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Generalized (p,q)-Fibonacci-Like Sequences and Their Properties

Yasemin Taşyurdu

Correspondence: Department of Mathematics, Faculty of Science and Art, Erzincan Binali Yıldırım University, 24100 Erzincan, Turkey. E-mail: ytasyurdu@erzincan.edu.tr

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Abstract

In this paper, we define the generalized (p,q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$ associated with the (p,q)-Fibonacci and the (p,q)-Lucas sequences and then we get some fundamental identities and sums formulas involving odd and even terms of the generalized (p,q)-Fibonacci-Like sequences. We obtain the Binet's formula to find the nth general term of generalized (p,q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$. Also, the generating functions of these sequences are presented and proved.

Keywords: (p,q)-Fibonacci sequence, (p,q)-Lucas sequence, Binet's formula

1. Introduction

The Fibonacci and Lucas sequences are well-known examples of the most interesting second order recurrence sequences in all of mathematics. Many authors have studied the Fibonacci and Lucas sequences, some of whom introduced new sequences related to it as well as proving many identities for them (Horadam, 1961; Koshy, 2001; Gupta *et al*, 2012). The Fibonacci sequence $\{F_n\}$ is defined by recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial values $F_0 = 0$, $F_1 = 1$ for $n \geq 2$, and Lucas sequence $\{L_n\}$ is defined by recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial values $L_0 = 2$, $L_1 = 1$ for $n \geq 2$. Then, the k-Fibonacci sequences and k-Lucas sequences are introduced by recurrence relations

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad F_{k,0} = 0, F_{k,1} = 1, \quad n \geq 2,\ k \geq 1$$

and

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \quad L_{k,0} = 2, L_{k,1} = k, \quad n \geq 2,\ k \geq 1$$

respectively (Falcon 2001; Falcon & Plaza, 2007). Most of the authors introduced Fibonacci pattern based sequences in many ways which are known as Fibonacci-Like sequences (Harne, 2014; Singh, *et al*, 2010, 2014; Wani *et al*, 2016). Some authors defined various k-Fibonacci-like sequences. It has obtained a lot of study on the new family of k-Fibonacci sequences, k-Lucas sequences (Panwar, 2014; Tasyurdu, 2016; Wani *et al*, 2018).

The (p,q)-Fibonacci sequences and the (p,q)-Lucas sequences are defined by second-order recurrence relations

$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}, \quad F_{p,q,0} = 0, F_{p,q,1} = 1, \quad n \geq 2 \tag{1}$$

and

$$L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}, \quad L_{p,q,0} = 2, L_{p,q,1} = p, \quad n \geq 2 \tag{2}$$

respectively. It is considered some properties of the (p,q)-Fibonacci sequences and the (p,q)-Lucas sequences (Suvarnamani & Tatong, 2015, 2016). The corresponding characteristic equation of the equation (1) and (2) is

$$r^2 - pr - q = 0$$

and its roots are $r_1 = \frac{p+\sqrt{p^2+4q}}{2}$ and $r_2 = \frac{p-\sqrt{p^2+4q}}{2}$. Then the Binet's formulas for the (p,q)-Fibonacci sequences and the (p,q)-Lucas sequences are given respectively by

$$F_{p,q,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$$

$$L_{p,q,n} = r_1^n + r_2^n.$$

Also, the roots r_1 and r_2 verifies the relations

$$r_1 + r_2 = p$$

$$r_1 r_2 = -q$$

$$r_1 - r_2 = \sqrt{p^2 + 4q}.$$

In this paper, we introduce the generalized (p,q)-Fibonacci-Like sequences associated with the (p,q)-Fibonacci sequences and the (p,q)-Lucas sequences and present some identities of the generalized (p,q)-Fibonacci-Like sequences.

2. Results

2.1 Generalized (p,q)-Fibonacci-Like Sequences $\{S_{p,q,n}\}$

In this section, we define the generalized (p,q)-Fibonacci-Like sequences associated with the (p,q)-Fibonacci sequence and the (p,q)-Lucas sequence and obtain the Binet's formula to find the nth term in the sequences $\{S_{p,q,n}\}$. Also, we found generating functions of the generalized (p,q)-Fibonacci-Like sequences.

Definition 2.1. For $p \geq 1, q \geq 1$ and $k \geq 0$, the generalized (p,q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$ are defined by recurrence relation

$$S_{p,q,n} = pS_{p,q,n-1} + qS_{p,q,n-2}, \quad n \geq 2 \tag{3}$$

with initial conditions $S_{p,q,0} = 2k,\ S_{p,q,1} = 1 + kp$.

The first values of the generalized (p,q)-Fibonacci-Like sequences are

$$2k, 1 + kp, p + kp^2 + 2kq, p^2 + kp^3 + 3kpq + q, \ldots$$

Definition 2.1 is the general model of the different sequences presented by many authors. We can give with the following corollary.

Corollary 2.1. Particular cases of Definition 2.1 are

- If $k = 0$, the (p,q)-Fibonacci sequences $\{F_{p,q,n}\}_{n\geq 0}$ in (Suvarnamani & Tatong, 2015) are obtained
$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}, \quad F_{p,q,0} = 0, F_{p,q,1} = 1, \quad n \geq 2$$
- If $k = 0$, $p = a$ and $q = b$, the generalize the Fibonacci sequences $\{\mathcal{F}_n\}_{n\geq 0}$ in (Klaman and Mena, 2002) are obtained
$$\mathcal{F}_n = a\mathcal{F}_{n-1} + b\mathcal{F}_{n-2}, \quad \mathcal{F}_0 = 0,\ \mathcal{F}_1 = 1, \quad n \geq 2$$
- If $k = 0$ and $p = q = 1$, the Fibonacci sequence $\{F_n\}_{n\geq 0}$ (Horadam, 1961) is obtained
$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0,\ F_1 = 1, \quad n \geq 2$$
- If $k = 0$, $p = 2$ and $q = 1$, the Pell sequence $\{P_n\}_{n\geq 0}$ in (Horadam, 1971) is obtained
$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0,\ P_1 = 1, \quad n \geq 2$$
- If $k = 0$, $p = 1$ and $q = 2$, the Jacobsthal sequence $\{J_n\}_{n\geq 0}$ in (Horadam, 1996) is obtained
$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0,\ J_1 = 1, \quad n \geq 2$$
- If $k = 1$, $p = 1$ and $q = 1$, the Fibonacci-Like sequence $\{S_n\}_{n\geq 0}$ in (Singh, *et al*, 2010) is obtained
$$S_n = S_{n-1} + S_{n-2}, \quad S_0 = 2,\ S_1 = 2, \quad n \geq 2$$
- If $k = 1$, $p = 1$ and $q = 2$, the sequence $\{V_n\}_{n\geq 0}$ in (Gupta *et al*, 2012) is obtained
$$V_n = V_{n-1} + 2V_{n-2}, \quad V_0 = 2,\ V_1 = 2, \quad n \geq 2$$

The relation between (p,q)-Fibonacci sequences, (p,q)-Lucas sequences and the generalized (p,q)-Fibonacci-Like sequences can be written as follows

$$S_{p,q,n} = F_{p,q,n} + kL_{p,q,n}, \quad n \geq 0.$$

The corresponding characteristic equation of the equation (3) is

$$r^2 - pr - q = 0$$

and its roots are

$$r_1 = \frac{p + \sqrt{p^2 + 4q}}{2}$$

$$r_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

By using these two roots, we obtain Binet's formula of the generalized (p, q)-Fibonacci-Like sequences is

$$S_{p,q,n} = \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) + k(r_1^n + r_2^n) \quad (4)$$

where $r_1 + r_2 = p,\ r_1 r_2 = -q$ and $r_1 - r_2 = \sqrt{p^2 + 4q}$.

Now, we will give generating functions for the generalized (p, q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$. A generating function $g(x)$ is a formal power series

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \quad (5)$$

whose coefficients give the sequence $\{a_0, a_1, \dots\}$ given a generating function is the analytic expression for the nth term in the corresponding series.

Theorem 2.1. The generating functions of the generalized (p, q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$ are

$$g(x) = \sum_{n=0}^{\infty} S_{p,q,n} x^n = \frac{2k + (1 - kp)x}{1 - px - qx^2}.$$

Proof. The generalized (p, q)-Fibonacci-Like sequences can be considered as the coefficients of the power series of the corresponding generating function in equation (5). Let us suppose that generalized (p, q)-Fibonacci-Like sequences are coefficient of a potential series contered at the origin and consider the corresponding analytic function $g(x)$ such that

$$g(x) = S_{p,q,0} + S_{p,q,1}x + S_{p,q,2}x^2 + \cdots + S_{p,q,n}x^n + \cdots \quad (6)$$

Then we can write

$$pg(x)x = pS_{p,q,0}x + pS_{p,q,1}x^2 + pS_{p,q,2}x^3 + \cdots + pS_{p,q,n}x^{n+1} + \cdots \quad (7)$$

$$qg(x)x^2 = qS_{p,q,0}x^2 + qS_{p,q,1}x^3 + qS_{p,q,2}x^4 + \cdots + qS_{p,q,n}x^{n+2} + \cdots \quad (8)$$

From the equations (6), (7) and (8), we obtain

$$g(x)(1 - px - qx^2) = S_{p,q,0} + S_{p,q,1}x - pS_{p,q,0}x$$

where $S_{p,q,n} = pS_{p,q,n-1} + qS_{p,q,n-2},\ n \geq 0$ with initial conditions $S_{p,q,0} = 2k,\ S_{p,q,1} = 1 + kp$ from equation (3). So the generating functions of the generalized (p, q)-Fibonacci-Like sequences is

$$g(x) = \sum_{n=0}^{\infty} S_{p,q,n} x^n = \frac{2k + (1 - kp)x}{1 - px - qx^2}. \qquad \blacksquare$$

2.2 Some Identities of the Generalized (p, q)-Fibonacci-Like Sequences $\{S_{p,q,n}\}$

In this section, we obtain some fundamental identities of the generalized (p, q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$ like Cassini's identity, Catalan's identity, Vajda's identity and d'Ocagne's identity and introduce sums formulas of the first n terms with odd and even indices of the generalized (p, q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$.

Theorem 2.2. (Cassini's identity) Let p, q, k and n be positive integers. For $n \geq 1$, we get

$$S_{p,q,n+1}S_{p,q,n-1} - S_{p,q,n}^2 = (-1)^n q^{n-1}\left(1 - k^2(p^2 + 4q)\right).$$

Proof. Let p, q, k and $n \geq 1$ be positive integers. By using the equation (4), we have

$$\begin{aligned} S_{p,q,n+1}S_{p,q,n-1} - S_{p,q,n}^2 &= \left[\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) + k(r_1^{n+1} + r_2^{n+1})\right]\left[\left(\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\right) + k(r_1^{n-1} + r_2^{n-1})\right] \\ &\quad - \left[\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) + k(r_1^n + r_2^n)\right]^2 \\ &= \frac{-r_1^{n-1}r_2^{n+1} - r_1^{n+1}r_2^{n-1} + 2r_1^n r_2^n}{(r_1 - r_2)^2} + k\left(\frac{2r_1^{2n} - 2r_2^{2n} - 2r_1^{2n} + 2r_2^{2n}}{r_1 - r_2}\right) \end{aligned}$$

$$+k^2(r_1^{n+1}r_2^{n-1}+r_1^{n-1}r_2^{n+1}-2r_1^nr_2^n)$$

$$=\frac{r_1^{n-1}r_2^{n-1}(-1)(r_1^2-2r_1r_2+r_2^2)}{(r_1-r_2)^2}+k^2\left(r_1^{n-1}r_2^{n-1}(r_1^2-2r_1r_2+r_2^2)\right)$$

$$=\frac{(-1)^nq^{n-1}(r_1-r_2)^2}{(r_1-r_2)^2}+k^2(-q)^{n-1}(r_1-r_2)^2$$

$$=(-1)^nq^{n-1}+k^2(-q)^{n-1}(p^2+4q)$$

$$=(-1)^nq^{n-1}\left(1-k^2(p^2+4q)\right).$$ ■

Using Theorem 2.2, we obtain the Cassini's identity for each sequence in Corollary 2.1. So, we can write following remark.

Remark 2.1. We have the following Cassini's identities for all sequences in Corollary 2.1:

- The (p,q)-Fibonacci sequences $\{F_{p,q,n}\}_{n\geq 0}$: $F_{p,q,n+1}F_{p,q,n-1}-F_{p,q,n}^2=(-1)^nq^{n-1}$
- The generalize the Fibonacci sequences $\{\mathcal{F}_n\}_{n\geq 0}$: $\mathcal{F}_{n+1}\mathcal{F}_{n-1}-\mathcal{F}_n^2=(-1)^nb^{n-1}$
- The Fibonacci sequence $\{F_n\}_{n\geq 0}$: $F_{n+1}F_{n-1}-F_n^2=(-1)^n$
- The Pell sequence $\{P_n\}_{n\geq 0}$: $P_{n+1}P_{n-1}-P_n^2=(-1)^n$
- The Jacobsthal sequence $\{J_n\}_{n\geq 0}$: $J_{n+1}J_{n-1}-J_n^2=(-1)^n2^{n-1}$
- The Fibonacci-Like sequence $\{S\}_{n\geq 0}$: $S_{n+1}S_{n-1}-S_n^2=4(-1)^{n+1}$
- The sequence $\{V_n\}_{n\geq 0}$: $V_{n+1}V_{n-1}-V_n^2=(-1)^{n+1}2^{n+2}$

Theorem 2.3. Let p,q,k and n be positive integers. For $n\geq 2$, we have

$$S_{p,q,n-2}S_{p,q,n+1}-S_{p,q,n-1}S_{p,q,n}=p(-q)^{n-2}\left(-1+k^2(p^2+4q)\right).$$

Proof. Let p,q,k and $n\geq 2$ be positive integers. By using the equation (4), we have

$$S_{p,q,n-2}S_{p,q,n+1}-S_{p,q,n-1}S_{p,q,n}=\left[\left(\frac{r_1^{n-2}-r_2^{n-2}}{r_1-r_2}\right)+k(r_1^{n-2}+r_2^{n-2})\right]\left[\left(\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}\right)+k(r_1^{n+1}+r_2^{n+1})\right]$$

$$-\left[\left(\frac{r_1^{n-1}-r_2^{n-1}}{r_1-r_2}\right)+k(r_1^{n-1}+r_2^{n-1})\right]\left[\left(\frac{r_1^{n}-r_2^{n}}{r_1-r_2}\right)+k(r_1^{n}+r_2^{n})\right]$$

$$=\frac{-r_1^{n-2}r_2^{n+1}-r_1^{n+1}r_2^{n-2}+r_1^nr_2^{n-1}+r_1^{n-1}r_2^n}{(r_1-r_2)^2}+k\left(\frac{2r_1^{2n-1}-2r_2^{2n-1}-2r_1^{2n-1}+2r_2^{2n-1}}{r_1-r_2}\right)$$

$$+k^2(r_1^{n-2}r_2^{n+1}+r_1^{n+1}r_2^{n-2}-r_1^nr_2^{n-1}-r_1^{n-1}r_2^n)$$

$$=\frac{r_1^{n-2}r_2^{n-2}(-1)(r_1^3+r_2^3-r_1^2r_2-r_1r_2^2)}{(r_1-r_2)^2}+k^2\left(r_1^{n-2}r_2^{n-2}(r_1^3+r_2^3-r_1^2r_2-r_1r_2^2)\right)$$

$$=(-q)^{n-2}(-1)(r_1+r_2)+k^2((-q)^{n-2}(r_1+r_2)(r_1-r_2)^2)$$

$$=(-q)^{n-2}(-1)p+k^2\left((-q)^{n-2}p(p^2+4q)\right)$$

$$=p(-q)^{n-2}\left(-1+k^2(p^2+4q)\right).$$ ■

Theorem 2.4. (Catalan's identity) Let p,q,k,m and n be positive integers. For $m\geq n$, we have

$$S_{p,q,m-n}S_{p,q,m+n}-S_{p,q,m}^2=(-q)^{m-n}\left(-1+k^2(p^2+4q)\right)F_{p,q,n}^2.$$

Proof. Let p,q,k and $m\geq n$ be positive integers. By using the equation (4), we have

$$S_{p,q,m-n}S_{p,q,m+n}-S_{p,q,m}^2=\left[\left(\frac{r_1^{m-n}-r_2^{m-n}}{r_1-r_2}\right)+k(r_1^{m-n}+r_2^{m-n})\right]\left[\left(\frac{r_1^{m+n}-r_2^{m+n}}{r_1-r_2}\right)+k(r_1^{m+n}+r_2^{m+n})\right]$$

$$-\left[\left(\frac{r_1^{m}-r_2^{m}}{r_1-r_2}\right)+k(r_1^{m}+r_2^{m})\right]^2$$

$$= \frac{-r_1^{m-n} r_2^{m-n}(r_1^{2n} - 2r_1^n r_2^n + r_2^{2n})}{(r_1 - r_2)^2} - k\left(\frac{2r_1^{2m} - 2r_2^{2m} - 2r_1^{2m} + 2r_2^{2m}}{r_1 - r_2}\right)$$

$$+k^2\left(r_1^{m-n} r_2^{m-n}(r_1^{2n} - 2r_1^n r_2^n + r_2^{2n})\right)$$

$$= \frac{(-q)^{m-n}(-1)(r_1^n - r_2^n)^2}{(r_1 - r_2)^2} + k^2\left((-q)^{m-n}(r_1^n - r_2^n)^2 \frac{(r_1 - r_2)^2}{(r_1 - r_2)^2}\right)$$

$$= (-q)^{m-n}(-1)F_{p,q,n}^2 + k^2\left((-q)^{m-n}(p^2 + 4q)F_{p,q,n}^2\right)$$

$$= (-q)^{m-n}\left(-1 + k^2(p^2 + 4q)\right)F_{p,q,n}^2. \quad \blacksquare$$

Using Theorem 2.4, we obtain the Catalan's identity for each sequence in Corollary 2.1. So, we can write following remark.

Remark 2.2. We have the following Catalan's identities for all sequences in Corollary 2.1:

- The (p, q)-Fibonacci sequences $\{F_{p,q,n}\}_{n\geq 0}$: $F_{p,q,m-n}F_{p,q,m+n} - F_{p,q,m}^2 = (-q)^{m-n}(-1)F_{p,q,n}^2$
- The generalize the Fibonacci sequences $\{\mathcal{F}_n\}_{n\geq 0}$: $\mathcal{F}_{m-n}\mathcal{F}_{m+n} - \mathcal{F}_m^2 = (-b)^{m-n}(-1)F_{a,b,n}^2$
- The Fibonacci sequence $\{F_n\}_{n\geq 0}$: $F_{m-n}F_{m+n} - F_m^2 = (-1)^{m-n+1}F_{1,1,n}^2$
- The Pell sequence $\{P_n\}_{n\geq 0}$: $P_{m-n}P_{m+n} - P_m^2 = (-1)^{m-n+1}F_{2,1,n}^2$
- The Jacobsthal sequence $\{J_n\}_{n\geq 0}$: $J_{m-n}J_{m+n} - J_m^2 = (-2)^{m-n}(-1)F_{1,2,n}^2$
- The Fibonacci-Like sequence $\{S_n\}_{n\geq 0}$: $S_{m-n}S_{m+n} - S_m^2 = (-1)^{m-n}4F_{1,1,n}^2$
- The sequence $\{V_n\}_{n\geq 0}$: $V_{m-n}V_{m+n} - V_m^2 = (-1)^{m-n}2^{m-n+3}F_{1,2,n}^2$

Theorem 2.5. (Vajda's identity) Let p, q, k, m, n and t be positive integers. Then we have

$$S_{p,q,m+n}S_{p,q,m+t} - S_{p,q,m}S_{p,q,m+n+t} = (-q)^m\left(1 - k^2(p^2 + 4q)\right)F_{p,q,n}F_{p,q,t}.$$

Proof. Let p, q, k, m, n and t be positive integers. By using the equation (4), we have

$S_{p,q,m+n}S_{p,q,m+t} - S_{p,q,m}S_{p,q,m+n+t}$

$$= \left[\left(\frac{r_1^{m+n} - r_2^{m+n}}{r_1 - r_2}\right) + k(r_1^{m+n} + r_2^{m+n})\right]\left[\left(\frac{r_1^{m+t} - r_2^{m+t}}{r_1 - r_2}\right) + k(r_1^{m+t} + r_2^{m+t})\right]$$

$$-\left[\left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right) + k(r_1^m + r_2^m)\right]\left[\left(\frac{r_1^{m+n+t} - r_2^{m+n+t}}{r_1 - r_2}\right) + k(r_1^{m+n+t} + r_2^{m+n+t})\right]$$

$$= \frac{r_1^m r_2^m(-r_1^n r_2^t - r_1^t r_2^n + r_1^{n+t} + r_2^{n+t})}{(r_1 - r_2)^2} - k\left(\frac{2r_1^{2m+n+t} - 2r_2^{2m+n+t} - 2r_1^{2m+n+t} + 2r_2^{2m+n+t}}{r_1 - r_2}\right)$$

$$+k^2\left(r_1^m r_2^m + (r_1^n r_2^t + r_1^t r_2^n - r_1^{n+t} - r_2^{n+t})\right)$$

$$= \frac{(-q)^m\left(-r_2^t(r_1^n - r_2^n) + r_1^t(r_1^n - r_2^n)\right)}{(r_1 - r_2)^2} + k^2(-q)^m\left(r_2^t(r_1^n - r_2^n) - r_1^t(r_1^n - r_2^n)\right)$$

$$= \frac{(-q)^m(r_1^t - r_2^t)(r_1^n - r_2^n)}{(r_1 - r_2)^2} + k^2(-1)(-q)^m(r_1^t - r_2^t)(r_1^n - r_2^n)\frac{(r_1 - r_2)^2}{(r_1 - r_2)^2}$$

$$= (-q)^m F_{p,q,n}F_{p,q,t} + k^2(-1)(-q)^m(p^2 + 4q)F_{p,q,n}F_{p,q,t}$$

$$= (-q)^m\left(1 - k^2(p^2 + 4q)\right)F_{p,q,n}F_{p,q,t}. \quad \blacksquare$$

Using Theorem 2.5, we obtain the Vajda's identity for each sequence in Corollary 2.1. So, we can write following remark.

Remark 2.3. We have the following Vajda's identities for all sequences in Corollary 2.1:

- The (p, q)-Fibonacci sequences $\{F_{p,q,n}\}_{n\geq 0}$: $F_{p,q,m+n}F_{p,q,m+t} - F_{p,q,m}F_{p,q,m+n+t} = (-q)^m F_{p,q,n}F_{p,q,t}$
- The generalize the Fibonacci sequence $\{\mathcal{F}_n\}_{n\geq 0}$: $\mathcal{F}_{m+n}\mathcal{F}_{m+t} - \mathcal{F}_m\mathcal{F}_{m+n+t} = (-b)^m F_{a,b,n}F_{a,b,t}$
- The Fibonacci sequence $\{F_n\}_{n\geq 0}$: $F_{m+n}F_{m+t} - F_mF_{m+n+t} = (-1)^m F_{1,1,n}F_{1,1,t}$

- The Pell sequence $\{P_n\}_{n\geq 0}$: $P_{m+n}P_{m+t} - P_mP_{m+n+t} = (-1)^m F_{2,1,n}F_{2,1,t}$
- The Jacobsthal sequence $\{J_n\}_{n\geq 0}$: $J_{m+n}J_{m+t} - J_mJ_{m+n+t} = (-2)^m F_{1,2,n}F_{1,2,t}$
- The Fibonacci-Like sequence $\{S_n\}_{n\geq 0}$: $S_{m+n}S_{m+t} - S_mS_{m+n+t} = (-1)^{m+1}4F_{1,1,n}F_{1,1,t}$
- The sequence $\{V_n\}_{n\geq 0}$: $V_{m+n}V_{m+t} - V_mV_{m+n+t} = (-1)^{m+1}2^{m+3}F_{1,2,n}F_{1,2,t}$

Theorem 2.6. Let p, q, k, m and n be positive integers. Then we have

$$S_{p,q,m}S_{p,q,n+1} + qS_{p,q,m-1}S_{p,q,n} = \left(1 + k^2(p^2+4q)\right)F_{p,q,m+n} + 2kL_{p,q,m+n}.$$

Proof. Let p, q, k, m and n be positive integers. By using the equation (4), we have

$$\begin{aligned}
S_{p,q,m}S_{p,q,n+1} + qS_{p,q,m-1}S_{p,q,n} &= \left[\left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right) + k(r_1^m + r_2^m)\right]\left[\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) + k(r_1^{n+1} + r_2^{n+1})\right] \\
&\quad + q\left[\left(\frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}\right) + k(r_1^{m-1} + r_2^{m-1})\right]\left[\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) + k(r_1^n + r_2^n)\right] \\
&= \frac{(r_1^{m+n+1} + r_2^{m+n+1} - r_1^{n+1}r_2^m - r_1^m r_2^{n+1}) + (-r_1r_2)(r_1^{m+n-1} + r_2^{m+n-1} - r_1^{m-1}r_2^n - r_1^n r_2^{m-1})}{(r_1 - r_2)^2} \\
&\quad + k\left(\frac{2r_1^{m+n+1} - 2r_2^{m+n+1} + (-r_1r_2)2r_1^{m+n-1} - 2r_2^{m+n-1}}{r_1 - r_2}\right) \\
&\quad + k^2\left((r_1^{m+n+1} + r_2^{m+n+1} + r_1^{n+1}r_2^m + r_1^m r_2^{n+1}) + (-r_1r_2)(r_1^{m+n-1} + r_2^{m+n-1} + r_1^{m-1}r_2^n + r_1^n r_2^{m-1})\right) \\
&= \frac{r_1^{m+n+1} + r_2^{m+n+1} - r_1^{m+n}r_2 - r_1r_2^{m+n}}{(r_1 - r_2)^2} + k\left(\frac{2r_1^{m+n+1} - 2r_2^{m+n+1} - 2r_1^{m+n}r_2 + 2r_1r_2^{m+n}}{r_1 - r_2}\right) \\
&\quad + k^2(r_1^{m+n+1} + r_2^{m+n+1} - r_1^{m+n}r_2 - r_1r_2^{m+n}) \\
&= \frac{(r_1^{m+n} - r_2^{m+n})(r_1 - r_2)}{(r_1 - r_2)^2} + k\left(\frac{2r_1^{m+n}(r_1 - r_2) + 2r_2^{m+n}(r_1 - r_2)}{r_1 - r_2}\right) + k^2(r_1^{m+n} - r_2^{m+n})(r_1 - r_2) \\
&= \frac{r_1^{m+n} - r_2^{m+n}}{r_1 - r_2} + k\left(\frac{2(r_1^{m+n} + r_2^{m+n})(r_1 - r_2)}{r_1 - r_2}\right) + k^2\left(\frac{(r_1^{m+n} - r_2^{m+n})(r_1 - r_2)^2}{r_1 - r_2}\right) \\
&= F_{p,q,m+n} + k\left(2L_{p,q,m+n}\right) + k^2(p^2+4q)F_{p,q,m+n} \\
&= \left(1 + k^2(p^2+4q)\right)F_{p,q,m+n} + 2kL_{p,q,m+n}.
\end{aligned}$$

■

Theorem 2.7. (d'Ocagne's identity) Let p, q, k, m and n be positive integers. For $m \geq n$, we have

$$S_{p,q,m}S_{p,q,n+1} - S_{p,q,m+1}S_{p,q,n} = (-q)^n\left(1 - k^2(p^2+4q)\right)F_{p,q,m-n}.$$

Proof. Let p, q, k and $m \geq n$ be positive integers. By using the equation (4), we have

$$\begin{aligned}
S_{p,q,m}S_{p,q,n+1} - S_{p,q,m+1}S_{p,q,n} &= \left[\left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right) + k(r_1^m + r_2^m)\right]\left[\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) + k(r_1^{n+1} + r_2^{n+1})\right] \\
&\quad - \left[\left(\frac{r_1^{m+1} - r_2^{m+1}}{r_1 - r_2}\right) + k(r_1^{m+1} + r_2^{m+1})\right]\left[\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) + k(r_1^n + r_2^n)\right] \\
&= \frac{-r_1^m r_2^{n+1} - r_1^{n+1}r_2^m + r_1^{m+1}r_2^n + r_1^n r_2^{m+1}}{(r_1 - r_2)^2} + k\left(\frac{2r_1^{m+n+1} - 2r_2^{m+n+1} - 2r_1^{m+n+1} + 2r_2^{m+n+1}}{r_1 - r_2}\right) \\
&\quad + k^2(r_1^m r_2^{n+1} + r_1^{n+1}r_2^m - r_1^{m+1}r_2^n - r_1^n r_2^{m+1}) \\
&= \frac{(r_1^m r_2^n - r_1^n r_2^m)(r_1 - r_2)}{(r_1 - r_2)^2} - k^2(r_1^m r_2^n - r_1^n r_2^m)(r_1 - r_2) \\
&= \frac{r_1^n r_2^n(r_1^{m-n} - r_2^{m-n})}{r_1 - r_2} - k^2\frac{r_1^n r_2^n(r_1^{m-n} - r_2^{m-n})(r_1 - r_2)^2}{r_1 - r_2}
\end{aligned}$$

$$= (-q)^n F_{p,q,m-n} - k^2(-q)^n F_{p,q,m-n}(p^2+4q)$$

$$= (-q)^n\left(1-k^2(p^2+4q)\right)F_{p,q,m-n}. \quad \blacksquare$$

Using Theorem 2.7, we obtain the d'Ocagne's identity for each sequence in Corollary 2.1. So, we can write following remark.

Remark 2.4. We have the following d'Ocagne's identities for all sequences in Corollary 2.1:

- The (p,q)-Fibonacci sequences $\{F_{p,q,n}\}_{n\geq 0}$: $F_{p,q,m}F_{p,q,n+1} - F_{p,q,m+1}F_{p,q,n} = (-q)^n F_{p,q,m-n}$
- The generalize the Fibonacci sequence $\{\mathcal{F}_n\}_{n\geq 0}$: $\mathcal{F}_m\mathcal{F}_{n+1} - \mathcal{F}_{m+1}\mathcal{F}_n = (-b)^n F_{a,b,m-n}$
- The Fibonacci sequence $\{F_n\}_{n\geq 0}$: $F_mF_{n+1} - F_{m+1}F_n = (-1)^n F_{1,1,m-n}$
- The Pell sequence $\{P_n\}_{n\geq 0}$: $P_mP_{n+1} - P_{m+1}P_n = (-1)^n F_{2,1,m-n}$
- The Jacobsthal sequence $\{J_n\}_{n\geq 0}$: $J_mJ_{n+1} - J_{m+1}J_n = (-2)^n F_{1,2,m-n}$
- The Fibonacci-Like sequence $\{S_n\}_{n\geq 0}$: $S_mS_{n+1} - S_{m+1}S_n = (-1)^{n+1}4F_{1,1,m-n}$
- The sequence $\{V_n\}_{n\geq 0}$: $V_mV_{n+1} - V_{m+1}V_n = (-1)^{n+1}2^{n+3}F_{1,2,m-n}$

Now, we give sums formulas of terms of the generalized (p,q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$.

Theorem 2.8. Sums of the first n terms of the generalized (p,q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$ are

$$\sum_{i=0}^{n} S_{p,q,i} = \frac{1+k(2-p) - S_{p,q,n+1} - qS_{p,q,n}}{1-p-q}.$$

Proof. Let n be positive integer. By using the equation (4), we have

$$\sum_{i=0}^{n} S_{p,q,i} = \sum_{i=0}^{n}\left[\left(\frac{r_1^i - r_2^i}{r_1-r_2}\right) + k\left(r_1^i + r_2^i\right)\right]$$

$$= \frac{1}{r_1-r_2}\left(\sum_{i=0}^{n} r_1^i - \sum_{i=0}^{n} r_2^i\right) + k\left(\sum_{i=0}^{n} r_1^i + \sum_{i=0}^{n} r_2^i\right)$$

$$= \frac{1}{r_1-r_2}\left(\frac{1-r_1^{n+1}}{1-r_1} - \frac{1-r_2^{n+1}}{1-r_2}\right) + k\left(\frac{1-r_1^{n+1}}{1-r_1} + \frac{1-r_2^{n+1}}{1-r_2}\right)$$

$$= \frac{1}{(1-r_1)(1-r_2)}\left(\frac{r_1-r_2}{r_1-r_2} - \frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2} + \frac{r_1^n - r_2^n}{r_1-r_2}(r_1r_2)\right)$$

$$+ \frac{k}{(1-r_1)(1-r_2)}\left(2-(r_1+r_2) - (r_1^{n+1}+r_2^{n+1}) + (r_1^n + r_2^n)(r_1r_2)\right)$$

$$= \frac{1}{(1-r_1)(1-r_2)}\left(\frac{r_1-r_2}{r_1-r_2} + 2k - k(r_1+r_2)\right)$$

$$+ \frac{1}{(1-r_1)(1-r_2)}\left[-\left(\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2} + k(r_1^{n+1}+r_2^{n+1})\right) + \left(\frac{r_1^n - r_2^n}{r_1-r_2} + k(r_1^n+r_2^n)(r_1r_2)\right)\right]$$

$$= \frac{1+k(2-p) - S_{p,q,n+1} - qS_{p,q,n}}{1-p-q}. \quad \blacksquare$$

Theorem 2.9. Sums of the first n terms with odd indices of the generalized (p,q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$ are

$$\sum_{i=0}^{n} S_{p,q,2i+1} = \frac{1+kp(1+q) - q - S_{p,q,2n+3} + q^2S_{p,q,2n+1}}{1-p^2-2q+q^2}.$$

Proof. Let n be positive integer. By using the equation (4), we have

$$\sum_{i=0}^{n} S_{p,q,2i+1} = \sum_{i=0}^{n}\left[\left(\frac{r_1^{2i+1}-r_2^{2i+1}}{r_1-r_2}\right)+k\left(r_1^{2i+1}+r_2^{2i+1}\right)\right]$$

$$=\frac{1}{r_1-r_2}\left(r_1\sum_{i=0}^{n} r_1^{2i}-r_2\sum_{i=0}^{n} r_2^{2i}\right)+k\left(r_1\sum_{i=0}^{n} r_1^{2i}+r_2\sum_{i=0}^{n} r_2^{2i}\right)$$

$$=\frac{1}{r_1-r_2}\left[r_1\left(\frac{1-r_1^{2(n+1)}}{1-r_1^2}\right)-r_2\left(\frac{1-r_2^{2(n+1)}}{1-r_2^2}\right)\right]+k\left[r_1\left(\frac{1-r_1^{2(n+1)}}{1-r_1^2}\right)+r_2\left(\frac{1-r_2^{2(n+1)}}{1-r_2^2}\right)\right]$$

$$=\frac{1}{(1-r_1^2)(1-r_2^2)}\left(\frac{(r_1-r_2)(1+r_1r_2)}{r_1-r_2}-\frac{r_1^{2n+3}-r_2^{2n+3}}{r_1-r_2}+\frac{r_1^{2n+1}-r_2^{2n+1}}{r_1-r_2}(r_1r_2)^2\right)$$

$$+\frac{k}{(1-r_1^2)(1-r_2^2)}\left((r_1+r_2)(1-r_1r_2)-(r_1^{2n+3}+r_2^{2n+3})+(r_1^{2n+1}+r_2^{2n+1})(r_1r_2)^2\right)$$

$$=\frac{1}{(1-r_1^2)(1-r_2^2)}\left(1+r_1r_2+k(r_1+r_2)(1-r_1r_2)\right)$$

$$-\frac{1}{(1-r_1^2)(1-r_2^2)}\left(\frac{r_1^{2n+3}-r_2^{2n+3}}{r_1-r_2}+k(r_1^{2n+3}+r_2^{2n+3})\right)$$

$$+\frac{1}{(1-r_1^2)(1-r_2^2)}\left(\frac{r_1^{2n+1}-r_2^{2n+1}}{r_1-r_2}+k(r_1^{2n+1}+r_2^{2n+1})\right)(r_1r_2)^2$$

$$=\frac{1+kp(1+q)-q-S_{p,q,2n+3}+q^2S_{p,q,2n+1}}{1-p^2-2q+q^2}.$$ ■

Theorem 2.10. Sums of the first n terms with even indices of the generalized (p,q)-Fibonacci-Like sequences $\{S_{p,q,n}\}$ are

$$\sum_{i=0}^{n} S_{p,q,2i}=\frac{p+k(2-p^2-2q)-S_{p,q,2n+2}+q^2S_{p,q,2n}}{1-p^2-2q+q^2}.$$

Proof. Let n be positive integer. By using the equation (4), we have

$$\sum_{i=0}^{n} S_{p,q,2i}=\sum_{i=0}^{n}\left[\left(\frac{r_1^{2i}-r_2^{21}}{r_1-r_2}\right)+k\left(r_1^{2i}+r_2^{2i}\right)\right]$$

$$=\frac{1}{r_1-r_2}\left(\sum_{i=0}^{n} r_1^{2i}-\sum_{i=0}^{n} r_2^{2i}\right)+k\left(\sum_{i=0}^{n} r_1^{2i}+\sum_{i=0}^{n} r_2^{2i}\right)$$

$$=\frac{1}{r_1-r_2}\left[\frac{1-r_1^{2(n+1)}}{1-r_1^2}-\frac{1-r_2^{2(n+1)}}{1-r_2^2}\right]+k\left[\frac{1-r_1^{2(n+1)}}{1-r_1^2}+\frac{1-r_2^{2(n+1)}}{1-r_2^2}\right]$$

$$=\frac{1}{(1-r_1^2)(1-r_2^2)}\left(\frac{r_1^2-r_2^2}{r_1-r_2}-\frac{r_1^{2n+2}-r_2^{2n+2}}{r_1-r_2}+\frac{r_1^{2n}-r_2^{2n}}{r_1-r_2}(r_1r_2)^2\right)$$

$$+\frac{k}{(1-r_1^2)(1-r_2^2)}(2-r_1^2-r_2^2-(r_1^{2n+2}+r_2^{2n+2})+(r_1^{2n}+r_2^{2n})(r_1r_2)^2)$$

$$=\frac{1}{(1-r_1^2)(1-r_2^2)}\left(r_1+r_2+k(2-r_1^2-r_2^2)\right)$$
$$-\frac{1}{(1-r_1^2)(1-r_2^2)}\left(\frac{r_1^{2n+2}-r_2^{2n+2}}{r_1-r_2}+k(r_1^{2n+2}+r_2^{2n+2})\right)$$
$$+\frac{1}{(1-r_1^2)(1-r_2^2)}\left(\frac{r_1^{2n}-r_2^{2n}}{r_1-r_2}+k(r_1^{2n}+r_2^{2n})\right)(r_1r_2)^2$$
$$=\frac{p+k(2-p^2-2q)-S_{p,q,2n+2}+q^2S_{p,q,2n}}{1-p^2-2q+q^2}. \quad \blacksquare$$

4. Discussion

In this study we defined new recurrence sequences called the generalized (p,q)-Fibonacci-Like sequences. Some properties involving terms of these sequences, Binet's formula and generating functions were presented. We obtained the fundamental relationship between these sequences and well-known the (p,q)-Fibonacci sequences, the generalize the Fibonacci sequences, the Fibonacci sequence, the Pell sequence, the Jacobsthal sequence, some Fibonacci-Like sequences. We gave some fundamental identities like Cassini's identity, Catalan's identity, Vajda's identity and d'Ocagne's identity for the generalized (p,q)-Fibonacci-Like sequences and their sums.

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References

Falcon, S. (2001). On the k-Lucas Numbers. *International Journal of Contemporary Mathematical Sciences*, *6*(21), 1039-1050.

Falcon, S., & Plaza, A. (2007). On the k-Fibonacci Numbers. *Chaos, Solitons and Fractals*, *32*(5), 1615-1624. https://doi.org/10.1016/j.chaos.2006.09.022

Gupta, V. K., Panwar, Y. K., & Sikhwal, O. (2012). Generalized Fibonacci Sequences. *Theoretical Mathematics & Applications*, *2*(2), 115-124.

Harne, S. (2014). Generalized Fibonacci-Like Sequence and Fibonacci Sequence. *International Journal of Contemporary Mathematical Sciences*, *9*(5), 235-241. https://doi.org/10.12988/ijcms.2014.4218

Horadam, A. F. (1961). A Generalized Fibonacci Sequence. *American Mathematical Monthly*, *68*(5), 455-459. https://doi.org/10.1080/00029890.1961.11989696

Horadam, A. F. (1971). Pell Identities. *The Fibonacci Quarterly*, *9*(3), 245-252, 263.

Horadam, A. F. (1996). Jacobsthal Representation Numbers. *The Fib. Quart*, *34*, 40-54.

Kalman, D., & Mena, R. (2002). The Fibonacci Numbers–Exposed. *The Mathematical Magazine*, 2.

Koshy, T. (2001). *Fibonacci and Lucas numbers with applications*, New York, Wiley-Interscience. https://doi.org/10.1002/9781118033067

Panwar, Y. K., Rathore, G. P. S., & Chawla, R. (2014). On the k-Fibonacci-like numbers. *Turkish J. Anal. Number Theory*, *2*(1), 9-12. https://doi.org/10.12691/tjant-2-1-3

Singh, B., Sikhwal, O., & Bhatnagar, S. (2010). Fibonacci-Like Sequence and its Properties. *Int. J. Contemp. Math. Sciences*, *5*(18), 859-868.

Singh, M., Sikhwal, O., & Gupta, Y. K. (2014). Identities of Generalized Fibonacci-Like Sequence. *Turkish Journal of Analysis and Number Theory*, *2*(5), 170-175. https://doi.org/10.12691/tjant-2-5-3

Suvarnamani, A., & Tatong, M. (2015). Some Properties of (p,q)-Fibonnacci Numbers, *Science and Technology RMUTT Journal*, *5*(2), 17-21.

Suvarnamani, A., & Tatong, M. (2016). Some Properties of (p,q)-Lucas Numbers. *Kyungpook Mathematical Journal*, *56*(2), 367-370. https://doi.org/10.5666/KMJ.2016.56.2.367

Tasyurdu, Y., Cobanoğlu, N., & Dilmen, Z. (2016). On The a New Family of k-Fibonacci Numbers, *Erzincan University*

Journal of Science and Thechnology, *9*(1), 95-101. https://doi.org/10.18185/eufbed.01209

Wani, A. A., Catarino, P., & Rafiq, R. U. (2018). On the Properties of *k*-Fibonacci-Like Sequence. *International Journal of Mathematics And its Applications*, *6*(1-A), 187-198.

Wani, A. A., Rathore, G. P. S., & Sisodiya, K. (2016). On The Properties of Fibonacci-Like Sequence. *International Journal of Mathematics Trends and Technology*, *29*(2), 80-86. https://doi.org/10.14445/22315373/IJMTT-V29P513