Determination of Time-Dependent Coefficients in Time-Fractional Diffusion Equations by Variational Iteration Method

Mimi Li[1], Gongsheng Li[1], Zhiyuan Li[1] & Xianzheng Jia[1]

[1] School of Mathematics and Statistics, Shandong University of Technology, Zibo, China

Correspondence: Gongsheng Li, School of Mathematics and Statistics, Shandong University of Technology, Zibo, China. E-mail: ligs@sdut.edu.cn

Abstract

Inverse problems of determining time-dependent coefficients in partial differential equations are difficult to deal with in general cases. The variational iteration method is introduced to determine the time-dependent coefficient in the fractional diffusion equation as well as the solution of the forward problem. By utilizing the additional condition and the property of the fractional derivative, an expression of the unknown is derived by which a nonlinear dynamical differential equation is obtained. The variational iteration method is applied to solve the nonlinear system and the time-dependent coefficient can be reconstructed in a semi-analytical form. Such method can give explicit expression of the solution in the meaning of approximation, or exact solution to the inverse problem in some cases. Several examples are presented to demonstrate feasibility and effectiveness of the proposed method for inverse time-dependent coefficient problems in the fractional diffusion equations.

Keywords: time fractional diffusion equation, inverse problem, time-dependent coefficient, variational iteration method, series of approximate solutions

1. Introduction

The fractional differential equations and their applications have attracted much attention during the last two decades. If the usual integer-order derivative in a classical diffusion equation is replaced with a fractional derivative of any real number in time or space, a time or space fractional diffusion equation is established, which has more and more applications in the fields of physics, mechanics, environmental science and hydrology, etc. On concrete applications of mathematical models and simulation, inverse problems are always encountered with in which we need to identity and determine some unknowns in the model using suitable additional information. As for inverse problems arising in the time/space fractional diffusion equations, we refer to Chen, Liu, Jiang, Turner and Burrage (2016), Cheng, Nakagawa, Yamamoto and Yamazaki (2009), Chi, Li and Jia (2011), Jin and Rundell (2015), Li, Zhang, Jia and Yamamoto (2013), Li, Imanuvilov and Yamamoto(2016), Liu and Yamamoto(2010), Liu, Rundell and Yamamoto (2016), Luchko, Rundell and Yamamoto(2013), Miller and Yamamoto (2013), Sakamoto and Yamamoto (2011), Wei, L. Li and S. Li (2016), Yamamoto and Zhang (2012), and recently see Fan, Liu and Jiang (2018), Kian, Oksanen, Soccorsi and Yamamoto (2018), Ruan, Zhang and Wang (2018), Sun, Li and Jia (2017), Sun, Yan and Wei (2019) and the references therein. It is noted that almost inverse problems in the fractional diffusion equations are concerned with the fractional orders, the source term, or the space-dependent coefficient, it is still difficult for inverse problems of determining time-dependent coefficients in partial differential equations in general cases, especially for fractional diffusion models with time-dependent coefficients.

On the other hand, the inversion methods for solving inverse problems are often proposed based on the minimization for the cost functional and regularization strategies. Such kind of algorithms involve in gradient-type iterations and computations of the forward problem repeatedly, and the computational cost is expansive. From viewpoint of engineering application it is important and convenient to get an approximate solution with an explicit expansion for a mathematical model. The main motivation of this work is to drive an inversion method which is independent upon the gradient computation and can give a fast and accurate approximation for the solution of an inverse time-dependent coefficient problem.

As we know the variational iteration method is a semi-analytical method to get a solution of a linear/nonlinear algebra/differential equation arising from mathematical physics and engineering science, see the original work by He (1997, 2000, 2007), etc. Such variational iteration method can provide us with a possibility to analyze some nonlinear models because of its no specific requirements for the nonlinear operators, see Chang (2016), Nadeem, Li and Ahmad (2019), Tapaswini, Chakraverty and Behera (2015), Zhao, Zhu, Yu and Xu (2010) for example, and it was employed to find the exact solution or approximate solutions of fractional differential equations with effectiveness, see Biazar, Gholamin and

Hosseini (2010), Dehghan and Tatari (2006), Inc (2008), S. Kumar, A. Kumar and Argyros (2017), Wu (2012), Yulita (2009), etc.

By the variational iteration method, we consider the inverse problem of determining the time-dependent diffusion coefficient $a(t)$ in the time-fractional diffusion equation

$$\partial_t^\alpha u - a(t)\triangle u = f(x,t), \tag{1.1}$$

and the inverse problem of determining the average flow velocity $v(t)$ in the mobile/immobile time-fractional diffusion equation

$$u_t + r\partial_t^\alpha u - a(t)\triangle u + v(t)\nabla u = f(x,t), \tag{1.2}$$

respectively, where $\partial_t^\alpha u$ denotes the α-order Caputo's fractional derivative of $u = u(x,t)$ at $t > 0$ for $0 < \alpha < 1$, see the definition of fractional calculus by Kilbas, Srivastara and Trujillo (2006), Podlubny (1999).

For the inverse problem of determining the diffusion coefficient $a(t)$ in Eq.(1.1) using the boundary measurement, a uniqueness and a fixed point iteration algorithm are presented by Zhang (2016). As for Eq.(1.2), there are a few of researches on numerical methods for the forward problem, see Jiang and Liu (2017), Liu, Meerschaert, McGough, Zhuang and Liu (2013), Q. Liu, F. Liu, Turner, Anh and Gu (2014), Zhang, Liu, Phanikumar and Meerschaert (2013) for example, however, the related inverse problems seem to be paid less attention in the literatures.

This paper will give numerical determination for the above inverse problems of determining time-dependent coefficients using the variational iteration method. Although the variational iteration method has been utilized to solve nonlinear problems not only for classical integer-order diffusion equations but also for fractional diffusion equations, there are few researches on inverse coefficient problems by the method. One purpose of this work is try to apply the variational iteration method to solve some inverse coefficient problems in the fractional diffusion equations, especially for determination of a time-dependent coefficient which is difficult to cope with in general cases. We will perform the approximate inversions for the considered inverse time-dependent problems using the variational iteration method, and the exact solution can be obtained in some cases. The rest of the paper is organized as follows.

In Section 2, the variational iteration method for solving a forward problem and an inverse coefficient problem in a differential equation is introduced. In Section 3 by using the properties of fractional calculus, the variational iteration method is applied to solve the inverse problem of determining a time-dependent diffusion coefficient in the time-fractional diffusion equation based on the additional data at one interior-point, and in Section 4 a time-dependent flow velocity in the mobile/immobile time fractional advection-dispersion equation is recovered also by the variational iteration method. Finally concluding remarks are given in Section 5.

2. The Variational Iteration Method

2.1 The Variational Iteration Method for Forward Problem

Consider a second-order partial differential equation in an abstract form

$$F(u, u_t, u_{tt}, u_x, u_{xx}) = 0, \tag{2.1}$$

with specified initial boundary conditions. The variational iteration method changes the equation to a correction functional in $t-$direction in the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s) F(u_n, u_{ns}, u_{nss}, \tilde{u}_{nx}, \tilde{u}_{nxx}) ds, \tag{2.2}$$

where λ is a general Lagrange multiplier which is identified optimally by variational theory, u_n is the n-th approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. Let the variation $\delta u_{n+1} = 0$ and an optimal value of λ is solved, and then an iteration formula for solving the equation (2.1) is constructed by which an approximate analytical solution can be obtained.

Rewrite (2.2) with

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t \lambda(t,s) F(u_n, u_{ns}, u_{nss}, u_{nx}, u_{nxx}) ds. \tag{2.3}$$

Roughly speaking, the optimal value of λ must be chosen to minimize the residual such that the limitation of u_n exists as $n \to \infty$. On the other hand, suppose that the function F is continuous and the forward problem has unique solution. Then $u^* = \lim_{n\to\infty} u_n$ is just the solution satisfying $F(u^*, u_t^*, u_{tt}^*, u_x^*, u_{xx}^*) = 0$.

Actually, by (2.2) define a mapping

$$A[u] = u(x,t) + \int_0^t \lambda(t,s) F(u, u_s, u_{ss}, u_x, u_{xx}) ds. \tag{2.4}$$

A sufficient condition for the convergence is the operator A is a contraction, and its fixed point is also the solution of the equation (2.2).

2.2 The Variational Iteration Method for Inverse Coefficient Problem

Consider a general differential equation involving an unknown coefficient function $p = p(t)$ given as

$$F(u, u_t, u_x, u_{xx}, p) = f(x, t), \ x \in \Omega, \ 0 < t < T, \tag{2.5}$$

with specified initial boundary conditions. Suppose that we have some additional information given at one point $x^* \in \bar{\Omega}$

$$u(x^*, t) = E(t), x^* \in \bar{\Omega}, \tag{2.6}$$

We are to determine the time-dependent coefficient $p = p(t)$ in (2.5) with the overposed condition (2.6) using the variational iteration method.

Firstly we have to deduce an expression of the coefficient $p = p(t)$ by the forward problem and the additional condition, denoted by

$$p(t) = G(u, u_t, u_x, u_{xx}, f, E), \tag{2.7}$$

which is a key step for the application of the variational iteration method in the inverse coefficient problem.

Next, substituting (2.7) into the equation (2.5), there has

$$F(u, u_t, u_x, u_{xx}, G(u, u_t, u_x, u_{xx}, f, E)) = 0. \tag{2.8}$$

which is always a nonlinear equation on $u = u(x, t)$.

Utilizing the variational iteration method to the equation (2.8) with suitable initial iteration, an approximate solution, or the exact solution of $u = u(x, t)$ in some cases can be obtained, and the coefficient $p(t)$ is determined by (2.7).

3. Inversion for Diffusion Coefficient in Time-FDE

Consider the time fractional diffusion equation with time-dependent diffusion coefficient

$$\partial_t^\alpha u - a(t)u_{xx} = f(x, t), \ 0 < x < 1, \ 0 < t < T, \tag{3.1}$$

with specified initial boundary conditions, where ∂_t^α is Caputo's derivative for $\alpha \in (0, 1)$.

The inversion problem is determine the diffusion coefficient $a = a(t)$ given the interior measurement at a fixed point $x^* \in (0, 1)$.

$$u(x^*, t) = E(t), \ 0 < t \le T. \tag{3.2}$$

3.1 Transformation of the Inverse Coefficient Problem

For the aim of utilizing the variational iteration method to solve the inverse coefficient problem we need some preliminary knowledge of the fractional calculus, see Kilbas, Srivastara and Trujillo(2006), Podlubny(1999) for example.

The α-order Riemann-Liouville integral operator I^α, defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t - \tau)^{1-\alpha}} d\tau. \tag{3.3}$$

The α-order ($0 < \alpha < 1$) Riemann-Liouville derivative, $^{RL}D_t^\alpha f$, defined by

$$^{RL}D_t^\alpha f(t) = \frac{d}{dt}(I^{1-\alpha} f(t)). \tag{3.4}$$

The α-order ($0 < \alpha < 1$) Caputo's derivative, $\partial_t^\alpha f$, defined by

$$\partial_t^\alpha f(t) = I^{1-\alpha}(\frac{df}{dt}). \tag{3.5}$$

There hold for $0 < \alpha < 1$

$$\partial_t^\alpha I^\alpha f(t) = f(t),$$

and

$$I^\alpha \partial_t^\alpha f(t) = f(t) - f(0).$$

Now consider the equation (3.1) again. Since

$$I^{\alpha}\partial_t^{\alpha}u(x,t) = u(x,t) - u(x,0), \tag{3.6}$$

By using the additional information we have

$$E(t) - u(x^*,0) = I^{\alpha}[a(t)u_{xx}(x^*,t) + f(x^*,t)]. \tag{3.7}$$

Then it holds

$$
\begin{aligned}
\partial_t^{\alpha}E(t) &= \partial_t^{\alpha}[E(t) - u(x^*,0)] \\
&= \partial_t^{\alpha}\{I^{\alpha}[a(t)u_{xx}(x^*,t) + f(x^*,t)]\} \\
&= a(t)u_{xx}(x^*,t) + f(x^*,t).
\end{aligned} \tag{3.8}
$$

Suppose that $u_{xx}(x^*,t) \neq 0$, we get

$$a(t) = \frac{\partial_t^{\alpha}E(t) - f(x^*,t)}{u_{xx}(x^*,t)}. \tag{3.9}$$

On the other hand, by taking $1 - \alpha$-order Riemann-Liouville differential operator on equation (3.1) there holds

$$^{RL}D_t^{1-\alpha}\,\partial_t^{\alpha}u(x,t) = {}^{RL}D_t^{1-\alpha}\,[a(t)u_{xx} + f(x,t)]. \tag{3.10}$$

Noting that $^{RL}D_t^{1-\alpha} = \frac{\partial}{\partial t}I^{\alpha}$, we have

$$\frac{\partial u}{\partial t} = {}^{RL}D_t^{1-\alpha}\,[a(t)u_{xx} + f(x,t)]. \tag{3.11}$$

Substituting (3.9) into (3.11) we have

$$\frac{\partial u}{\partial t} = {}^{RL}D_t^{1-\alpha}\,[\frac{\partial_t^{\alpha}E(t) - f(x^*,t)}{u_{xx}(x^*,t)}u_{xx}(x,t) + f(x,t)]. \tag{3.12}$$

3.2 Inversion for the Diffusion Coefficient

Now we are ready to employ the variational iteration method to solve the inverse coefficient problem. By the VIM, define a corrected functional along t-direction given as

$$
\begin{aligned}
u_{n+1}(x,t) = \ & u_n(x,t) + \int_0^t \lambda(\tau)\{\frac{\partial u_n}{\partial \tau}(x,\tau) \\
& - {}^{RL}D_{\tau}^{1-\alpha}\,[\frac{\partial_{\tau}^{\alpha}E(\tau) - f(x^*,\tau)}{u_{nxx}(x^*,\tau)}\tilde{u}_{nxx}(x,\tau) + f(x,\tau)]\}d\tau,
\end{aligned} \tag{3.13}
$$

where $\lambda(\tau)$ is the general Lagrange multiplier, and $\tilde{u}_n$ is a confined variational which satisfies $\delta\tilde{u}_n = 0$ as discussed by He (2007).

Noting that $\delta\tilde{u}_n = 0$, we take the first-variation on u_n for (3.13), and let $\delta u_{n+1} = 0$, there holds

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda(\tau)\frac{\partial(\delta u_n)}{\partial \tau}d\tau = 0. \tag{3.14}$$

By integration by parts, and thanks to $\delta u_n(x,0) = 0$ there must have

$$
\begin{cases}
\lambda'(\tau) = 0, \\
1 + \lambda(t) = 0,
\end{cases}
$$

from which we get $\lambda(\tau) = -1$ for $\tau \in [0,t]$. Thus by (3.13) we have the iteration formula

$$
\begin{aligned}
u_{n+1}(x,t) = \ & u_n(x,t) - \int_0^t \{\frac{\partial u_n}{\partial \tau}(x,\tau) \\
& - {}^{RL}D_{\tau}^{1-\alpha}\,[\frac{\partial_{\tau}^{\alpha}E(\tau) - f(x^*,\tau)}{u_{nxx}(x^*,\tau)}u_{nxx}(x,\tau) + f(x,\tau)]\}d\tau,
\end{aligned} \tag{3.15}
$$

for $n = 0, 1, \cdots$. With the initial value function $u_0(x)$ as the initial iteration, we can get a series of $u_n(x,t)$ by (3.15) which are the approximate solutions of the forward problem, and then the diffusion coefficient $a(t)$ is computed by (3.9).

3.2.1 Example 3.1

Let the exact solution of the equation (3.1) be $u(x,t) = x^3 + t^2$, and the diffusion coefficient be $a(t) = 1 + t$, and the source be

$$f(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 6x(1+t).$$

Suppose that the additional condition is given at $x = \frac{1}{2}$:

$$E(t) = u(\frac{1}{2}, t) = \frac{1}{8} + t^2. \tag{3.16}$$

By the variational iteration formula (3.15), we have

$$
\begin{aligned}
u_{n+1} &= u_n - \int_0^t \{\frac{\partial u_n}{\partial \tau} \\
&- {}^{RL}D_\tau^{1-\alpha}[\frac{\partial_\tau^\alpha E(\tau) - f(x^*, \tau)}{u_{xx}(x^*, \tau)} u_{xx} + f(x, \tau)]\} d\tau \\
&= u_0(x) + \int_0^t {}^{RL}D_\tau^{1-\alpha}[\frac{\partial_\tau^\alpha E(\tau) - f(x^*, \tau)}{u_{xx}(x^*, \tau)} u_{xx} + f(x, \tau)] d\tau
\end{aligned}
\tag{3.17}
$$

Choosing the initial iteration $u_0 = u(x, 0) = x^3$, we get by computation

$$
\begin{aligned}
u_1 &= u_0 + \int_0^t {}^{RL}D_\tau^{1-\alpha}[\frac{2}{\Gamma(3-\alpha)}\tau^{2-\alpha}] d\tau \\
&= u_0 + \int_0^t {}^{RL}D_\tau^{1-\alpha}\partial_\tau^\alpha(\tau^2) d\tau \\
&= x^3 + t^2,
\end{aligned}
\tag{3.18}
$$

which is the exact solution of the forward problem. Then the inversion coefficient is worked out by (3.9) given as

$$
\begin{aligned}
a(t) &= \frac{\partial_t^\alpha E(t) - f(\frac{1}{2}, t)}{u_{xx}(\frac{1}{2}, t)} \\
&= \frac{\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - [\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 3(t+1)]}{3} \\
&= t + 1,
\end{aligned}
\tag{3.19}
$$

which is just the exact the diffusion coefficient.

3.2.2 Example 3.2

Let the exact solution be $u(x, t) = \exp(-t)(x^2 + 1)$, and $a(t) = \exp(t)$, and the additional condition also given at $x = \frac{1}{2}$

$$E(t) = u(\frac{1}{2}, t) = \frac{5}{4}\exp(-t). \tag{3.20}$$

By the variational iteration method as utilized in the above, we have by setting $u_0 = u(x, 0) = x^2 + 1$

$$
\begin{aligned}
u_1 &= u_0 + \int_0^t {}^{RL}D_\tau^{1-\alpha}[\partial_\tau^\alpha E(\tau) - f(\frac{1}{2}, \tau) + f(x, \tau)] d\tau \\
&= u_0 + \int_0^t {}^{RL}D_\tau^{1-\alpha}\partial_\tau^\alpha[E(\tau) + (x^2 + 1)e^{-\tau} - \frac{5}{4}e^{-\tau}] d\tau \\
&= (x^2 + 1)\exp(-t),
\end{aligned}
\tag{3.21}
$$

which is the exact solution of the forward problem, and the inversion diffusion coefficient is computed also by (3.9)

$$
\begin{aligned}
a(t) &= \frac{\partial_t^\alpha E(t) - f(\frac{1}{2}, t)}{u_{xx}(\frac{1}{2}, t)} \\
&= \frac{\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - [\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2]}{2\exp(-t)} \\
&= \exp(t).
\end{aligned}
\tag{3.22}
$$

4. Inversion for Average Velocity in Mobile/Immobile Fractional Transport Model

Consider the time fractional diffusion equation in a bounded domain $(0, l)$ for $l > 0$

$$u_t + r\partial_t^\alpha u - a(t)u_{xx} + v(t)u_x = f(x, t),\ 0 < x < l,\ 0 < t < T, \tag{4.1}$$

with the initial value condition:

$$u(x, 0) = u_0(x),\ 0 \le x \le l, \tag{4.2}$$

and the specified Neumann boundary conditions.

Suppose that the average flow velocity $v(t)$ in the model is unknown, we are to determine it using the out-flow measurements at $x = l$:

$$u(l, t) = E(t),\ 0 < t \le T. \tag{4.3}$$

In addition, in order to utilizing the variational iteration method to solve the inverse coefficient problem, we need the conditions that the right-boundary function $u_x(l, t) \ne 0$ for $t \in [0, T]$, and the coefficient $r > 0$ in (4.1).

By the equation (4.1) we have

$$\partial_t^\alpha u = \frac{1}{r}[f(x,t) - u_t + a(t)u_{xx} - v(t)u_x]. \tag{4.4}$$

Similarly as done in the subsection 3.1, noting the condition $u_x(l,t) \neq 0$, by utilizing the properties of the fractional calculus and the additional condition, we get an expression of the average flow velocity given by

$$v(t) = \frac{f(l,t) - E'(t) + a(t)u_{xx}(l,t) - r\partial_t^\alpha E(t)}{u_x(l,t)}. \tag{4.5}$$

Substituting (4.5) into equation (4.1), we have

$$
\begin{aligned}
& u_t(x,t) + r\partial_t^\alpha u(x,t) - a(t)u_{xx}(x,t) \\
+ \ & \frac{f(l,t) - E'(t) + a(t)u_{xx}(l,t) - r\partial_t^\alpha E(t)}{u_x(l,t)}u_x(x,t) - f(x,t) \\
= \ & 0.
\end{aligned}
\tag{4.6}
$$

Now we are ready to employ the variational iteration method to solve the inverse coefficient problem. By utilizing the VIM to the equation (4.6), there is

$$
\begin{aligned}
u_{n+1}(x,t) \ = \ & u_n(x,t) - \int_0^t [u_{n\tau}(x,\tau) + r\partial_\tau^\alpha u(x,\tau) - a(\tau)u_{nxx}(x,\tau) \\
& + \frac{f(l,\tau) - E'(\tau) + a(\tau)u_{nxx}(l,\tau) - r\partial_\tau^\alpha E(\tau)}{u_{nx}(l,\tau)}u_{nx}(x,\tau) - f(x,\tau)]d\tau,
\end{aligned}
\tag{4.7}
$$

for $n = 0, 1, \cdots$. With the initial value function $u_0(x)$ as the initial iteration, we can get a series of $u_n(x,t)$ by (4.7) which are the approximate solutions of the forward problem, and then the average flow velocity $v(t)$ is computed by (4.5).

4.1 Example 4.1

Let $l = 1$ and $T = 1$, and the exact solution of the forward problem be

$$u(x,t) = \exp(-x)(t^2 + 1), \tag{4.8}$$

and the exact flow velocity be

$$v(t) = \frac{1}{1 + t^2}. \tag{4.9}$$

In addition, the time-scale coefficient $r = \frac{1}{2}$, the diffusion coefficient $a(t) = 1 + \frac{t}{1+t^2}$, and the additional information $E(t) = \exp(-1)(t^2 + 1)$, and the source term

$$f(x,t) = \exp(-x)(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - t^2 + t - 2). \tag{4.10}$$

We are to reconstruct the solution $u(x,t)$ and the flow velocity $v(t)$ by the variational iteration method.

Noting that

$$E'(t) = 2t\exp(-1), \partial_t^\alpha E(t) = \exp(-1)\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)},$$

we have by the variational iteration formula (4.7) by choosing $u_0 = \exp(-x)$

$$
\begin{aligned}
u_1 \ = \ & u_0 - \int_0^t [-a(\tau)\exp(-x) + \frac{f(1,\tau) - E'(\tau) + a(\tau)\exp(-1) - r\partial_\tau^\alpha E(\tau)}{\exp(-1)}\exp(-x) - f(x,\tau)]d\tau \\
= \ & u_0 + \int_0^t [(\tau^2 + \tau + 2)\exp(-x) + f(x,\tau)]d\tau \\
= \ & \exp(-x)[(t^2 + 1) + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)}].
\end{aligned}
\tag{4.11}
$$

Consequently by the iteration process follows that

$$u_2 = \exp(-x)[(t^2 + 1) - \frac{t^{4-2\alpha}}{2\Gamma(5-2\alpha)}];$$

$$u_3 = \exp(-x)[(t^2 + 1) + \frac{t^{5-3\alpha}}{2^2\Gamma(6-3\alpha)}];$$

$$\cdots\cdots;$$

and

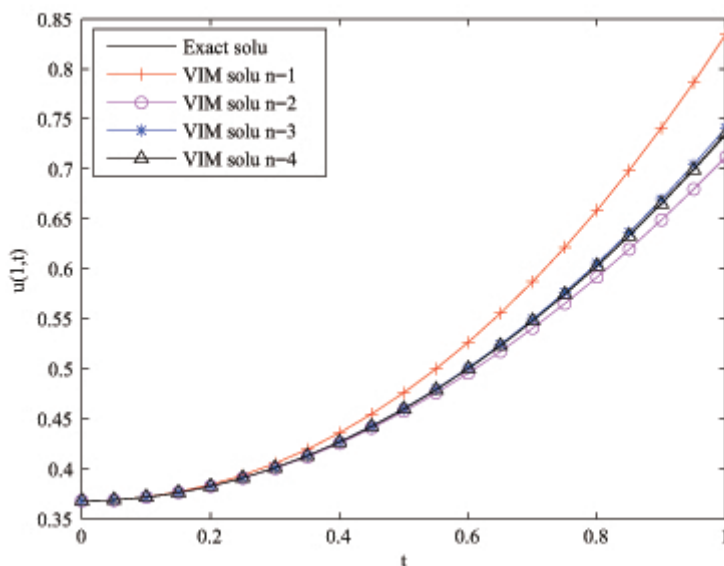$$u_n = \exp(-x)[(t^2 + 1) + (-1)^{n-1}\frac{t^{n(1-\alpha)+2}}{2^{n-1}\Gamma(n(1-\alpha)+3)}]. \tag{4.12}$$

Figure 1. Exact solution and approximate solutions by VIM in Ex.4.1

For any given $\alpha \in (0, 1)$ and $T < \infty$, there holds

$$\lim_{n \to \infty} \frac{T^{n\alpha}}{2^n \Gamma(n\alpha)} = 0. \tag{4.13}$$

Therefore we get

$$u(x, t) = \lim_{n \to \infty} u_n(x, t) = \exp(-x)(t^2 + 1), \tag{4.14}$$

which is the exact solution of the forward problem.

Furthermore, we have by (4.5)

$$
\begin{aligned}
v_n(t) &= \frac{f(1,t) - E'(t) + a(t)\frac{\partial^2 u_n}{\partial x^2}(1,t) - r\partial_t^\alpha E(t)}{\frac{\partial u_n}{\partial x}(1,t)} \\
&= \frac{f(1,t) - 2t \exp(-1) + a(t)u_n(1,t) - \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\exp(-1)}{-u_n(1,t)}.
\end{aligned}
\tag{4.15}
$$

Utilizing (4.12), (4.10) and noting $\frac{t^2+t+1}{t^2+1} \leq \frac{3}{2}$ and (4.13), we finally get

$$v_n(t) = \frac{1 + \frac{t^2+t+1}{t^2+1}\frac{(-1)^n t^{n(1-\alpha)+2}}{2^{n-1}\Gamma(n(1-\alpha)+3)}}{t^2 + 1 + \frac{(-1)^{n-1}t^{n(1-\alpha)+2}}{2^{n-1}\Gamma(n(1-\alpha)+3)}} \to \frac{1}{t^2 + 1}(n \to \infty), \tag{4.16}$$

which is just the exact flow velocity. Moreover, the approximate solutions of $u_n(x, t)$ $(n = 1, 2, 3, 4)$ by the variational iteration method, and the exact solution $u(x, t)$ for $\alpha = 0.4$ and $x = 1$ are plotted in Fig.1, respectively, where VIM denotes the variational iteration method.

4.2 Example 4.2

Also let $l = 1$ and $T = 1$, and the exact solution in this example be

$$u(x, t) = x^2 + (t + 1)^2, \tag{4.17}$$

and the exact velocity be $v(t) = \frac{t^2}{2} + 0.01$. The model coefficients $r = \frac{1}{2}$ and $a(t) = 0.01$, and the source term

$$f(x, t) = 1.98 + \frac{t^{2-\alpha}}{\Gamma(3 - \alpha)} + \frac{t^{1-\alpha}}{\Gamma(2 - \alpha)} + 2t + t^2 x + 0.02x, \tag{4.18}$$

and we are to determine the average flow velocity $v(t)$ using the additional information $E(t) = 1 + (t + 1)^2$.
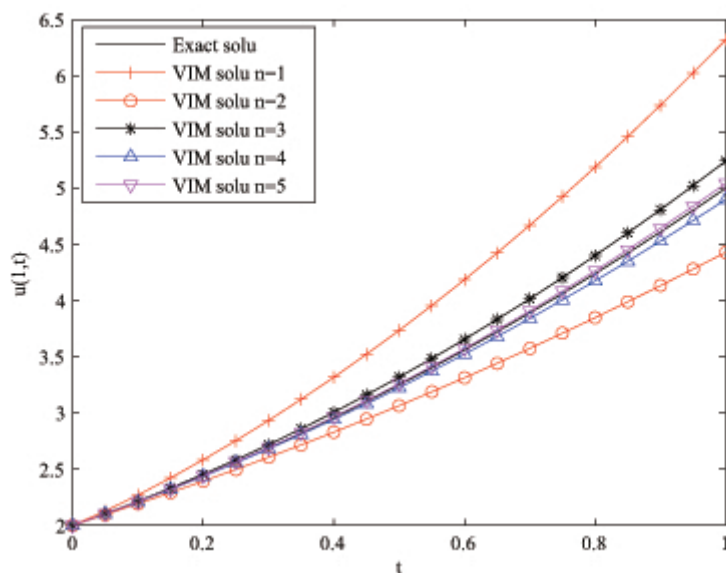
Figure 2. Exact and approximate solutions by VIM in Ex.4.2

Similarly as done in Ex. 4.1, we get a series of the approximated solutions by choosing the initial iteration $u_0 = x^2 + 1$ given as follows:

$$u_1 = x^2 + (t+1)^2 + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)};$$

$$u_2 = x^2 + (t+1)^2 - \frac{t^{3-2\alpha}}{2\Gamma(4-2\alpha)} - \frac{t^{4-2\alpha}}{2\Gamma(5-2\alpha)};$$

$$\cdots\cdots;$$

$$u_n = x^2 + (t+1)^2 + \frac{(-1)^{n-1}t^{n(1-\alpha)+1}}{2^{n-1}\Gamma(n(1-\alpha)+2)} + \frac{(-1)^{n-1}t^{n(1-\alpha)+2}}{2^{n-1}\Gamma(n(1-\alpha)+3)}. \tag{4.19}$$

It is obvious that there holds

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = x^2 + (t+1)^2. \tag{4.20}$$

Moreover, we have by (4.5) and (4.20)

$$
\begin{aligned}
v(t) &= \frac{f(1,t) - E'(t) + a(t)u_{xx}(1,t) - r\partial_t^\alpha E(t)}{u_x(1,t)} \\
&= \frac{f(1,t) - 2(t+1) + 0.01 \times 2 - \frac{1}{2}[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)}]}{2} \\
&= \frac{t^2}{2} + 0.01,
\end{aligned} \tag{4.21}
$$

which is just the exact flow velocity. As done in Ex.4.1, the approximate solutions of $u_n(x,t)(n = 1, \cdots, 5)$ by VIM, and the exact solution $u(x,t)$ for $\alpha = 0.8$ and $x = 1$ are plotted in Fig.2, respectively.

It can be seen from Figures 1-2 that the approximate solutions by the VIM are convergent to the exact solutions only by several iteration steps, which demonstrates a fast convergence of the VIM algorithm.

5. Conclusion

We deal with determination problems of time-dependent coefficients in the time fractional diffusion equations using the variational iteration method. The variational iteration method is well-studied in giving semi-analytical approximate solutions for linear/nonlinear algebraic/differential equations, it is also effective for solving some nonlinear inverse time-dependent coefficient problems in fractional differential equations. We will consider the convergence of the method and give more applications in other inverse problems.

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