}\|^* \left(\left\|N_P^{-1}\right\|^* \left\|(u_1 + A_{1P}^{-1}u_2) - (u_1^{'} + A_{1P}^{-1}u_2^{'}\right\| + \left\|A_{1P}^{-1}u_2 - A_{1P}^{-1}u_2^{'}\right\|\right) \\ &\leq \left\|u_2 - u_2^{'}\right\| + \|A_{1P}\|^* \left(\left\|N_P^{-1}\right\|^* \left\|u_1 - u_1^{'}\right\| + \left\|N_P^{-1}\right\|^* \left\|A_{1P}^{-1}u_2 - A_{1P}^{-1}u_2^{'}\right\| + \left\|A_{1P}^{-1}u_2 - A_{1P}^{-1}u_2^{'}\right\|\right) \\ &\leq \left\|u_2 - u_2^{'}\right\| + \|A_{1P}\|^* \left\|N_P^{-1}\right\|^* \left\|u_1 - u_1^{'}\right\| + \left\|N_P^{-1}\right\|^* \|A_{1P}\|^* \left\|A_{1P}^{-1}\right\|^* \left\|u_2 - u_2^{'}\right\| + \|A_{1P}\|^* \left\|A_{1P}^{-1}\right\|^* \left\|u_2 - u_2^{'}\right\| \\ &\leq \rho \|A_{1P}\|^* \left\|u_1 - u_1^{'}\right\| + \left(1 + (\rho + 1)\|A_{1P}\|^* \left\|A_{1P}^{-1}\right\|^*\right) \left\|u_2 - u_2^{'}\right\| \\ &= \lambda_{21} \left\|u_1 - u_1^{'}\right\| + \lambda_{22} \left\|u_2 - u_2^{'}\right\|, \end{aligned}$$

where $\lambda_{21} = \rho \|A_{1P}\|^*$ and $\lambda_{22} = 1 + (\rho + 1)\|A_{1P}\|^* \left\|A_{1P}^{-1}\right\|^*$. Therefore $[A_1, A_2]$ and $\left[A_1, A_2^0\right]$ are Lipschitz continuous for both inputs.

(ii) Let $(u_1, u_2) \mapsto (e_1, e_2) \in H_e^2$ and $(u_1, u_2) \mapsto (e_1^0, e_2^0) \in H_e^2$, where $(u_1, u_2) \in H_P^2$ with $\|u_1\|, \|u_2\| \leq r$, and (e_1, e_2), (e_1^0, e_2^0) the respective solutions of $[A_1, A_2]$ and $\left[A_1, A_2^0\right]$. Set $w = u_1 + A_{1P}^{-1}u_2$, then

$$\begin{aligned} \left\|A_{1P}N^{-1}w\right\| &\leq \|A_{1P}\| \rho (\|u_1\| + \mu_{A_{1P}}^{-1} \|u_2\|) \\ &\leq \|A_{1P}\| \rho (1 + \mu_{A_{1P}}^{-1}) r, \end{aligned}$$

therefore, $A_{1P}N^{-1}w \in B_{\|A_{1P}\|\rho(1+\mu_{A_{1P}}^{-1})r.}$. As

$$\begin{aligned} \left\|e_1 - e_1^0\right\| &= \left\|M_{u_2}^{-1}u_1 - M_{u_2}^{0-1}u_1\right\| \\ &= \left\|N^{-1}(u_1 + A_{1P}^{-1}u_2) - A_{1P}^{-1}u_2 - N^{0-1}(u_1 + A_{1P}^{-1}u_2) + A_{1P}^{-1}u_2\right\| \\ &= \left\|\left(N^{-1} - N^{0-1}\right) w\right\| \\ &= \left\|N^{0-1}\left(N^0 - N\right) N^{-1} w\right\| \\ &= \left\|N^{0-1}\left(A_{2P}^0 A_{1P} - A_{2P}A_{1P}\right) N^{-1} w\right\| \\ &= \left\|N^{0-1}\left(A_{2P}^0 - A_{2P}\right) A_{1P} N^{-1} w\right\| \\ &\leq a_2 \left\|N^{0-1}\right\| \|A_{1P}\| \left\|N_P^{-1}\right\|^* \|w\| \\ &\leq \eta a_2 \rho \|A_{1P}\| \|w\|. \end{aligned}$$

We deduce that

$$\left\|e_1 - e_1^0\right\| \leq \lambda \|w\| \leq \lambda \|u_1\| + \lambda \mu_{A_{1P}}^{-1} \|u_2\|$$

where $w = u_1 + A_{1P}^{-1}u_2$ and $\lambda = \eta a_2 \rho \|A_{1P}\|$. On the other hand

$$\begin{aligned}\left\|e_2 - e_2^0\right\| &= \left\|A_{1P}M_{u_2}^{-1}u_1 - A_{1P}M_{u_2}^{0-1}u_1\right\| = \left\|A_{1P}\left(N^{-1} - N^{0-1}\right)w\right\| \\ &\leq \|A_{1P}\| \left\|N^{0-1}\left(N^0 - N\right)N^{-1}w\right\| \\ &= \|A_{1P}\| \left\|N^{0-1}\left(A_{2P}^0 A_{1P} - A_{2P}A_{1P}\right)N^{-1}w\right\| \\ &\leq \|A_{1P}\| \left\|N^{0-1}\left(A_{2P}^0 - A_{2P}\right)A_{1P}N^{-1}w\right\| \\ &\leq \|A_{1P}\| \left\|N^{0-1}\right\| \left\|A_{2P}^0 - A_{2P}\right\| \|A_{1P}\| \left\|N_P^{-1}\right\|^* \|w\|,\end{aligned}$$

from where

$$\begin{aligned}\left\|e_2 - e_2^0\right\| &\leq \lambda \|A_{1P}\| \|w\| \\ &\leq \lambda \|A_{1P}\| \|u_1\| + \lambda \mu_{A_{1P}}^{-1} \|A_{1P}\| \|u_2\|.\end{aligned}$$

4. Conclusion

(*Vaclav Dolezal*, 1979), introduced the notion of feedback systems in general, and established normality and linearization results on a Hilbert space. The notion of extended Hilbert space has also been introduced and one result of normality on this space has been demonstrated (theorem 4.1). The importance of this theory and its fields of application is examined by the author in a series of publications of which the most interesting are cited in the references below. Considering the imprtance of extended spaces. In the present work, we have been interested, in the formulation and the establishment, of the results of normalities and linearizations, on this space.

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Appendix

This section is devoted to the reminders of the results of ((*Vaclav Dolezal*, 1979), corollary 1, 2, 3, 5; lemma 6, 7, 8, 10 & theorem 8) relating to our work.

Corollary 4.1. The $FS\ [A_1, A_2]$, where A_1 and A_2 are two operators on H, is normal if and only if M_a is bijective, for all a in H. In this case, for (u_1, u_2) in H^2 the solution is given by

$$(e_1, e_2) = (M_{u_2}^{-1} u_1, u_2 + A_1 M_{u_2}^{-1} u_1). \tag{3}$$

Corollary 4.2. Let A_1 and A_2 be two operators on H, of which A_2 is linear. $[A_1, A_2]$ is normal if the operator $I + A_2A_1$ is bijective. In this case, for (u_1, u_2) in H^2 the solution is given by

$$\begin{aligned} e_1 &= (I + A_2A_1)^{-1}(u_1 - A_2u_2); \\ e_2 &= u_2 + A_1(I + A_2A_1)^{-1}(u_1 - A_2u_2). \end{aligned} \tag{4}$$

Lemma 4.1. Let $N \in M$, with $\mu_N > 0$; if N is hemicontinuous, then N is invertible, $N^{-1} \in Lip$, $\mu_{N^{-1}} \geq 0$ and $\left\|N^{-1}\right\|^* \leq \mu_N^{-1}$. If $N \in Lip$ then $\mu_{N^{-1}} \geq \mu_N \|N\|^{*-2}$.

Lemma 4.2. Let $A_2 \in M$ be a hemicontinuous operator and $A_1 \in Lip$ with $\mu_{A_1} > 0$. If $\mu_{A_2} + \mu_{A_1} \|A_1\|^{*-2} > 0$, then the operator $I + A_2A_1$ is invertible, $(I + A_2A_1)^{-1} \in Lip$ and

$$\left\|(I + A_2A_1)^{-1}\right\|^* \leq \mu_{A_1}^{-1} \left(\mu_{A_2} + \mu_{A_1} \|A_1\|^{*-2}\right)^{-1}.$$

If A_1 and A_2 are causal, $(I + A_2A_1)^{-1}$ is also causal.

Lemma 4.3. Let $A_2 \in \mathcal{M}$ be a linear operator, such that $\mu_{A_2} > 0$ and $A_1 \in Lip$ a hemicontinuous operator with $\mu_{A_1} \leq 0$. If $\mu_{A_2} + \mu_{A_1} \|A_2\|^2 > 0$, then $I + A_2A_1$ is invertible, $(I + A_2A_1)^{-1} \in Lip$ and

$$\left\|(I + A_2A_1)^{-1}\right\|^* \leq \|A_1\| \left(\mu_{A_2} + \mu_{A_1} \|A_2\|^2\right)^{-1}.$$

If A_1 and A_2 are causal, $(I + A_2A_1)^{-1}$ is also causal.

Lemme 4.4. Let $A_2 \in \mathcal{M}$ be hemicontinuous with $\mu_{A_2} \leq 0$ and $A_1 \in \mathcal{M}$ a linear operator with $\mu_{A_1} > 0$. If $\mu_{A_1} + \mu_{A_2} \|A_1\|^2 > 0$, then $I + A_2A_1$ is invertible, $(I + A_2A_1)^{-1} \in Lip$ and

$$\left\|(I + A_2A_1)^{-1}\right\|^* \leq \|A_1\| \left(\mu_{A_1} + \mu_{A_2} \|A_1\|^2\right)^{-1}.$$

If, in addition de plus, A_1 and A_2 are causal, then $(I + A_2A_1)^{-1}$ is also causal.

Lemma 4.5. If $A_1, A_2 : H_e \mapsto H_e$ are two operators, where A_1 is linear and invertible, and the operator $N = I + A_2A_1$ is invertible, then for the given a in H_e, the operator $M_a = I + A_2(a + A_1)$ is invertible and the inverse M_a^{-1} is given by:

$$M_a^{-1}x = N^{-1}(x + A_1^{-1}a) - A_1^{-1}a, \ \forall x \in H_e. \tag{5}$$

Lemma 4.6. Let $A : H_e \mapsto H_e$ be causal, and for every $P \in P$, $A_p : H_P \to H_P$ the restriction of PA to H_P. Then A is invertible and the inverse $A^{-1} : H_e \to H_e$ is causal iff A_p is invertible for each $P \in P$. In that case

$$PA^{-1} = A_P^{-1}P, \ \forall P \in \mathcal{P}$$

and A_P^{-1} is the restriction of PA^{-1} to H_P

Théorème 4.1. Let $A_1, A_2 : H_e \to H_e$ be tow causal operators. For all $P \in P$, A_{1P} and A_{2P} the restriction of PA_1 and PA_2 to H_P. Suppose that for each $P \in P$

(i) $A_{1P} \in Lip$ and $\mu_{A_{1P}} > 0$;

(ii) $A_{2P} \in M$ and it is hemicontinuous;

(iii) $\mu_{A_{2P}} + \mu_{A_{1P}} \|A_{1P}\|^{*-2} > 0$.

Then, the $FS\ [A_1, A_2]$ over H_e is normal and causal.

Moreover if, there exists $\lambda > 0$ such that, for all $P \in P$

$$\mu_{A_{1P}}^{-1}\left(\mu_{A_{2P}} + \mu_{A_{1P}}\|A_{1P}\|^{*-2}\right)^{-1} \leq \lambda,$$

and, if $(u_{1,}u^*) \mapsto (e_{1,}e_2) \in H_e^2$; $(u_{1,}^{'}u^*) \mapsto (e_{1,}^{'}e_2^{'}) \in H_e^2$; $u_1 - u_1^{'} \in H$, then $e_1 - e_1^{'} \in H$,

$$\left\|e_1 - e_1^{'}\right\| \leq \lambda\left\|u_1 - u_1^{'}\right\|.$$

If, in addition there existes $k > 0$ such that, for all $P \in P$, $\|A_{1P}\|^* \leq k$, also $e_2 - e_2^{'} \in H$,

$$\left\|e_2 - e_2^{'}\right\| \leq \lambda k\left\|u_1 - u_1^{'}\right\|.$$