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Nonlinear Hybrid Procedures for Solving Some Nonlinear Equations

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Abstract

The goal of this paper is to study a classic problem used in several fields such as science and engeneering. It's solving nonlinear equations:

$$f(x) = 0 \tag{1}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is considered of class C^1 on the interval I containing x^* the solution of (1) .

We propose in this research a new method for solving nonlinear equations. A convergence acceleration result is established and numerical examples are given.

Keywords: hybrid procedures, nonlinear equations

1. Introduction

Nonlinear equations are widely used to mathematically model many scientific computing problems. A large number of works have studied the resolution of nonlinear equations using Newton's method and its variants (Chun, 2007; Dennis & Schnabel, 1983; Frontini & Sormani, 2003; Ortega & Rheinboldt, 1970). Newton's method is defined as follows (Dennis & Schnabel, 1983; Ortega & Rheinboldt):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_0 \in I$$

and

$$|x_{n+1} - x^*| \leq C|x_n - x^*|^2$$

The advantage is its quadratic convergence (Dennis & Schnabel, 1983; Ortega & Rheinboldt), but its major drawback is calculating the derivative for each iteration (Dennis & Schnabel, 1983). When the function f is not explicitly defined, the derivative is not always obtainable.

Our objective is to use hybrid procedures (Brezinski & Chehab, 1998) to build a method whose convergence is at least quadratic, without computing the derivative for each iteration. Hybrid procedures were used by C. Brezinski and J.-P. Chehab in (Brezinski & Chehab, 1998) for solving fixed point problems. They were also studied by B. Rhanizar, In (Rhanizar, 1999) for solving optimization problems.

Let x' and x'' be two approximate solutions for the equation (1), hybrid procedures construct a new approximate solution y defined by:

$$y = \alpha x'' + (1 - \alpha)x'$$

The parameter α is chosen to minimise the absolute value of the residual ρ defined by:

$$\rho = \alpha f(x'') + (1 - \alpha) f(x')$$

Let x_n be an approximate solution to the equation (1), we consider the new approximate solution y_n defined by:

$$y_n = x_n + h f(x_n), \quad n \geq 0 \ \ with \quad 0 < |h| \leq 1$$

We assume that:

$$x_{n+1} = \alpha_n x''_n + (1 - \alpha_n)x'_n$$

where the approximate solutions x'_n and x''_n are given by: $x'_n = x_n$ and $x''_n = y_n$

It is easy to see that the approximation defined by hybrid procedures will be given by:

$$x_{n+1} = x_n - h\frac{f(x_n)}{f(x_n + hf(x_n)) - f(x_n)} f(x_n) \tag{2}$$

2. Convergence Study

We have the following theorem:

Theorem 1 *Let f be of class C^2 on the interval I containing x^* and suppose that there exists m, M and K such that:*

$$m \leq |f^{'}(x)| \leq M \; et \; |f^{''}(x)| \leq K \quad \forall x \in I$$

Then:

1. *There exists a neighborhood of x^* such that, for every x_0 belonging to this neighborhood the sequence (x_n) converges to x^*.*
2. $\exists C_1 > 0$ *such that:*

$$|x_{n+1} - x^*| \leq C_1 |x_n - x^*|^2$$

where $C_1 = \frac{K}{m}(1 + |h|M)$

Proof. Using the mean value theorem to f, we have:

$$f(x_n) = f^{'}(\beta_n)(x_n - x^*) \tag{3}$$

$$f(x_n + hf(x_n)) - f(x_n) = f^{'}(\alpha_n)hf(x_n) \tag{4}$$

where:

$$\beta_n = x^* + s_n(x_n - x^*), \qquad s_n \in]0,\ 1[\tag{5}$$

$$\alpha_n = x_n + t_n hf(x_n), \qquad t_n \in]0,\ 1[\tag{6}$$

$$= x_n + t_n hf^{'}(\beta_n)(x_n - x^*).$$

And by (2) and (4), we deduced:

$$(x_{n+1} - x^*) = (x_n - x^*) - \frac{f(x_n)}{f^{'}(\alpha_n)}$$

Then by (3) we have:

$$(x_{n+1} - x^*)f^{'}(\alpha_n) = (x_n - x^*)f^{'}(\alpha_n) - f^{'}(\beta_n)(x_n - x^*)$$

By applying the mean value theorem to $f^{'}$ and by using (5) and (6) we get:

$$(x_{n+1} - x^*)f^{'}(\alpha_n) = (x_n - x^*)^2 f^{''}(\gamma_n)(1 - s_n + t_n hf^{'}(\beta_n))$$

which gives:

$$x_{n+1} - x^* = (x_n - x^*)^2 \frac{f^{''}(\gamma_n)}{f^{'}(\alpha_n)}(1 - s_n + t_n hf^{'}(\beta_n))$$

i.e.

$$|x_{n+1} - x^*| \leq C_1 |x_n - x^*|^2 \tag{7}$$

where: $C_1 = \frac{K}{m}(1 + |h|M)$

We deduced then by induction that:

$$C_1|x_n - x^*| \leq [C_1|x_O - x^*|]^{2^n} \tag{8}$$

where x_0 is chosen to verify: $C_1|x_O - x^*| < 1$

Thus, from (7) and (8), the sequence x_n converges to the solution of the equation (1) and the convergence is quadratic.

3. A Composite Method

We set:

$$y_n = x_{n+1} \tag{9}$$

where x_{n+1} is given by (2) and let we apply one more time hybrid procedures, we obtain then a new approximation $x_{n+1} = \alpha_n x_n^{''} + (1-\alpha_n)x_n^{'}$

where the approximate solutions $x_n^{'}$ and $x_n^{''}$ are given by: $x_n^{'} = x_n$ and $x_n^{''} = y_n$

The equation (2) implies that the approximate solution defined by hybrid procedures will be given by:

$$x_{n+1} = x_n - h\frac{f(x_n)}{f(x_n + hf(x_n)) - f(x_n)}\frac{f(x_n)}{f(y_n) - f(x_n)} \tag{10}$$

We have the following theorem:

Theorem 2 *Under the assumptions of theorem 1, we have:*

1. *There exists a neighborhood of x^* such that, for every x_0 belonging to this neighborhood the sequence (x_n) converges to x^*.*
2. $\exists C_2 > 0$ *such that:*

$$|x_{n+1} - x^*| \simeq C_2|(x_n - x^*)|^3$$

where $C_2 = \frac{K^2}{2m^2}(1 + |h|M)$

Proof. By (2), (9) and (10), we have:

$$(x_{n+1} - x^*) = \frac{(x_n - x^*)f(y_n) - (y_n - x^*)f(x_n)}{f(y_n) - f(x_n)} \tag{11}$$

Then, by applying Taylor formula to f for the order 2, we have:

$$f(x_n) = f^{'}(x^*)(x_n - x^*) + \frac{f^{''}(x^*)}{2}(x_n - x^*)^2 + o(|(x_n - x^*)|^2)$$

and

$$f(y_n) = f^{'}(x^*)(y_n - x^*) + \frac{f^{''}(x^*)}{2}(y_n - x^*)^2 + o(|(y_n - x^*)|^2)$$

By applying the mean value theorem to f, replacing $f(x_n)$ and $f(y_n)$ by their values in (11), and then using Theorem 1 we have:

$$x_{n+1} - x^* = (y_n - x^*)(x_n - x^*)\frac{f^{''}(x^*)}{2f^{'}(\delta_n)} + \epsilon_n \tag{12}$$

where:

$$\epsilon_n = \frac{(x_n - x^*)o(|(y_n - x^*)|^2) - (y_n - x^*)o(|(x_n - x^*)|^2)}{f^{'}(\delta_n)(y_n - x_n)}$$

Setting $y_n - x_n = (y_n - x^*) - (x_n - x^*)$, factoring by $(x_n - x^*)$, and using theorem 1, we obtain:

$$\epsilon_n = \frac{o(|(y_n - x^*)|^2) - (y_n - x^*)\frac{o(|(x_n - x^*)|^2)}{x_n - x^*}}{f'(\delta_n)(\frac{y_n - x^*}{x_n - x^*} - 1)} \rightarrow 0 \quad (13)$$

Therefore, by theorem 1, we have: $|y_n - x^*| \leq C_1|x_n - x^*|^2$ and $m \leq |f'(\delta_n)|$. Thus by using (12) and (13), we finally have:

$$|x_{n+1} - x^*| \simeq C_2|(x_n - x^*)|^3$$

where $C_2 = \frac{K^2}{2m^2}(1 + |h|M)$

3. Numerical Experiments

In this section, we will give some numerical experiments for the purpose of comparing this new method based on nonlinear hybrid procedures with Newton's method and Secant method. These methods will be denoted respectively by **M.PHNL** , **M.NWT.** and **M.SEC.**. For the different examples, the comparison will be summarized in tables which give the number of iterations, the absolute value of the associated residue and the time to converge to the optimal solution for each method.

Example 1.

We consider the function:

$$f(x) = e^x - 1.5 - tan^{-1}(x)$$

where $x_0 = -7,\ \epsilon = 10^{-16}$

Table 1. Numerical results of example 1

Iterations	M.PHNL	M.NEW.	M.SEC
1	0.0223	0.0223	0.0223
2	0.0043	0.0044	0.0187
3	2.2608D-04	2.3902D-04	0.0013
4	7.1204D-07	7.9958D-07	1.1673D-04
5	7.1008D-16	9.0083D-12	2.1567D-06
6		0	3.5415D-09
7			1.0747D-13
8			0
CPU	0.010168 s	0.010284 s	0.0103 44 s

Example 2.

We consider the function:

$$f(x) = cos(x) - xe^x + x^2$$

where $x_0 = 0,\ \epsilon = 10^{-16}$

Table 2. Numerical results of example 2

Iterations	M.PHNL	M.NEW.	M.SEC
1	1	1	1
2	0.2684	1.1780	0.0163
3	0.0026	0.2218	7.7020D-04
4	2.4830D-09	0.0134	4.0829D-06
5	1.1102D-16	5.7482D-05	1.0179D-09
6		1.0692D-09	1.2768D-15
7		1.1102D-16	1.1102D-16
CPU	0.008762 s	0.008949	0.010741

Example 3.

We consider the function:

$$f(x) = sin(2x) - 1 + x$$

where $x_0 = 0.7,\ \epsilon = 10^{-16}$

Table 3. Numerical results of example 3

Iterations	M.PHNL	M.NEW.	M.SEC
1	0.4435	0.4435	0.4435
2	0.0014	0.0222	0.4707
3	4.3761D-10	9.6837D-05	0.0956
4	5.5511D-16	1.9069D-09	0.0148
5		5.511D-16	3.0644D-04
6			9.1552D-07
7			5.7072D-11
8			5.5511D-16
CPU	0.013363 s	0.011394 s	0.018114 s

Example 4. (Annuity Rate)

The objective is to compute the average annuity rate I of an investment fund for many years. We invest in a fund of $V = 1000$ *euros* annually. After 5 years we have an amount of $M = 6000$ *euros*. The relation between those three parameters M, V, I and the number of years n is as follows:

$$M - V\frac{1+I}{I}[(1+I)^n - 1] = 0$$

The problem is represented as a nonlinear equation, of which we are not able to found an exact solution. We consider the function:

$$f(I) = 6000 - 1000\frac{1+I}{I}((1+I)^5 - 1)$$

where: $I_0 = 0.05,\ \epsilon = 10^{-16}$

Table 4. Numerical results of example 4

Iterations	M.PHNL	M.NEW.	M.SEC
1	0.4435	0.4435	0.4435
2	0.0014	0.0222	0.3689
3	4.3761D-10	9.6837D-05	6.7256D-04
4	5.5511D-16	1.9069D-09	1.8270D-08
5		5.511D-16	2.7284D-12
6			1.0913D-11
7			4.5474D-12
8			8.1854D-12
9			1.8189D-12
10			9.0949D-13
CPU	0.014224 s	0.015270 s	0.008676 s

4. Conclusions

Throughout these examples, the methods based on nonlinear hybrid procedures give better results and have shown their efficiency. Moreover, they have a convergence at least quadratic, don't use the derivative in each iteration and easy to manipulate.

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