Journal of Mathematics Research; Vol. 12, No. 2; April 2020
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Hopfian and Cohopfian Objects in the Categories of $Gr(A-Mod)$ and $COMP(Gr(A-Mod))$

Seydina Ababacar BALDE [1], Mohamed Ben Faraj BEN MAAOUIA[1] & Ahmed Ould CHBIH [1]

[1] Gaston Berger University, Saint-Louis, Senegal

Correspondence: Mohamed Ben Faraj BEN MAAOUIA, Gaston Berger University, Saint-Louis, Senegal.
E-mail: maaouiaalg@hotmail.com

Received: January 1, 2020 Accepted: January 29, 2020 Online Published: February 19, 2020

doi:10.5539/jmr.v12n2p17 URL: https://doi.org/10.5539/jmr.v12n2p17

Abstract

We study in this work the notions of hopficity and cohopficity in the categories

$AGr(A-Mod)$ and $COMP(AGr(A-Mod))$ of associate complex to a graded left A-module and we show that:

1. Let M a graded left $A-$module, N a graded submodule of M, M_* be a complex associate to M. Suppose that M_* be a quasi-projective and N_* be a completely invariant and essential sub-complex of M_* associate to N. Then N_* is cohopfian if, and only, if M_* is cohopfian.
2. Let M a graded left $A-$module, N a graded submodule of M, M_* quasi-injective and N_* a completely invariant and superfluous sub-complex of M_*. Then M_* is cohopfian if, and only, if M_*/N_* is cohopfian.

Keywords: Hopfian complex, cohopfian complex, chain complex, sequence complex, quasi-injective chain complex, quasi-projective chain complex

1. Introduction

In this paper, the ring A is supposed to be associatif, unitary and not necessairly commutative, every left A-module is unifere.

The aim of this article is to study the hopfian and cohopfian objets in the category $Gr(A-Mod)$ of graded left $A-$modules and in the category $COMP(AGr(A-Mod))$of associated complex of graded left $A-$modules. In particular we give conditions over M_* and N_* such that M_*/N_* be cohopfian (respectively is hopfian) and conditions over M_*/N_* and N_* such that M_* be hopfian.

We define $AGr(A-Mod)$ and $COMP(AGr(A-Mod))$:

1. The category of graded of left A-modules denoted $AGr(A-Mod)$ where :
   (a) The objects are the graded left $A-$modules;
   (b) The morphisms are the graded morphisms..
2. the category of complexes associate of graded left A-modules denoted $COMP(AGr(A-Mod))$ where
   (a) the objects are the complex sequences associate of graded left A-modules ;
   (b) the morphisms are the complex chains associate of graded morphisms.

We note that $COMP(AGr(A-Mod))$ is a sub-category of $COMP(A-Mod)$.

The principal results of this article is given in the third section, which are:

1. Let M be a graded left A-module. then M is hopfian(resp. cohopfian) if for any $n\in\mathbb{Z}$, $M(n)$ is hopfian(resp. cohopfian).
2. Let M a graded left $A-$module, N a graded submodule of M, M_* be a complex associate to M and N_* be a complex associate to N. If M_*/N_* is hopfian, then M_* is hopfian.

3. Let M a graded left $A-$module, N a graded submodule of M, M_* be a complex associate to M and N_* be a complex associate to N. Suppose that N_* be a completely invariant and superfluous sub-complex of M_*. Then M_* is hopfian if, and only, if M_*/N_* is hopfian.

4. Let M a graded left $A-$module, N a graded submodule of M, M_* quasi-injective and N_* a completely invariant and superfluous sub-complex of M_*. Then M_* is cohopfian if, and only, if M_*/N_* is cohopfian.

5. Let M a graded left $A-$module, N a graded submodule of M, M_* be a complex associate to M. Suppose that N_* be a complex associate to N, M_* be a quasi-projective and N_* be a completely invariant and essential sub-complex of M_*. Then N_* is cohopfian if, and only, if M_* is cohopfian.

6. Let M a graded left $A-$module, N a graded submodule of M, M_* be a complex associate to M. If M_*/N_* is hopfian for all nonzero sub-complex N_* associate to N, then M_* is hopfian.

7. Let M a graded left $A-$module, M_* be a complex associate to M. If M_*/N_* is hopfian for all sub-complex N_* associate to N, then M_* is hopfian.

8. (p) denotes the following property :

   $\ll$*any epimorphism of sub-complex N_* of M_* (where M is an object of $COMP(AGr(A-Mod))$)*is an isomorphism $\gg$

   Let M a graded left $A-$module, M_* a complex associate to M. If M_* is an hopfian quasi-projective , then M_* owns the property (p).

2. Preliminaries

Definition 1 *Lets A be a ring and is a family $\{A_n\}_{n\in\mathbb{Z}}$ of sub-group of A. If*

1. $A = \bigoplus_{n\in\mathbb{Z}} A_n$;
2. $A_n \cdot A_m \subset A_{n+m}, \forall\, n,\ m \in \mathbb{Z}$.

Then we say that A is a graded ring. Else, if $A_n = 0, \forall n < 0$.Then A is called positively graded ring.

In all that follows, A and M are supposed unitary.

Definition 2 *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring and M be a left $A-$module, then M is called a graded left $A-$module if there exist a sequence $(M_n)_{n\in\mathbb{Z}}$ of sub-group of M such that:*

1. $M = \bigoplus_{n\in\mathbb{Z}} M_n$;
2. $A_n \cdot M_d \subset M_{n+d}, \forall\, n,\ d \in \mathbb{Z}$.

Definition 3 *Lets $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded left $A-$module and N is a sub-module of M, then N is called a graded sub-module of M, if $\forall x = \sum_{n\in\mathbb{Z}} x_n \in N$, with $x_n \in M_n$, then $x_n \in N$, $\forall n \in \mathbb{Z}$.*

Proposition 1

Lets $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is graded left $A-$module , then for all $n \in \mathbb{Z}$ fixed, we have

$$M(n) = \bigoplus_{k\geq n} M_k = \bigoplus_{k\in\mathbb{N}} M_{n+k}$$

is a graded sub-module of M and :

$$\cdots M(n+2) \subset M(n+1) \subset M(n) \subset \cdots.$$

Proof

Let $n \in \mathbb{Z}$ fixed, $M(n) = \bigoplus_{k\geq n} M_k$ is a sub-group of M and

$$A_s \cdot M(n)_k = A_s \cdot M_{n+k} \subset M_{n+k+s} = M_{n+(k+s)} = M(n)_{k+s}.$$

Else

$$M(n) = \bigoplus_{k\geq n} M_k = M_n \bigoplus M(n+1) = \bigoplus_{k\in\mathbb{N}} M_k.$$

Hence $M(n+1) \subset M(n)$. Thus

$$\cdots M(n+2) \subset M(n+1) \subset M(n) \subset \cdots.$$

Definition 4 *Lets* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$, $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *are two graded left* $A-$*modules and* $f : M \longrightarrow N$ *is a morphism of left* $A-$*modules, then* f *is called a graded morphism if for any* $m \in M_s$ *then* $f(m) \in N_{s+k}$.

Theorem and Definition 1 *Let* A *be a graded ring, the category of graded left* $A-$*module is the category denoted by* $Gr(A-Mod)$*-Mod whose*

1. *The objects are the graded left* $A-$*modules;*
2. *The morphisms are the graded morphisms.*

Proof

See (OULD CHBIH et al., 2015).

Definition 5 *A complex sequence* $(C,d) : \ldots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} \ldots$ *is a sequence of morphisms of* $A-$ *modules satisfying* $d_n \circ d_{n+1} = 0$*, for all* $n \in \mathbb{Z}$.

Definition 6 *A complex chain* $f : (C,d) \to (C',d')$ *is a sequence of homomorphisms* $(f_n : C_n \longrightarrow C'_n)_{n\in\mathbb{Z}}$ *of* $A-$ *modules making the following diagram commute :*

$$\begin{array}{ccccccccc}
(C,d): \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
\downarrow f & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\
(C',d'): \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

i.e $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$*, for all* $n \in \mathbb{Z}$.

Proposition and Definition 1 *Let* A *be a ring, then the category of complexes of left* $A-$*modules is the category denoted* $COMP(A-Mod)$ *whose :*

1. *The objects are the sequences complex;*
2. *The morphisms are the complex chains.*

Proof

See (Dade E. C. 1980).

Proposition 2 *Lets* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring and* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *be a graded left* $A-$*module, then we have* M_* *:*

$$M_* : \cdots \to M(n+1) \overset{d_{n+1}}{\to} M(n) \overset{d_n}{\to} M(n-1) \to \cdots$$

is an associate complex sequence of a grade $A-$*module* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *with* $M(n) = \bigoplus_{k\in\mathbb{Z}} M_{n+k}$ *and*

$$d_n : M(n) \longrightarrow M(n-1)$$

$$x \longmapsto y$$

with $(y, z) \in M_n \times M(n+1)$.

Proof

We have $M(n) = \bigoplus_{k\in\mathbb{Z}} M_{n+k} = \bigoplus_{k\geq n} M_k = M_n \bigoplus M_{n+1}$ and

$$M(n-1) = M_{n-1} \bigoplus M(n) = M_{n-1} \bigoplus M_n \bigoplus M(n+1).$$

Let $x \in M(n) \Longrightarrow \exists!(y, z) \in M_n \times M(n+1)$ such that $x = y + z$.

Put

$$d_n : M(n) \longrightarrow M(n-1)$$

$$x = y + z \longmapsto y$$

so $Im(d_n) = M_n$; on the other hand

$$d_{n-1} : M(n-1) \longrightarrow M(n-2)$$

$$w = u + v \longmapsto u$$

with $(u, v) \in M_{n-1} \times M(n)$ so $ker(d_{n-1}) = M(n)$ so $Im(d_n) \subset \ker(d_{n-1})$ so

$$d_{n-1} \circ d_n = 0$$

thus

$$M_* : \cdots \to M(n+1) \overset{d_{n+1}}{\to} M(n) \overset{d_n}{\to} M(n-1) \to \cdots$$

is a complex sequence.

Proposition 3 *Let M be a graded left A-module, N a graded submodule of M, M_* the complex associate to M and for all $n \in \mathbb{Z}$, $N(n)$ is a submodule of $M(n)$. Then*

$$N_* : \cdots \to N(n+1) \overset{\delta_{n+1}}{\to} N(n) \overset{\delta_n}{\to} N(n-1) \to \cdots \text{ with } d_n(x) = \delta_n(n)(x)$$

is a sub-complex of M_*

Proof

We have $\delta_n : N(n+1) \longrightarrow N(n)$

let $x, y \in N(n+1) : x = y$

then $d_n(x) = d_n(y)$

$\Longrightarrow \delta_n(x) = \delta_n(y)$

$\Longrightarrow$ is well define.

Let's calculate $\delta_n \circ \delta_{n+1}$

Let $x \in N(n+1)$, we have :

$\delta_n \circ \delta_{n+1}(x) = \delta_n(\delta_{n+1}(x))$

$= \delta_n(d_{n+1}(x)) = d_n(d_{n+1}(x)) = d_n \circ d_{n+1}(x) = 0$

Thus $\delta_n \circ \delta_{n+1} = 0$

hence N_* is a sub-complex of M_*.

Proposition 4 *Lets $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$module $f : M \longrightarrow N$ is a graded morphism of a graded left $A-$modules, then for all $n \in \mathbb{Z}$*

$$f(n) : M(n) \longrightarrow N(n)$$

$$m \quad \longmapsto \quad = f(m)$$

is graded morphism.

Proof

We have $M(n)$ is a graded sub-module of left $A-$module M and $f : M \longrightarrow N$ is graded morphism, then let $m \in M(n)$, so

$$m = \sum_{i \in \mathbb{Z}} m_{i+n} \Longrightarrow f(n)(m) = f(m) = f(\sum_{i \in \mathbb{Z}} m_{i+n}) = \sum_{i \in \mathbb{Z}} f(m_{i+n})$$

or $f(m_{i+n}) \in N_{i+n+k} = (N(n))_{i+k}$ thus f is graded morphism of a graded left $A-$modules.

Proposition 5 *Lets* $A = \bigoplus_{n \in \mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n \in \mathbb{Z}} M_n$*,* $N = \bigoplus_{n \in \mathbb{Z}} N_n$ *are two graded left* $A-$*modules and* $f : M = \bigoplus_{n \in \mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n \in \mathbb{Z}} N_n$ *is a graded morphism of a graded* $A-$*modules, then :*

$$\begin{array}{ccccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \dots \\
\downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \downarrow f(n-1) & & \\
N_* : \cdots & \longrightarrow & N(n+1) & \xrightarrow{d'_{n+1}} & N(n) & \xrightarrow{d'_n} & N(n-1) & \longrightarrow & \cdots
\end{array}$$

is an associate chain complex f_* *of graded morphism* f

Proof

Let $x \in M(n+1) \Longrightarrow \exists!(y, z) \in M_{n+1} \times M(n+2)$ such that $x = y + z$ then

$$(f(n) \circ d_{n+1})(x) = f(n)[d_{n+1}(x)] = f[d_{n+1}(x)] = f[y] = f(y)$$

and

$$(d'_{n+1} \circ f(n+1))(x) = d'_{n+1}[f(n+1)(x)] = d'_{n+1}[f(x)] = d'_{n+1}[f(y+z)] = d'_{n+1}[f(y) + f(z)] = f(y)$$

$$\Longrightarrow (f(n) \circ d_{n+1})(x) = (d'_{n+1} \circ f(n+1))(x), \ \ \forall\ x\ \in M(n+1)$$

so

$$f(n) \circ d_{n+1} = d'_{n+1} \circ f(n+1)$$

thus f_* is a complex chain.

Theorem 1 *Let* $A = \bigoplus_{n \in \mathbb{Z}} A_n$ *be a graded ring. We called* $COMP(AGr(A - Mod))$ *the category of associate complex of a graded left* $A-$*modules whose :*

1. *The objets are the associate complex sequences of a graded left* $A-$*modules;*
2. *The morphisms are the associate complex chains of a graded morphism.*

Proof

1. Lets M_* and N_* two complex sequences associate of a graded left A-module M and N respectively.
   Put $Hom_{COMP(AGr(A-Mod))}(M_*, N_*)$ = the class of complex chains associate of graded morphism of $M \longrightarrow N$. Then $Hom_{COMP(Gr(A-Mod))}(M_*, N_*)$ is a set, because the class of complex chain of $M_* \longrightarrow N_*$ of the category $COMP(A - Mod)(M_*, N_*)$ is a set (it suffices to remark also the class of graded of $M \longrightarrow N$ is a set).
2. $\forall\ \ f_* \in Hom_{COMP(Gr(A-Mod))}(M_*, N_*)$; $g_* \in Hom_{COMP(Gr(A-Mod))}(N_*, P_*)$ and $h_* \in Hom_{COMP(Gr(A-Mod))}(P_*, Q_*)$ we have :

$$\begin{array}{ccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \dots \\
\downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \\
N_* : \cdots & \longrightarrow & N(n+1) & \xrightarrow{d'_{n+1}} & N(n) & \xrightarrow{d'_{n+}} & \dots \\
\downarrow g_* & & \downarrow g(n+1) & & \downarrow g(n) & & \\
P_* : \cdots & \longrightarrow & P(n+1) & \xrightarrow{d''_{n+1+k+r}} & P(n) & \xrightarrow{d''_{n+k+r}} & \dots \\
\downarrow h_* & & \downarrow h(n+1) & & \downarrow h(n) & & \\
Q_* : \cdots & \longrightarrow & Q(n+1) & \xrightarrow{d'''_{n+1}} & Q(n) & \xrightarrow{d'''_n} & \dots
\end{array}$$

So $(h_* \circ g_*) \circ f_* = h_* \circ (g_* \circ f_*)$;

3. Let M_* the object of $COMP(Gr(A-Mod))$ we have :

$$1_{M_*} : M_* \longrightarrow M_*$$

$$\begin{array}{ccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \dots \\ \downarrow 1_{M_*} & & \downarrow 1(n+1) & & \downarrow 1(n) & & \\ M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \dots \end{array}$$

1_{M_*} verified $f_* \circ 1_{M_*} = f_* \quad \forall\ f_* \in Hom_{COMP(Gr(A-Mod))}(M_*, N_*)$.
Furthermore $1_{M_*} \circ g_* = g_* \quad \forall\ g_* \in Hom_{COMP(Gr(A-Mod))}(N_*, M_*)$.

4. $\forall (M_*, N_*) \neq (M'_*, N'_*) \Longrightarrow Hom_{COMP(Gr(A-Mod))}(M_*, N_*) \neq Hom_{COMP(Gr(A-Mod))}(M'_*, N'_*)$

Thus $COMP(Gr(A-Mod))$ is a category.

Remark 1 *$COMP(AGr(A-Mod))$ is a sub-category of $COMP(A-Mod)$*

Proposition and Definition 2 *Lets $(C,d) : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$ be a complex sequence of left A--modules and (E_n) be a family of left A-modules such that for all n, E_n is a submodule of C_n. If $d_n(E_n) \subset E_{n-1}$, then the complex sequence of induced morphisms $(d_n : E_n \longrightarrow E_{n-1})$ is complex sequence of left A-modules called sub-complex of C*

Proof

Let $\delta_n : E_n \longrightarrow E_{n-1}$ be the induced morphism of d_n, indeed let $x \in E_n$, hence $d_n(x) \in E_n$, then δ_n is well defined.

δ_n is a morphism, since it's composed of two morphisms.

Let's verify that $\delta_n \circ \delta_{n+1} = 0$

Let be $x \in E_{n+1}$, hence $\delta_{n+1}(x) = d_{n+1}(x) \in E_n$ and $\delta((d_{n+1})(x) = d_n \circ d_{n+1}(x) = 0$, then $\delta_n \circ \delta_{n+1} = 0, \forall n \in \mathbb{Z}$.

Proposition and Definition 3 *Lets C be a complex and E be a sub-complex of C. Suppose that $K = (K_n)_{n \in \mathbb{Z}}$, where $K_n = C_n/E_n$. Then K is called quotient complex of C by E and denoted C/E*

Proof

See (Diallo E. O. 2014).

Theorem 2 *let $(C,d), (C',d')$ two objects of $COMP(A-Mod)$ and $f : C \longrightarrow C'$ be a complex chain. Then f is a monomorphism of $COMP(A-Mod)$ if, and only, if $ker(f) = 0$.*

Proof

If f is a monomorphism of C into C', hense $f \circ u = f \circ v \Longrightarrow u = v$, then $\forall n \in \mathbb{Z}$, $f_n \circ u_n = f_n \circ_n \Longrightarrow u_n = v_n$, so f_n is a monomorphism, thus $ker(f)_n = 0$, hence $ker(f) = 0$.

Suppose that $ker(f) = 0$, hence $ker(f)$ is a zero complex, so each terms is zero, thus f_n is a monomorphism of left A--modules then $\forall n \in \mathbb{Z}$ it gives if $f_n \circ u_n = f_n \circ v_n$ hence $u_n = v_n$, so $(f \circ u)_n = (f \circ v)_n \Longrightarrow (u)_n = (v)_n$, finally, f is a monomorphism of complex chains.

Theorem 3 *lets $(C,d), (C',d')$ two objects of $COMP(A-Mod)$ and $f : (C,d) \longrightarrow (C',d')$ be a complex chain. Then f is an epimorphism of $COMP(A-Mod)$ if, and only, if $Im(f) = C'$.*

Proof

If f is an epimorphism of $COMP(A-Mod)$, then $u \circ f = v \circ f \Longrightarrow u = v$, hence for all $n \in \mathbb{Z}$, $(u \circ f)_n = (v \circ f)_n \Longrightarrow u_n = v_n$, f_n is an epimorphism of left A-modules, then $Im(f)_n = C'_n$, $\forall n \in \mathbb{Z}$, so $Im(f) = C'$.
Suppose that $Im(f) = C'$, hence $\forall n \in \mathbb{Z}$, $Im(f)_n = C'_n$, then f_n is an epimorphism of left A-modules so $\forall n \in \mathbb{Z}$, $u_n \circ f_n = v_n \circ f_n$, then $(u \circ f)_n = (v \circ f_n) \Longrightarrow u_n = v_n$ then f is an epimorphism of complex chains.

3. Hopfian and Cohopfian Objects, Quasi-Injective and Quasi-Projective Objects in the Category COMP(Gr(A-Mod))

Definition 1 *Let M_* an object of $COMP(AGr(A-Mod))$. Then M_* is said to be hopfian (resp. cohopfian) if any epimorphism (resp. monomorphism) f_* of M_* is an automorphism.*

Proposition 1 *Let M be a graded left A-module and f a graded endomorphism of M. If f is an epimorphism (resp. monomorphism, resp. isomorphism) then $f(n)$ is an epimorphism (resp. monomorphism, resp. isomorphism) .*

Proof

Is evident, since $f(n)$ is the induce of f.

Proposition 2 *Let M be a graded left A-module. then M is hopfian(resp. cohopfian) in $AGr(A-Mod)$ if for any $n \in \mathbb{Z}$, $M(n)$ is hopfian(resp. cohopfian) $AGr(A-Mod)$.*

Proof

let $g : M(n) \longrightarrow M(n)$ a graded epimorphism (resp. a monomorphism), since $M = M(n)\bigoplus_{k>n} M_{n+k}$.

Put $f = g + id_{M_{n+k}}$ where $f \in End(M)$ is an epimorphism (resp. monomorphism). Since M hopfian (resp. cohopfian) this implies f is an isomorphism, thus g is an isomorphism, then $M(n)$ is hopfian (resp. cohopfian)

Definition 2 *Let M be a graded left A-module and N a graded submodule of M. Then N is called a completely invariant if for any graded endomorphism f of M, we have $f(N) \subset N$.*

Definition 3 *Let M_* an objet of $COMP(AGr(A-Mod)$ and N_* a sub-complex of M_*. N_* is called completely invariant if for any endomorphism f_* of M_*, $f(N_*)$ is a of M_*.*

Proposition 3 *Let $M_* : \ldots M(n+1) \xrightarrow{d_{n+1}} M(n) \xrightarrow{d_n} M(n-1) \longrightarrow \ldots$ be an objet of $COMP(AGr(A-Mod))$ and $N_* : \ldots N(n+1) \xrightarrow{\delta_{n+1}} N(n) \xrightarrow{\delta_n} N(n-1) \longrightarrow \ldots$ be a subcomplex of M_*. Then N_* completely invariant in $AGr(A-Mod)$ if, and only if, for all $n \in \mathbb{Z}$, $N(n)$ is a completely invariant sub-module of $M(n)$.*

Proof

Let $N(n)$ be a completely invariant in $M(n)$, $\forall\ n \in \mathbb{Z}$ then we have :

$f(n)(N(n)) \subset N(n) \Longrightarrow f_*(N_*) \subset N_*, \forall\ \ n \in \mathbb{Z}$

Suppose that N_* completely invariant in M_*, then

$f(n)(n) \subset N(n), \forall\ \ n \in \mathbb{Z}$

Thus $N(n)$ is completely invariant in $M(n)$.

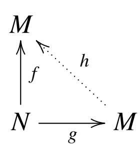
Definition 4 *Lets M be a left A-module and N be a sub-module of M. Then N is called essential in M if for any nonzero submodule K of M, we have $K \cap N \neq 0$; M is called essential extension of E.*

Definition 5 *Lets $M_* : \ldots M(n+1) \xrightarrow{d_{n+1}} M(n) \xrightarrow{d_n} M(n-1) \longrightarrow \ldots$ be a complex sequence of left A-modules and $N_* : \ldots N(n+1) \xrightarrow{\delta_{n+1}} N(n) \xrightarrow{\delta_n} N(n-1) \longrightarrow \ldots$ be a subcomplex of M_*. Then N_* is called essential of M_* if $N(n)$ is an essential submodule of $M(n)$, $\forall n \in \mathbb{Z}$*

Definition 6 *Lets M be a left A-module and N be a submodule of M. Then N is called superfluous in M if for any submodule K of M, we have $K + N = M$, then $K = M$.*

Definition 7 *Lets $M_* : \ldots M(n+1) \xrightarrow{d_{n+1}} M(n) \xrightarrow{d_n} M(n-1) \longrightarrow \ldots$ be an objet of $COMP(AGr(A-Mod))$ and $N_* : \ldots N(n+1) \xrightarrow{\delta_{n+1}} N(n) \xrightarrow{\delta_n} N(n-1) \longrightarrow \ldots$ be a subcomplex of M_*. Then N_* is called superfluous of M_* if $N(n)$ is superfluous submodule of $M(n)$, $\forall n \in \mathbb{Z}$.*

Definition 8 *Let M a graded left A-module. Then M is called quasi-injective, if for any graded monomorphism $g : N \longrightarrow M$ of graded left A-modules and for any graded morphism $f : N \longrightarrow M$, there exists a graded endomorphism h of M such $f = h \circ g$.*



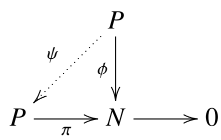
Definition 9 *Lets M_* and N_* are two objets of $COMP(AGr(A-Mod))$. M_* is called quasi-injective, if for any monomorphism $g : N_* \longrightarrow M_*$ and for any complex chain $f : N_* \longrightarrow M_*$, there exists a complex chain $h : M_* \longrightarrow M_*$ verifying $f = h \circ g$*

Theorem 1 *Given $M_* : \ldots M(n+1) \xrightarrow{d_{n+1}} M(n) \xrightarrow{d_n} M(n-1) \longrightarrow \ldots$ an object of $COMP(Gr(A-Mod))$. M_* is* *quasi-injective in $COMP(Gr(A - Mod))$ if, and only, if for all $n \in \mathbb{Z}$, $M(n)$ is a quasi-injective in $COMP(Gr(A - Mod))$.*

Proof

See (OULD CHBIH et al., 2015).

Definition 10 *Let P a graded left A-module. Then P is called quasi-projective, if for any graded left A-module N and any graded epimorphism $\pi : P \longrightarrow N$ of graded left A-modules and for any graded morphism $\phi : P \longrightarrow N$, there exists a graded endomorphism ψ of P such $\phi = \pi \circ \psi$.*



Definition 11 *Lets M_* and N_* are two objects of $COMP(Gr(A - Mod))$. M_* is called quasi-projective, if for any epimorphism $g : M_* \longrightarrow N_*$ and for any complex chain $f : M_* \longrightarrow N_*$, there exists a complex chain $h : M_* \longrightarrow M_*$ verifying $f = h \circ g$.*

Theorem 2 *Let be $M_* : \ldots M(n+1) \xrightarrow{d_{n+1}} M(n) \xrightarrow{d_n} M(n-1) \longrightarrow \ldots$ an object of $COMP(Gr(A - Mod))$. M_* is quasi-projective if, and only, if for all $n \in \mathbb{Z}$, $M(n)$ is a quasi-injective left A-modules*

Proof

See (OULD CHBIH et al., 2015).

Proposition 4 *Lets M be a graded left $A-$module, N be a graded submodule of M, M_* be a complex associate to M and N_* be a complex associate to N. If M_*/N_* is hopfian, then M_* is hopfian.*

Proof

If M_*/N_* is hopfian and M_* is not hopfian, hence there exists $n \in \mathbb{Z}$ such $f(n) : M(n) \longrightarrow M(n)$ be an epimorphism and no monomorphic, i.e $ker(f)(n) \neq 0$. Put $N(n) = ker(f)(n)$. Thus $f(n)$ induce an isomorphism $\overline{f}(n) : M(n)/N(n) \longrightarrow M(n)$. If $\pi(n) : M(n) \longrightarrow M(n)/N(n)$ the canonical surjection, $\pi(n) \circ f(n) : M(n)/N(n) \longrightarrow M(n)/N(n)$ is an epimorphism and no monomorphic, this is a contradiction because all epimorphism of $M(n)/N(n)$ is an isomorphism.

Lemma 1 *if $f : M \longrightarrow N$ and $g : N \longrightarrow M$ such that $f \circ g = id_N$, then $M = ker(f) \oplus Img$.*

Proof

Let $m \in ker(f) \cap Img \Longrightarrow f(m) = 0$ and there exists $n \in N$, $g(n) = m \Longrightarrow f(m) = f \circ g(n) = n$ and $f(m) = 0 \Longrightarrow n = 0 \Longrightarrow g(0) = m \Longrightarrow ker(f) \cap Img = 0$.

Let $m \in M$, show that $m - g \circ f(m) \in ker(f)$.

We have $f(m - g \circ f(m)) = f(m) - (f \circ g)(f(m)) = f(m) - f(m) = 0$.

Since $m = g \circ f(m) + (m - g \circ f(m))$.

More than $g \circ f(m) \in Img$ and $m - g \circ f(m) \in ker(f)$, hence $M = ker(f) + Img$. Thus $M = ker(f) \oplus Img$.

Proposition 5 *Lets M be a left graded module, N a graded submodule of M. If M_* quasi-projective and N_* is a completely invariant and essential subcomplex of M_*. Then N_* is cohopfian if, and only, if M_* is cohopfian.*

Proof

Suppose that M_*/N_* is hopfian and let $f_* : M_* \longrightarrow M_*$ be an epimorphism.

As N_* is completely invariant, then $\forall n \in \mathbb{Z}$, $f(n)(N(n)) \subset N(n)$, implies $f(n)$ induce an epimorphism $\overline{f}(n) : M(n)/N(n) \longrightarrow M(n)/N(n)$, since M_*/N_* is hopfian, then $\overline{f}(n)$ is an automorphism. Put $K(n) = ker(f)(n)$ and $\pi(n) : M(n) \longrightarrow M(n)/N(n)$ be the canonical projection, we have :

$\overline{f}(n) \circ \pi(n)(K(n)) = \pi(n) \circ f(n)(K) = 0$
Indeed, $\forall x \in K(n)$, we have :

$\overline{f}(n) \circ \pi(n)(x) = \pi(n) \circ f(n)(x)$, so

$\pi(n) \circ f(n)(x) = \pi(n)(f(n)(x) = \pi(n)(0) = 0$, hence $\overline{f}(n) \circ \pi(n)(K) = 0$.

we have :

$\overline{f}(n) \circ \pi(n)(K(n)) = \pi(n) \circ f(n)(K(n)) = 0 \Longrightarrow \overline{f}(n)(\pi(n)(K(n))) = 0 \Longrightarrow \pi(n)(K(n)) \subset N(n) \Longrightarrow K(n) \subset N(n)$

Since M_* is quasi- projective, there exists an endomorphism $s(n) : M(n) \longrightarrow M(n)$ such that $f(n) \circ s(n) = id(n)_M$, which implies $M = K(n) \oplus Ims(n)$, or $K(n) = N(n)$ and $N(n)$ superfluous in $M(n), \forall n \in \mathbb{Z}$, then $M(n) = Ims(n)$, so $K(n) = ker(f)(n) = 0, \forall n \in \mathbb{Z}$, hence $f(n)$ is an monomorphism, $\forall n \in \mathbb{Z}$

At last, f_* is an monomorphism, thus M_* is hopfian.

Reciprocally,

If M_* is hopfian and show that M_*/N_* est hopfian.

Let $\varphi(n) : M_*/N_* \longrightarrow M_*/N_*$ an epimorphism of complex chains, as M_* is quasi-projective, hence $\forall n \in \mathbb{Z}, M(n)$ is projective. We Consider $\pi(n) : M(n) \longrightarrow M(n)/N(n)$, then there exists $f(n) \in End(M(n))$ such that $\pi(n) \circ f(n) = \varphi(n) \circ \pi$.

Since $\varphi(n)$ is an epimorphism, $\forall \bar{x} \in M(n)/N(n), \exists \bar{y} \in M(n)/N(n)$ such that

$\varphi(n)(\bar{y}) = \bar{x} = \varphi(n)(\pi(n)(y)$

$\pi(n) \circ f(n)(y) = \varphi(n) \circ \pi(n)(y)$

$\pi(n)(f(n)) = \varphi(\bar{y})$

$\Longrightarrow \varphi(n)(\bar{y}) = \overline{f}(n)(\bar{y}) = \bar{x}$

$\Longrightarrow \overline{f(n)(y) - x} = \bar{0}$

$\Longrightarrow f(n) - x = 0$

$\Longrightarrow f(n) - x \in N(n),$

then $M = Im(f)(n) + N$, as N is superfluous, then $Im(f)(n) = M(n) \Longrightarrow f(n)$ is an epimorphism. Thus, $f(n)$ is an automorphism, because $M(n)$ is hopfian for all $n \in \mathbb{Z}$.

Hence the restriction of $f(n)$ over $N(n)$ is an automorphism of $N(n)$.

If $\varphi(\bar{x}) = \overline{f}(n)(x) = \bar{0}$, then $f(n)(x) \in N(n)$, or $N(n)$ is completely invariant, then $x \in N$, so $\bar{x} = \bar{0} \Longrightarrow ker\varpi(n) = N = \bar{0} \Longrightarrow \varphi(n)$ is an monomorphism $\Longrightarrow \varphi(n)$ is an automorphism for all $n \in \mathbb{Z}$, finally $M(n)/N(n)$ is hopfian.

Proposition 6 *Lets M a graded left $A-$module, N a graded submodule of M, M_* quasi-injective and N_* a completely invariant and superfluous subcomplex of M_*. Then M_* is cohopfian if, and only, if M_*/N_* is cohopfian.*

Proof

Suppose that M_* is cohopfian and let $f_* : N_* \longrightarrow N_*$ be a monomorphism. As M_* is quasi-injective, then for all $n \in \mathbb{Z}$ is quasi-injective. Then there exists $g(n) \in End(M(n))$ $\forall n \in \mathbb{Z}$ such that $g_{|N(n)} = f(n)$.

$g(n)$ is injective because N_* is essential in M_*, hence $\forall n \in \mathbb{Z}$, $N(n)$ is essential, and since M_* is cohopfian, g is invertible. Let $x \in N(n)$, there exists $y \in M$ such that $x = g(n)(y)$. Or $g^{-1}(n) \in End(N(n))$ and $N(n)$ is completely invariant, thus $y = g^{-1}(x) \in N$, hence $f(n)$ is an epimorphism for all $n \in \mathbb{Z}$. Thus f_* is an automorphism, consequently N_* is cohopfian.

Reciprocally, suppose that N_* is cohopfian an let $f_* : M_* \longrightarrow M_*$ be a monomorphism. Then $f_{*|N_*}$ is an injective endomorphism of N_*, hence for all $n \in \mathbb{Z}, f(n)_{|N(n)}$ is an injective endomorphism of $N(n)$. Thus, $f(n) \in Aut(N(n))$, hence $f(n)(N(n)) = N(n)$. As $M(n)$ is quasi-injective, then there exists $L(n)$ a submodule of $M(n)$ such that $M(n) = f(M(n)) \oplus L(n)$. Thus, we have $0 = f(n)(N) \cap L(n) = N(n) \cap L(n)$, as $N(n)$ is essential, then $L(n) = 0$, hence $M(n) = f(n)(M(n))$, thus $f(n)$ is an epimorphism for all $n \in \mathbb{Z} \Longrightarrow f_*$ is a graded epimorphism complex chain, consequently, M_* is cohopfian.

Proposition 7 *Let M a graded left $A-$module. If M_*/N_* is hopfian for all submodule N of M, then M_* is hopfian.*

Proof

Suppose on the contrary M_* is not hopfian. Then there exists $n \in \mathbb{Z}$ such that $f(n) : M(n) \longrightarrow M(n)$ is an epimorphism which is not an automorphism. Put $N(n) = ker(f)(n)$, thus $N(n) \neq 0$ and $f(n)$ induces an isomorphism $\overline{f}(n) : M(n)/N(n) \longrightarrow M(n)/N(n)$.

If $\pi(n) : M(n)/N(n) \longrightarrow M(n)/N(n)$ denotes the canonical quotient map, then $\pi(n) \circ \overline{f}(n) : M(n)/N(n) \longrightarrow M(n)/N(n)$ is an epimorphism which is not an isomorphism $\forall n \in \mathbb{Z}$, contradicting the hopfian nature of $M(n)/N(n)$.

Proposition 8 *Let M a graded left $A-$module. If all owns submodule N of M is cohopfian, then M_* itself is cohopfian.*

Proof

Suppose on the contrary M_* is not cohopfian. Then there exists $n \in \mathbb{Z}$ such that the injective map $f(n) : M(n) \longrightarrow M(n)$

is not an automorphism. Put $N(n) = Im(f)(n)$, thus $N(n) \subset M(n)$ and $f(n)$ induces an isomorphism $\overline{f}(n) : M(n) \longrightarrow N(n)$ s an injective map which is not an isomorphism, contradicting the cohopfian nature of $N(n)$ for all $n \in \mathbb{Z}$.

Remark 1 *(p) denotes the following property: $\ll$any epimorphism of subcomplex N_* of M_* (where M is an object of $COMP(Gr(A - Mod))$)is an isomorphism $\gg$*

Proposition 9 *Let M a graded left $A-$module. If M_* is hopfian quasi-injective object of $COMP(AGr(A - Mod))$, then M_* owns the property (p).*

Proof

Let N_* be a subcomplex of M_* and $f_* : N_* \longrightarrow M_*$ be an graded epimorphism complex chain. Since M_* is quasi-injective, then for all $n \in \mathbb{Z}$, $M(n)$ is quasi-injective, thus there exists $\tilde{f}(n) \in End(M(n))$ such that $\tilde{f}(n)_{|N(n)} = f(n)$. Or $f(n)$ is surjective, then for all $x \in M(n)$, there exists $y \in N(n)$ such that $x = f(n)(y)$, hence $\tilde{f}(n)$ is surjective, as $M(n)$ is hopfian, thus $\tilde{f}(n) \in Aut(N(n))$ for all $n \in \mathbb{Z}$. We deduce that $f(n)$ is a monomorphism of $N(n)$ into $M(n)$, for all $n \in \mathbb{Z}$. Consequently, f_* is a graded isomorphism of complex chain.

References

Abu-Dawwas, R. (2010). More on crossed product over the support of graded rings. *International Mathematical Forum, 5*(63), 3121-3126.

Abu-Dawwas, R., & Refai, M. (2010). Further results on graded Prime submodules. *International Journal of Algebra, 4*(28), 1413-1419.

Artin, M. (1966). *Commutative rings*, course notes, M.I.T.

Atiyah, M. F., & Macdonald, I. G. (1969). *Introduction to commutative algebra.* Addison - Wesley Publishing Company, University of OXFORD.

Dade, E. C. (1980). Group graded rings and modules. *Math. Z., 174*, 241-262.

Diallo, E. O. (2014). *Thèse de Doctorat unique, Faculté des Sciences et Techniques.* Universit Cheikh Anta DIOP, Dakar.

Diallo, E. O., Maaouia, M. F., & Sanghare, M. (2013). Hopfian Objects, Cohopfian Objects in the category of complexes of left A-Modules. *International Mathematical Forum, 8*(39), 1903-1920. http://dx.doi.org/10.12988/imf.2013.37128

Eelbert, M. P. (1984). Localisation in duo - Ring. *Publ. Math. Debrecen, 31*, 47-52.

Faye, D., Maaouia, M. F., & Sanghare, M. (2016). Localization in a Duo-Ring and Polynomials Algebra. In Gueye C., & Molina, M. (Eds), *Non-Associative Algebra and Operator Theory.* Springer Proceedings in Mathematics and Statistics, Vol 160. Springer, Cham. http://doi.org/10.1007/978-3-319-32902-4_13

Hmainou, A., Kaidi, A., & Sanchez, E. C. (2006). *Generalising Fitting modules and rings*. Journal of Algebra.

Maaouia, M. F. (2011). *These d'etat, Faculté des Sciences et Techniques.* Universit Cheikh Anta DIOP, Dakar.

Maaouia, M. F., & Sanghare, M. (2012). Module de Fractions, Sous-modules $S-$saturée et Foncteur S^{-1}. *International Journal of Algebra*, *6*(16), 775-798.

Maaouia, M. F., & Sanghare, M. (2012). Anneau de valuation non Nécessairement commutatif et duo-anneau de D-edekind. *Global Journal of Pures and Applied Mathematics, 8*(1), 49-63.

Maaouia, M. F., & Sanghare, M. (2009). Localisation dans les duo - anneaux. *Afrika Mathematika, 3*(20), 163-179.

Mendelson, M. (1970). *Graded rings, modules and algebras*. M.I.T.

Nastasescu, C. (1983). Strongly graded rings of finite groups. *Communication in Algebra, 11*(10), 1033-1071.

Nastasescu, C., & Van Oystaeyen, F. (1982). Graded ring theory. *Mathematical Library, 28*, North Holland, Amesterdam.

OULD, C. A. (2016). *Thèse de Doctorat unique,Unit de Formation de la Recherche de Sciences Appliquées et de Technologie.* Universit Gaston Berger, Saint-Louis.

OULD, C. A., Maaouia, M. F., & Sanghare, M. (2018). LOCALISATION DES DUO-ANNEAUX GRADUS ET GRADUATION DES MODULES DE FRACTIONS SUR DES DUO-ANNEAUX GRADUS. *Annales Mathmatiques Africaines, 7*(39), 73-88.

OULD, C. A., Maaouia, M. F., & Sanghare, M. (2016). Factorization of graded modules of Fraction. *International Mathematical Forum, 11*(22), 1067-1088. https://doi.org/10.12988/imf.2016.67100

OULD, C. A., Maaouia, M. F., & Sanghare, M. (2015). Graduation of Module of Fraction on a Graded Domain Ring Not

Necessarily Commutative. *International Journal of Algebra, 9*(10), 457-474. http://dx.doi.org/10.12988/ija.2015.5852

Renault, G. (1975). *Algèbre non commutative*, Gauthier-Villars, Paris - Bruxelles - Montréal .

Copyrights