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Three Positive Solutions for Boundary Value Problem of Fractional q-Differential Equation

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Abstract

In this paper, we study the boundary value problem of a Riemann-Liouville fractional q-difference equation. By applying the Leggett-Williams fixed point theorem and the properties of the Green's function, three positive solutions are obtained.

Keywords: fractional q-integral of the Riemann-Liouville, fractional q-derivative of the Riemann-Liouville, Green's function, fixed point theorem, fractional q-difference equation, positive solutions

MSC: 39A12; 26A33

1. Introduction

The q-difference calculus was firstly developed by (Jackson, 1908) and (Jackson, 1910). The fractional q-difference calculus had its origin in the works by (Al-Salam, 1966) and (Agarwal, 1996). With the wide application in various fields, in recent years, the research on the qualitative properties of solutions of fractional differential equations has attracted great interest. Among them, there have been many results about the existence of solutions of fractional differential equations (see (Bai & Lü, 2005)-(Jleli & Samet, 2016) and references therein). In those works, the authors mainly apply the following methods: the monotone iterative method, the upper and lower solutions method, Krasnosel'skiĭ and Schauder fixed point theorem, Leggett-Williams fixed point theorem.

(Ferreira, 2010) studied the existence of nontrivial solutions for the following nonlinear fractional q-difference equation

$$(D_q^{\alpha}y)(x) = -f(x, y(x)), \quad x \in (0, 1), \ \ q \in (0, 1), \quad 1 < \alpha \le 2,$$

subjected to the boundary conditions

$$y(0) = 0, \quad y(1) = 0,$$

where $f : [0, 1]\times\mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. (Ferreira, 2011) went on studying the existence of positive solutions for the following nonlinear fractional q-difference equation

$$(D_q^{\alpha}y)(x) = -f(x, y(x)), \quad x \in (0, 1), \ \ q \in (0, 1), \ \ 2 < \alpha \le 3,$$

subjected to the boundary conditions

$$y(0) = D_q y(0) = 0, \quad D_q y(1) = \beta \ge 0,$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function.

By constructing a special cone and using Guo-Krasnosel'skiĭ fixed point theorem, (Li, Han, & Sun, 2013) investigate the existence of positive solutions for the following boundary value problem of nonlinear fractional q-difference equation with parameter

$$(D_q^{\alpha}u)(x) + \lambda f(u(x)) = 0, \quad x \in (0, 1),$$

subjected to the boundary conditions

$$u(0) = D_q u(0) = D_q u(1) = 0,$$

where $q \in (0,1),\ 2 < \alpha < 3,\ f : C((0,1),(0,\infty))$.

(Zhai & Ren, 2017) obtained the existence of positive and negative solutions for a fractional q-difference equation

$$(D_q^\alpha u)(t) + \lambda f(t,u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = D_q u(0) = D_q u(1) = 0, \tag{1.2}$$

where $0 < q < 1,\ 2 < \alpha < 3, f \in C([0,1] \times [0,+\infty)) \to [0,+\infty)$.

(Kang, Chen, Li, Cui, & Ma, 2019) gained the existence of three positive solutions for a Riemann-Liouville fractional q-difference equation boundary value problem (1.1)-(1.2).

(Jleli & Samet, 2016) studied the following fractional q-difference equation boundary value problem

$$\begin{aligned} &({}_aD_q^\alpha u)(t) + \varphi(t)u(t) = 0, \quad q \in [0,1),\ \ t \in (a,b),\ \ 1 < \alpha \le 2, \\ &u(a) = u(b) = 0, \end{aligned}$$

where ${}_aD_q^\alpha$ denotes the Riemann-Liouville fractional q-derivative of order α and $\varphi : [a,b] \to \mathbb{R}$ is a continuous function.

Inspired by papers (Kang et al., 2019), (Zhao, Sun, Han, & Li, 2011), (Zhang, 2016) and (Tariboon & Ntouyas, 2014), we study the following nonlinear boundary value problem for a fractional q-differential equation:

$$({}_aD_q^\alpha x)(t) + f(t,x(t)) = 0, \quad 0 < q < 1,\ \ t \in (a,b),\ \ a \ge 0,\ \ 2 < \alpha < 3, \tag{1.3}$$

$$x(a) = ({}_aD_q^\alpha x)(a) = ({}_aD_q^\alpha x)(b) = 0, \tag{1.4}$$

where ${}_aD_q^\alpha$ denotes the Riemann-Liouville fractional q-derivative of order α, $f \in C([a,b] \times [0,+\infty)) \to [0,+\infty)$.

2. Preliminaries

For the convenience the reader, we give some background materials from fractional q-calculus theory.

Let $q \in (0,1)$ and $a \in \mathbb{R}$. For $s \in \mathbb{R}$, we define

$$[s]_q = \frac{1-q^s}{1-q}.$$

The q-analogue of the power function $(s-t)^n$ with $n \in \mathbb{N}$ is

$$(s-t)_a^{(0)} = 1,\ \ (s-t)_a^{(n)} = \prod_{i=0}^{n-1}((s-a)-(t-a)q^i),\ \ n \in \mathbb{N},\ \ s,t \in \mathbb{R}.$$

If $\alpha \in \mathbb{R}$, then

$$(s-t)_a^{(\alpha)} = (s-a)^\alpha \prod_{i=0}^{\infty} \frac{(s-a)-(t-a)q^i}{(s-a)-(t-a)q^{\alpha+i}}.$$

In addition, from (Jleli & Samet, 2016), we have

$$(s-t)_a^{(\alpha)} = (s-a)^\alpha \left(1 - \frac{t-a}{s-a}\right)_0^{(\alpha)}, \quad s \ne a. \tag{2.1}$$

The q-gamma function is defined by

$$\Gamma_q(s) = \frac{(1-q)_0^{(s-1)}}{(1-q)^{s-1}}, \quad s \in \mathbb{R}\backslash\{0,-1,-2,\cdots\},$$

and satisfies $\Gamma_q(s+1) = [s]_q\Gamma_q(s)$.

Definition 2.1 Let $f : [a,b] \to \mathbb{R}$. The fractional q-integral of Riemann-Liouville type is given by

$$\begin{aligned} &({}_aI_q^0 f)(t) = f(t), \quad a \le t \le b, \\ &({}_aI_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-(qs-(1-q)a))_a^{(\alpha-1)} f(s)\,{}_ad_qs,\ \ \alpha > 0,\ \ a \le t \le b. \end{aligned}$$

The fractional q-derivative of Riemann-Liouville type is defined by

$$({}_aD_q^0 f)(t) = f(t), \quad a \le t \le b,$$
$$({}_aD_q^\alpha f)(t) = ({}_aD_q^\beta {}_aI_q^{\beta-\alpha} f)(t), \ \ \alpha > 0, \ \ a \le t \le b,$$

where β is the smallest integer greater than or equal to α.

Lemma 2.2 For any $s, t \in [a, b]$, the following formulas hold:

$$ {}_t({}_aD_q(t-s)_a^{(\alpha)}) = [\alpha]_q(t-s)_a^{(\alpha-1)},$$
$$ {}_s({}_aD_q(t-s)_a^{(\alpha)}) = -[\alpha]_q(t-(qs+(1-q)a))_a^{(\alpha-1)},$$

where ${}_i({}_aD_q)$ denotes the q-derivative with respect to the variable i.

Lemma 2.3 Let $f : [a, b] \to \mathbb{R}$ be a given function, $q \in (0, 1)$, we get some results as follows:

(i) If $({}_aD_q f)(t) \ge 0$, for $a < t \le b$, then f is an increasing function.

(ii) If $({}_aD_q f)(t) \le 0$, for $a < t \le b$, then f is a decreasing function.

Lemma 2.4 If f, g are q-integral on $[a, b]$, $f(s) \le g(s)$ for all $s \in [a, b]$, then

(i) $\int_a^b f(s)_a d_q s \le \int_a^b g(s)_a d_q s$,

(ii) $\left|\int_a^b f(s)_a d_q s\right| \le \int_a^b |f(s)|_a d_q s$.

From (Tariboon, Ntouyas, & Agarwal, 2015), let $f : [a, b] \to \mathbb{R}$, p be a positive integer, we obtain the following compound operation rule for q-integral and q-derivative:

$$ {}_aD_q^\alpha({}_aI_q^\alpha f)(t) = f(t), \quad \alpha > 0, \ \ a \le t \le b, \tag{2.2}$$
$$ {}_aI_q^\alpha {}_aI_q^\beta f(t) = {}_aI_q^{\alpha+\beta} f(t), \quad \alpha,\beta > 0, \ \ a \le t \le b, \tag{2.3}$$
$$ {}_aI_q^\alpha {}_aD_q^p f(t) = {}_aD_q^p {}_aI_q^\alpha f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} {}_aD_q^k f(a), \ \ a \le t \le b, \ \ \alpha > p-1. \tag{2.4}$$

3. Main Results

Lemma 3.1 If $x \in C[a, b]$ is a solution of boundary value problem (1.3)-(1.4), then x satisfies

$$x(t) = \int_a^b G(t, qs + (1-q)a) f(s, x(s))_a d_q s, \quad t \in [a, b],$$

where

$$G(t, qs+(1-q)a) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}}(t-a)^{\alpha-1} - (t-(qs-(1-q)a))_a^{(\alpha-1)}, & a \le qs+(1-q)a \le t \le b, \\ \frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}}(t-a)^{\alpha-1}, & a \le t \le qs+(1-q)a \le b, \end{cases}$$

Proof. By the definition of the fractional q-derivative, we have

$$ {}_aD_q^3 {}_aI_q^{3-\alpha} x(t) = -f(t, x(t)), \quad a < t < b,$$

then

$$ {}_aI_q^\alpha {}_aD_q^3 {}_aI_q^{3-\alpha} x(t) = -{}_aI_q^\alpha f(t, x(t)), \quad a < t < b.$$

Using (2.2), (2.3) and (2.4) with $p = 3$, we get

$$\begin{aligned} x(t) &= c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3} \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-(qs-(1-q)a))_a^{(\alpha-1)} f(s, x(s))_a d_q s, \end{aligned} \tag{3.1}$$

for some constants $c_1, c_2, c_3 \in \mathbb{R}$. Using the boundary value condition $x(a) = 0$, we must set $c_3 = 0$. Applying the Riemann-Liouville fractional q-derivative of order α to both sides of (3.1), we obtain

$$\begin{aligned}({}_aD_q^\alpha x)(t) &= c_1[\alpha-1]_q(t-a)^{\alpha-2} + c_2[\alpha-2]_q(t-a)^{\alpha-3} \\ &\quad - \frac{1}{\Gamma_q(\alpha)}\int_a^t [\alpha-1]_q(t-(qs-(1-q)a))_a^{(\alpha-2)} f(s,x(s))_a d_q s.\end{aligned}$$

By using the boundary value condition $({}_aD_q^\alpha x)(a) = 0$, we must set $c_2 = 0$. Then the boundary condition given by (1.4) yields

$$c_1[\alpha-1]_q(b-a)^{\alpha-2} - \frac{1}{\Gamma_q(\alpha)}\int_a^b [\alpha-1]_q(b-(qs-(1-q)a))_a^{(\alpha-2)} f(s,x(s))_a d_q s = 0,$$

so

$$c_1 = \frac{1}{(b-a)^{\alpha-2}\Gamma_q(\alpha)}\int_a^b (b-(qs-(1-q)a))_a^{(\alpha-2)} f(s,x(s))_a d_q s.$$

Thus, we have

$$\begin{aligned}x(t) &= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}\Gamma_q(\alpha)}\int_a^b (b-(qs-(1-q)a))_a^{(\alpha-2)} f(s,x(s))_a d_q s \\ &\quad - \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-(qs-(1-q)a))_a^{(\alpha-1)} f(s,x(s))_a d_q s \\ &= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}\Gamma_q(\alpha)}\int_a^t (b-(qs-(1-q)a))_a^{(\alpha-2)} f(s,x(s))_a d_q s \\ &\quad + \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}\Gamma_q(\alpha)}\int_t^b (b-(qs-(1-q)a))_a^{(\alpha-2)} f(s,x(s))_a d_q s \\ &\quad - \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-(qs-(1-q)a))_a^{(\alpha-1)} f(s,x(s))_a d_q s,\end{aligned}$$

which yields the desired result.

From (Ferreira, 2010), let $\alpha > 0$ and $s \le t \le z$, we get

$$(z-s)_0^{(\alpha)} \ge (z-t)_0^{(\alpha)}. \tag{3.2}$$

Lemma 3.2 Suppose $2 < \alpha < 3$, then the function G defined by Lemma 3.1 satisfies the following conditions:

(1) $G(t, qs+(1-q)a) \ge 0$ and $G(t, qs+(1-q)a) \le G(b, qs+(1-q)a)$ for all $a \le t, s \le b$,

(2) $\left(\frac{t-a}{b-a}\right)^{\alpha-1} G(b, qs+(1-q)a) \le G(t, qs+(1-q)a), \quad a \le t, s \le b.$

Proof. Let

$$\begin{aligned}g_1(t, qs+(1-q)a) &= \frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}}(t-a)^{\alpha-1} \\ &\quad - (t-(qs-(1-q)a))_a^{(\alpha-1)}, \quad a \le qs+(1-q)a \le t \le b, \\ g_2(t, qs+(1-q)a) &= \frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}}(t-a)^{\alpha-1}, \quad a \le t \le qs+(1-q)a \le b.\end{aligned}$$

It is clear that $g_2(t, qs+(1-q)a) > 0$.

When $a \le qs+(1-q)a \le t \le b$, from (2.1), we have

$$\begin{aligned}g_1(t, qs+(1-q)a) &= (1-\frac{q(s-a)}{b-a})_0^{(\alpha-2)}(t-a)^{\alpha-1} - (t-a)^{\alpha-1}(1-\frac{q(s-a)}{t-a})_0^{(\alpha-1)} \\ &= (t-a)^{\alpha-1}\left[\left(1-\frac{q(s-a)}{b-a}\right)_0^{(\alpha-2)} - \left(1-\frac{q(s-a)}{t-a}\right)_0^{(\alpha-1)}\right].\end{aligned}$$

On the other hand, since

$$\frac{q(s-a)}{b-a} \le \frac{q(s-a)}{t-a} \le 1, \quad a < s \le t \le b,$$

using (3.2), we obtain $g_1(t, qs + (1-q)a) \ge 0$, therefore, $G(t, qs + (1-q)a) \ge 0$.

Moreover, for fixed $s \in [a, b]$, using Lemma 2.2, we have

$$\begin{aligned}
{}_t({}_aD_q g_1(t, qs + (1-q)a)) &= [\alpha-1]_q \frac{(b - (qs - (1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}}(t-a)^{\alpha-2} - [\alpha-1]_q (t - (qs - (1-q)a))_a^{(\alpha-2)} \\
&= [\alpha-1]_q (t-a)^{\alpha-2}\left[\left(1 - \frac{q(s-a)}{b-a}\right)_0^{(\alpha-2)} - \left(1 - \frac{q(s-a)}{t-a}\right)_0^{(\alpha-2)}\right] \\
&\ge 0,
\end{aligned}$$

i.e., $g_1(t, qs + (1-q)a)$ is an increasing function of t. Obviously, $g_2(t, qs + (1-q)a)$ is increasing in t, therefore $G(t, qs + (1-q)a)$ is an increasing function of t for fixed $s \in [a, b]$. This concludes the proof of (1).

If $a \le qs + (1-q)a \le t \le b$, from (2.1), we have

$$\begin{aligned}
\frac{G(t, qs + (1-q)a)}{G(b, qs + (1-q)a)} &= \frac{(t-a)^{\alpha-1}\left[\left(1 - \frac{q(s-a)}{b-a}\right)_0^{(\alpha-2)} - \left(1 - \frac{q(s-a)}{t-a}\right)_0^{(\alpha-1)}\right]}{(b-a)^{\alpha-1}\left[\left(1 - \frac{q(s-a)}{b-a}\right)_0^{(\alpha-2)} - \left(1 - \frac{q(s-a)}{b-a}\right)_0^{(\alpha-1)}\right]} \\
&\ge \left(\frac{t-a}{b-a}\right)^{\alpha-1}.
\end{aligned}$$

If $a \le t \le qs + (1-q)a \le b$, then

$$\frac{G(t, qs + (1-q)a)}{G(b, qs + (1-q)a)} = \left(\frac{t-a}{b-a}\right)^{\alpha-1},$$

this finishes the proof of (2).

For the convenience, we denote

$$A = |G(t_2, qs + (1-q)a) - G(t_1, qs + (1-q)a)|.$$

Remark 3.3 The function $G(t, qs + (1-q)a)$ has some other properties. We can obtain the following inequalities:

(i) For $a \le t_1 \le t_2 \le qs + (1-q)a \le b$, we get

$$\begin{aligned}
A &= \left|\frac{(b - (qs - (1-q)a))_a^{(\alpha-2)}}{\Gamma_q(\alpha)(b-a)^{\alpha-2}}(t_2-a)^{\alpha-1} - \frac{(b - (qs - (1-q)a))_a^{(\alpha-2)}}{\Gamma_q(\alpha)(b-a)^{\alpha-2}}(t_1-a)^{\alpha-1}\right| \\
&= \frac{(b - (qs - (1-q)a))_a^{(\alpha-2)}}{\Gamma_q(\alpha)(b-a)^{\alpha-2}}|(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}| \\
&= \frac{1}{\Gamma_q(\alpha)}\left(1 - \frac{q(s-a)}{b-a}\right)_0^{(\alpha-2)}|(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}| \\
&\le \frac{1}{\Gamma_q(\alpha)}[(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}].
\end{aligned}$$

(ii) For $a \le t_1 \le qs + (1-q)a \le t_2 \le b$, we get

$$\begin{aligned}
A =& \left| \frac{1}{\Gamma_q(\alpha)} \left[\frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}} (t_2-a)^{\alpha-1} - (t_2-(qs-(1-q)a))_a^{(\alpha-1)} \right] \right.\\
&\left. - \frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{\Gamma_q(\alpha)(b-a)^{\alpha-2}} (t_1-a)^{\alpha-1} \right| \\
=& \frac{1}{\Gamma_q(\alpha)} \left| \frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}} [(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}] - (t_2-(qs-(1-q)a))_a^{(\alpha-1)} \right| \\
\le& \frac{1}{\Gamma_q(\alpha)} \left| \frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}} [(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}] \right| + |(t_2-(qs-(1-q)a))_a^{(\alpha-1)}| \\
\le& \frac{1}{\Gamma_q(\alpha)} [(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1} + (t_2-t_1)_a^{\alpha-1}].
\end{aligned}$$

(iii) For $a \le qs + (1-q)a \le t_1 \le t_2 \le b$, we get

$$\begin{aligned}
A =& \left| \frac{1}{\Gamma_q(\alpha)} \left[\frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}} (t_2-a)^{\alpha-1} - (t_2-(qs-(1-q)a))_a^{(\alpha-1)} \right] \right.\\
&\left. - \frac{1}{\Gamma_q(\alpha)} \left[\frac{(b-(qs-(1-q)a))_a^{(\alpha-2)}}{(b-a)^{\alpha-2}} (t_1-a)^{\alpha-1} - (t_1-(qs-(1-q)a))_a^{(\alpha-1)} \right] \right|,
\end{aligned}$$

by using the similar procedure of Remark 3.1 in [13] and the derivation process of (i)-(ii), we obtain

$$A \le \frac{1}{\Gamma_q(\alpha)} [(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1} + (t_2-(qs-(1-q)a))_a^{(\alpha-1)} - (t_1-(qs-(1-q)a))_a^{(\alpha-1)}].$$

Let $C[a,b]$ be the space of all continuous real functions defined on $[a,b]$ with the maximum norm $\|x\| = \max_{t\in[a,b]} |x(t)|$. It is well known that $C[a,b]$ is a Banach space under the sup norm. Define the cone $P \subset C[a,b]$ as following:

$$P = \{x \in C[a,b] : x(t) \ge 0, t \in [a,b]\}.$$

From Lemma 3.1, we know that $x(t)$ is a solution of boundary value problem (1.3)-(1.4) if and only if it satisfies

$$x(t) = \int_a^b G(t, qs + (1-q)a) f(s, x(s))_a d_q s, \quad t \in [a,b]. \tag{3.3}$$

Then, the positive solutions $x(t)$ of problem (1.3)-(1.4) are the fixed points of T in $C[a,b]$ that defined by

$$(Tx)(t) = \int_a^b G(t, qs + (1-q)a) f(s, x(s))_a d_q s, \quad t \in [a,b]. \tag{3.4}$$

Then we have the following results.

From (Zhai & Ren, 2017), let $\alpha \ge 0$, $\{s_m\}$ is a sequence and $s_m \to s$ as $m \to \infty$, we have

$$(s_m - t)^{(\alpha)} \overset{m\to\infty}{\longrightarrow} (s-t)^{(\alpha)}, \quad s, t \in \mathbb{R}. \tag{3.5}$$

Lemma 3.4 If the operator T defined by (3.4), then $T : P \to P$ is completely continuous.

Proof. The operator $T : P \to P$ is continuous in view of continuity of $G(t, qs + (1-q)a)$ and $f(t, x(t))$. So we only prove T is compact. Let $\Omega \subset P$ be bounded. There exist $g \in L^1[a,b]$, such that $f(t, x(t)) \ge g(t)$ for $t \in [a,b]$. Let $M = \int_a^b g(s)_a d_q s$, then $0 \leq M < +\infty$. From Lemma 3.2, we get

$$
\begin{aligned}
|Tx| &= \max_{t\in[a,b]} \left| \int_a^b G(t, qs + (1-q)a) f(s, x(s))_a d_q s \right| \\
&\leq \max_{qs+(1-q)a\in[a,b]} G(b, qs + (1-q)a) \int_a^b f(s, x(s))_a d_q s \\
&\leq \frac{(b-a)^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^b g(s)_a d_q s \\
&= \frac{M(b-a)^{\alpha-1}}{\Gamma_q(\alpha)}.
\end{aligned}
$$

This shows that the set $T(\Omega)$ is uniform bounded. After that, for given $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, we obtain

$$
\begin{aligned}
|Tx(t_2) - Tx(t_1)| &\leq \int_a^b A f(s, x(s))_a d_q s \\
&\leq \max_{qs+(1-q)a\in[a,b]} A \int_a^b g(s)_a d_q s \\
&= M \max_{qs+(1-q)a\in[a,b]} A.
\end{aligned}
$$

In view of Remark 3.3 and (3.5), one has $Tx(t_1) \to Tx(t_2)$ as $t_1 \to t_2$. By means of the Arzela-Ascoli theorem, $T : P \to P$ is completely continuous. The proof is completed.

From (Bai & Lü, 2005), the map θ is a said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that θ: $P \to [0, \infty)$ is continuous and

$$\theta(\lambda x + (1-\lambda)y) \geq \lambda\theta(x) + (1-\lambda)\theta(y), \quad x, y \in P, \quad \lambda \in [0, 1].$$

Next, we take $\nu \in (a, b)$, set $\mu = \frac{\nu-a}{b-a}$, then $0 < \mu < 1$. We set the nonnegative concave continuous function θ on P be defined by

$$\theta(x) = \min_{t\in[\nu,b]} x(t). \tag{3.6}$$

We denote

$$
\begin{aligned}
&P_r = \{x \in P : \|x\| < r\}, \quad \bar{P}_r = \{x \in P : \|x\| \leq r\}, \\
&P(\theta, r_1, r_2) = \{x \in P : \theta(x) \geq r_1, \|x\| \leq r_2\}, \\
&\rho^{-1} = \int_a^b G(b, qs + (1-q)a)_a d_q s, \quad \omega^{-1} = \int_\nu^b G(b, qs + (1-q)a)_a d_q s,
\end{aligned}
$$

where r, r_1, r_2 are positive constants.

For the convenience, we show the Leggett-Williams fixed point theorem as follows.

Lemma 3.5 Let P be a cone in a real Banach space E, and $r_3 > 0$ be a constant. Suppose θ is a nonnegative continuous concave functional on P such that $\theta(x) \leq \|x\|$, for all $x \in \bar{P}_{r_3}$. Let $T : \bar{P}_{r_3} \to \bar{P}_{r_3}$ be a completely continuous operator. Assume there are numbers r, r_1 and r_2 with $0 < r < r_1 < r_2 \leq r_3$ such that

(i) The set $\{x \in P(\theta, r_1, r_2) : \theta(x) > r_1\}$ is nonempty and $\theta(Tx) > r_1$ for all $x \in P(\theta, r_1, r_2)$;

(ii) $\|Tx\| < r$ for $x \leq r$;

(iii) $\theta(Tx) > r_1$ for all $x \in P(\theta, r_1, r_3)$ with $\|Tx\| > r_2$.

Then T has at least three fixed points x_1, x_2 and $x_3 \in \bar{P}_{r_3}$ with $\|x_1\| < r$, $r_1 < \theta(x_2)$, $r < \|x_3\|$ and $\theta(x_3) < r_1$.

Theorem 3.6 Suppose $f(t, x)$ is a nonnegative continuous function on $[a, b] \times [0, +\infty)$ and there exists $t_n \to a$ such that $f(t_n, x(t_n)) > 0, n = 1, 2, \cdots$, and there exist constants $0 < r < r_1$ such that

(H_1) $f(t, x) < \rho r$ for $(t, x) \in [a, b] \times [0, r]$;

(H_2) $f(t, x) \geq \frac{\omega}{\mu^{\alpha-1}} r_1$ for $(t, x) \in [\nu, b] \times [r_1, r_3]$, where $r_3 > \frac{r_1}{\mu^{\alpha-1}}$;

(H_3) $f(t,x) \le kx+\beta$ for $(t,x) \in [a,b]\times[0,+\infty)$, where k,β are positive numbers.

Then the boundary value problem (1.3)-(1.4) has at least three positive solutions x_1, x_2 and x_3.

Proof. Set $r_3 > \max\{\frac{\beta}{\rho-k}, \frac{r_1}{\mu^{\alpha-1}}\}$, then for $x \in \bar{P}_{r_3}$, we have from (3.4), (H_3) and Lemma 3.2,

$$
\begin{aligned}
\|Tx\| &= \max_{t\in[a,b]} \int_a^b G(t, qs+(1-q)a) f(s,x(s))_a d_q s \\
&= \int_a^b \max_{t\in[a,b]} G(t, qs+(1-q)a) f(s,x(s))_a d_q s \\
&= \int_a^b G(b, qs+(1-q)a) f(s,x(s))_a d_q s \\
&\le \int_a^b G(b, qs+(1-q)a)(kx(s)+\beta)_a d_q s \\
&\le (k\|x\|+\beta)\int_a^b G(b, qs+(1-q)a)_a d_q s \\
&= \frac{k\|x\|+\beta}{\rho} \\
&< r_3,
\end{aligned}
$$

i.e, $Tx \in P_{r_3}$. Therefore, $T : \bar{P}_{r_3} \to \bar{P}_{r_3}$ be a completely continuous operator. By (H_1), we can get

$$
\begin{aligned}
\|Tx\| &= \int_a^b G(b, qs+(1-q)a) f(s,x(s))_a d_q s \\
&< \rho r \int_a^b G(b, qs+(1-q)a)_a d_q s \\
&= r.
\end{aligned}
$$

Hence assumption (ii) of Lemma 3.5 is satisfied.

To check condition (i) of Lemma 3.5, we choose $x_0 = \frac{(\mu^{\alpha-1}+1)r_1}{\mu^{\alpha-1}}$ for $t \in [\nu,b], \mu \in (0,1)$. It is easy to see that $x_0 = \frac{(\mu^{\alpha-1}+1)r_1}{\mu^{\alpha-1}} \in P(\theta, r_1, \frac{1}{\mu^{\alpha-1}}r_1)$, $\theta(x_0) = \theta(\frac{(\mu^{\alpha-1}+1)r_1}{\mu^{\alpha-1}}) > r_1$, consequently, $\{x \in P(\theta, r_1, \frac{1}{\mu^{\alpha-1}}r_1) : \theta(x) > r_1\} \ne \emptyset$. Hence, if $x \in P(\theta, r_1, \frac{1}{\mu^{\alpha-1}}r_1)$, then $r_1 \le x(t) \le \frac{1}{\mu^{\alpha-1}}r_1$ for $t \in [\nu,b], \mu \in (0,1)$. Thus, from (H_2) and Lemma 3.2, we have

$$
\begin{aligned}
\theta(Tx) &= \min_{t\in[\nu,b]} \int_a^b G(t, qs+(1-q)a) f(s,x(s))_a d_q s \\
&> \int_\nu^b \min_{t\in[\nu,b]} G(t, qs+(1-q)a) f(s,x(s))_a d_q s \\
&\ge \int_\nu^b \min_{t\in[\nu,b]} \left(\frac{t-a}{b-a}\right)^{\alpha-1} G(b, qs+(1-q)a) f(s,x(s))_a d_q s \\
&= \int_\nu^b \left(\frac{\nu-a}{b-a}\right)^{\alpha-1} G(b, qs+(1-q)a) f(s,x(s))_a d_q s \\
&= \int_\nu^b \mu^{\alpha-1} G(b, qs+(1-q)a) f(s,x(s))_a d_q s \\
&\ge \frac{\omega}{\mu^{\alpha-1}} r_1 \int_\nu^b \mu^{\alpha-1} G(b, qs+(1-q)a)_a d_q s \\
&= r_1,
\end{aligned}
$$

i.e, $\theta(Tx) > r_1$ for all $x \in P(\theta, r_1, \frac{1}{\mu^{\alpha-1}}r_1)$. This shows that condition (i) of Lemma 3.5 is satisfied.

Finally, for $x \in P(\theta, r_1, r_3)$ with $\|Tx\| > \frac{1}{\mu^{\alpha-1}} r_1$, we obtain

$$\begin{aligned}\theta(Tx) &= \min_{t\in[\nu,b]} \int_a^b G(t, qs + (1-q)a) f(s, x(s))_a d_q s \\ &\geq \int_a^b \min_{t\in[\nu,b]} \left(\frac{t-a}{b-a}\right)^{\alpha-1} G(b, qs + (1-q)a) f(s, x(s))_a d_q s \\ &= \int_a^b \left(\frac{\nu-a}{b-a}\right)^{\alpha-1} G(b, qs + (1-q)a) f(s, x(s))_a d_q s \\ &= \mu^{\alpha-1} \int_a^b \max_{t\in[a,b]} G(t, qs + (1-q)a) f(s, x(s))_a d_q s \\ &= \mu^{\alpha-1} \|Tx\| \\ &> r_1.\end{aligned}$$

This confirms that condition (iii) of Lemma 3.5 is fulfilled. By Lemma 3.5, the boundary value problem (1.3)-(1.4) has at least three positive solutions x_1, x_2 and x_3.

Remark Compared with (Ferreira, 2011), $t \in [a, b]$ is wider than $t \in [0, 1]$. When $a = 0, b = 1$, lemma 3.1 and lemma 3.2 are consistent with the existing results.

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors contributions

YL carried out the main results and completed the corresponding proof. RX participated in Section 3, and also check and revise the whole text. All authors read and approved the final manuscript.

Data availability statement

We declare that the data in the paper can be used publicly.

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