Journal of Mathematics Research; Vol. 11, No. 5; October 2019
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Exact Solution of Linear and Nonlinear Wave Equations by Double Laplace and Iterative Method

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Received: July 4, 2019 Accepted: August 20, 2019 Online Published: September 16, 2019

doi:10.5539/jmr.v11n5p33 URL: https://doi.org/10.5539/jmr.v11n5p33

Abstract

A coupling of double Laplace transform with Iterative method is used to solve linear and nonlinear wave equations subject to initial and boundary conditions. The iteration process leads to disappearance of noise terms and exact solution is obtained at first iteration. Through several examples, the convenience and efficiency of the method is demonstrated, showing its usefulness to overcome difficulties associated with some existing techniques.

Keywords: double Laplace transform, Inverse double Laplace transform, iterative method, wave equations

1. Introduction

The wave equation is known as one of three fundamental equations in mathematical physics and occurs in many branches of physics, applied mathematics, and engineering (Eltayeb & Kiliciman, 2013).

Many attempts have been made to develop analytic and approximate methods to solve the linear and nonlinear wave equations (Wazwaz, 2007, 2001, Sadighi & Ghanji, 2007, Al-Jawary, 2015). Although such methods have been successfully applied but some difficulties have appeared , for examples, in calculating Adomian polynomials to handle the nonlinear terms in ADM (Wazwaz, 2001), calculation of Lagrange multiplier in VIM (Wazwaz, 2007, Sadighi & Ghanji, 2007), also application of homotopy perturbation method requires construction of a homotopy and solving the corresponding algebraic equations.

In this paper, double Laplace transform combined with iterative method developed by Dhunde and Waghmare is used to solve the linear and nonlinear wave equation. The method which demonstrates simplicity in its principles have been used to solve non-linear telegraph equation (Dhunde & Waghmare, 2016a) and non-linear Klein-Gordon equation (Dhunde & Waghmare, 2016b).

The method is simple to understand and easy to implement. It has no specific requirements such as linearization, perturbation, small parameters etc. for nonlinear operators as required by some existing techniques.

This paper has been organized as follows. Section 2 is about the theoretical background of the work. In section 3, the double Laplace transform coupled with iterative method is applied to a generalized non-linear wave equation, while Section 4 is about application to linear and nonlinear wave equations and finally in section 5 the discussion of result is presented.

2. Theoretical Background

Linear and nonlinear wave equations are given by (Wazwaz, 2001, Al-Jawary, 2015)

$$u_{tt}(x,t) = u_{xx}(x,t) + f(u) \quad ; \quad 0 < x < L \tag{2.1}$$

and

$$u_{tt}(x,t) = u_{xx}(x,t) + F(u) + g(x,t) \quad ; \quad 0 < x < L \tag{2.2}$$

respectively.

The functions $f(u), F(u)\ and\ g(x,t)$ are linear, nonlinear and source functions respectively.

The wave equation plays an important role in various physical problems. Study of the wave equation is needed in diverse areas of engineering and scientific applications.

Recall the following definitions (Eltayeb & Kiliciman, 2013, Eltayeb, 2017).

The double Laplace transform is defined by

$$L_xL_t[f(x,s)] = F(p,s) = \int_0^\infty e^{-px}\left(\int_0^\infty e^{-st} f(x,t)dt\right)dx, \tag{2.3}$$

Whenever the integral exist, where $f(x,t)$ is a function of two variables x and t defined in the positive quadrant of the $xt-plane$, $x,t>0$ and p,s are complex numbers.

Double Laplace transform for second partial derivative with respect to x is given by

$$L_{xx}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = p^2F(p,s) - pF(0,s) - \frac{\partial F}{\partial x}(0,s), \tag{2.4}$$

And double Laplace for second partial derivative with respect to t is given by

$$L_{tt}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = s^2F(p,s) - sF(p,0) - \frac{\partial F}{\partial t}(p,0)\,, \tag{2.5}$$

from (2.3) ; we deduce (Dhunde & Waghmare, 2016)

$$L_xL_t[f(x)g(t)] = \bar{f}(p)\bar{g}(s) = L_x[f(x)]L_t[g(t)] \tag{2.6}$$

Further the double Laplace transform of second order partial derivatives are given by

$$L_xL_t\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = p^2\bar{f}\,(p,s) - p\bar{f}\,(0,s) - \bar{f}_x(0,s) \tag{2.7}$$

$$L_xL_t\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = s^2\bar{f}\,(p,s) - s\bar{f}\,(p,0) - \bar{f}_t(p,0) \tag{2.8}$$

The inverse double Laplace transform

$L_x^{-1}L_t^{-1}\left[\bar{f}(p,s)\right] = f(x,t)$ is defined by the complex double integral formula

$$L_x^{-1}L_t^{-1}\left[\bar{f}(p,s)\right] = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}dp\left(\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\,\bar{f}\,(p,s)ds\right) \tag{2.9}$$

Where $\bar{f}(p,s)$ must be an analytical function for all p and s in the region defined by the inequalities $Re\,p \geq c$ and $Re\ s\ \geq d, where\ c\ and\ d$ are real constants to be chosen suitably.

3. Double Laplace Transform and Iterative Method

Now consider the non linear wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + Fu(x,t) + g(x,t); \quad 0<x<L \tag{3.1}$$

with the initial conditions

$$u(x,0) = h_1(x)\,; \qquad \frac{\partial u}{\partial t}(x,0) = h_2(x) \tag{3.2}$$

and the boundary conditions

$$u(0,t) = h_1(t)\,; \qquad u_x(0,t) = h_2(t) \tag{3.3}$$

where $u(x,t)$ represents the wave displacement at position x and time $t, F(u)$ is the nonlinear force where

$\frac{\partial F}{\partial u} \geq 0\,; g(x,t);\ h_1(x);\ h_2(x);\ h_1(t)\ and\ \ h_2(t)\ are\ known\ functions.$

Application of double Laplace transform on (3.1) gives

$$s^2\bar{u}(p,s) - s\bar{u}(p,0) - \bar{u}_t(p,0) = p^2\bar{u}(p,s) - p\bar{u}(0,s) - \bar{u}_x(0,s) +$$

$$L_xL_t[F\big(u(x,t)\big) + g(x,t)] \tag{3.4}$$

Application of single Laplace transform to (3.2) and (3.3) leads to

$$\bar{u}(p,0) = \bar{h}_1(p);\ \ \bar{u}_t(p,0) = \bar{h}_2(p);\ \ \bar{u}(0,s) = \bar{h}_1(s)\ and\ \ \bar{u}_x(0,s) = \bar{h}_2(s) \tag{3.5}$$

Substituting (3.5) in (3.4) and simplifying; we have

$$\bar{u}(p,s) = \frac{s\bar{h}_1(p)+\bar{h}_2(p)-p\bar{h}_1(s)-\bar{h}_2(s)}{s^2-p^2} + \frac{1}{s^2-p^2}\{L_xL_t[F\big(u(x,t)\big) + g(x,t)]\} \tag{3.6}$$

Application of inverse double Laplace transform to (3.6) leads to

$$u(x,t) = L^{-1}{}_xL^{-1}{}_t\left[\frac{s\bar{h}_1(p) + \bar{h}_2(p) - p\bar{h}_1(s) - \bar{h}_2(s)}{s^2 - p^2}\right]$$

$$+L^{-1}{}_xL^{-1}{}_t\left[\frac{1}{s^2-p^2}\{L_xL_t\big[F\big(u(x,t)\big) + g(x,t)\big]\}\right] \tag{3.7}$$

Applying the Iterative method

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{3.8}$$

Substituting (3.8) in (3.7), we obtain

$$\sum_{i=0}^{\infty} u_i(x,t) = L^{-1}{}_xL^{-1}{}_t\left[\frac{s\bar{h}_1(p) + \bar{h}_2(p) - p\bar{h}_1(s) - \bar{h}_2(s)}{s^2 - p^2}\right]$$

$$+L^{-1}{}_xL^{-1}{}_t\left[\frac{1}{s^2-p^2}\{L_xL_t[F(\textstyle\sum_{i=0}^{\infty} u_i(x,t)) + g(x,t)]\}\right] \tag{3.9}$$

The nonlinear term F is decomposed as

$$F(\textstyle\sum_{i=0}^{\infty} u_i(x,t)) = F\big(u_0(x,t)\big) + \sum_{i=1}^{\infty}[F\big(\sum_{k=0}^{i} u_k(x,t)\big) - F\big(\sum_{k=0}^{i-1} u_k(x,t)\big)] \tag{3.10}$$

Put (3.10) in (3.9) to obtain

$$\sum_{i=0}^{\infty} u_i(x,t) = L^{-1}{}_xL^{-1}{}_t\left[\frac{s\bar{h}_1(p) + \bar{h}_2(p) - p\bar{h}_1(s) - \bar{h}_2(s)}{s^2 - p^2}\right]$$

$$+L^{-1}{}_xL^{-1}{}_t\left[\frac{1}{s^2 - p^2}\{L_xL_t\big[F\big(u_0(x,t)\big) + g(x,t)\big]\}\right]$$

$$+L^{-1}{}_xL^{-1}{}_t\left[\frac{1}{s^2-p^2}\{L_xL_t\big[\textstyle\sum_{i=1}^{\infty} F(\sum_{k=0}^{i} u_k(x,t) - F(\sum_{k=0}^{i-1} u_k(x,t)))\big]\}\right] \tag{3.11}$$

we then define the recurrence relations as

$$u_0(x,t) = L^{-1}{}_xL^{-1}{}_t\left[\frac{s\bar{h}_1(p)+\bar{h}_2(p)-p\bar{h}_1(s)-\bar{h}_2(s)}{s^2-p^2}\right] \tag{3.12}$$

$$u_1(x,t) = L^{-1}{}_xL^{-1}{}_t\left[\frac{1}{s^2-p^2}\{L_xL_t\big[F\big(u_0(x,t)\big) + g(x,t)\big]\}\right] \tag{3.13}$$

and

$$u_{n+1}(x,t) = \ \ L^{-1}{}_xL^{-1}{}_t\left[\frac{1}{s^2-p^2}\{L_xL_t\big[\textstyle\sum_{i=1}^{\infty} F\big(\sum_{k=0}^{i} u_k(x,t) - F\big(\sum_{k=0}^{i-1} u_k(x,t)\big)\big)\big]\},\ n \geq 1\right] \tag{3.14}$$

The recurrence relation generates the solution of (3.1) in series form given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots + u_n(x,t) + \dots \tag{3.15}$$

4. Applications

Here, we applied double Laplace transform couple with the iterative method to solve five examples of wave equations. The aim is to demonstrate the simplicity and efficiency of the method.

Example 4.1. Consider the inhomogenous non-linear wave equation:

$$u_{tt}(x,t) = u_{xx}(x,t) + u + u^2 - xt - x^2t^2\,; \;\; 0 < x < \pi, \;\; t > 0 \tag{4.1}$$

with initial conditions

$$u(x,0) = 0\;; \qquad u_t(x,0) = x\,, \tag{4.2}$$

and boundary conditions

$$u(0,t) = 0\;; \qquad u(\pi,t) = \pi t\,, \tag{4.3}$$

applying the double Laplace transform on both sides of (4.1), we have

$$s^2\bar{u}(p,s) - s\bar{u}(p,0) - \bar{u}_t(p,0) - p^2\bar{u}(p,s) + p\bar{u}(0,s) + \bar{u}_x(0,s) = L_xL_t[u + u^2 - xt - x^2t^2] \tag{4.4}$$

further application of single Laplace transform to conditions (4.2) and (4.3) gives

$$\bar{u}(p,0) = 0;\;\; \bar{u}_t(p,0) = \frac{1}{p^2};\;\; \bar{u}(0,s) = 0 \; and \;\; \bar{u}_x(0,s) = \frac{1}{s^2} \tag{4.5}$$

Substitute (4.5) in (4.4) and simplify to obtain

$$\bar{u}(p,s) = \frac{1}{p^2s^2} + \frac{1}{(s-p)(s+p)}L_xL_t[u + u^2 - xt - x^2t^2] \tag{4.6}$$

applying inverse double Laplace transform on (4.6) , we obtain

$$u(x,t) = xt + {L_x}^{-1}{L_t}^{-1}\left\{\frac{1}{(s-p)(s+p)}L_xL_t[u + u^2 - xt - x^2t^2]\right\} \tag{4.7}$$

To apply the Iterative method, we substitute (3.8) in (4.7) and apply (3.12) – (3.14). thus the components of the solution are:

$$u_0(x,t) = xt\,, \tag{4.8}$$

$$u_1(x,t) = {L_x}^{-1}{L_t}^{-1}\left\{\frac{1}{s^2-p^2}L_xL_t[u_0 + {u_0}^2 - xt - x^2t^2]\right\} = 0 \tag{4.9}$$

$$u_{n+1}(x,t) = -{L_x}^{-1}{L_t}^{-1}\left\{\frac{1}{s^2-p^2}L_xL_t\left[\sum_{k=0}^{n}\big(u_k(x,t)\big)^2 - (\textstyle\sum_{k=0}^{n-1}u_k(x,t))^2\right]\right\}\;;\;\; n \ge 1 \tag{4.10}$$

$$u_2(x,t) = -{L_x}^{-1}{L_t}^{-1}\left\{\frac{1}{s^2-p^2}L_xL_t[(u_0(x,t) + u_1(x,t))^2 - (u_0(x,t))^2]\right\} \quad = 0 \tag{4.11}$$

Similarly, $u_3(x,t) = u_4(x,t) = u_5(x,t) = 0 \;\, and\ so\ on.$

Thus we obtain the solution of (4.1) as

$$u(x,t) = xt \tag{4.12}$$

which is the required solution.

Example 4.2. Consider the homogenous wave equation:

$$u_{tt}(x,t) = u_{xx}(x,t)\,; \;\; 0 < x < \pi, \;\; t > 0 \tag{4.13}$$

with initial conditions

$$u(x,0) = \sin x\,; \qquad u_t(x,0) = 0\,, \tag{4.14}$$

and boundary conditions

$$u(0,t) = 0\;; \qquad u(\pi,t) = 0\,; t \ge 0 \tag{4.15}$$

applying the double Laplace transform on both sides of (4.13), we have

$$s^2\bar{u}(p,s) - s\bar{u}(p,0) - \bar{u}_t(p,0) - p^2\bar{u}(p,s) + p\bar{u}(0,s) + \bar{u}_x(0,s) = 0 \tag{4.16}$$

further application of single Laplace transform to conditions (4.14) and (4.15) gives

$$\bar{u}(p,0) = \frac{1}{p^2+1};\;\; \bar{u}_t(p,0) = 0;\;\; \bar{u}(0,s) = 0 \; and \;\; \bar{u}_x(0,s) = \frac{s}{s^2+1} \tag{4.17}$$

Substitute (4.17) in (4.16) and simplify to obtain

$$\bar{u}(p,s) = \frac{s}{(p^2+1)(s^2+1)} \tag{4.18}$$

applying inverse double Laplace transform on (4.18) , we obtain

$$u(x,t) = {L_x}^{-1}{L_t}^{-1}\left\{\frac{s}{(p^2+1)(s^2+1)}\right\} = {L_x}^{-1}\left(\frac{1}{p^2+1}\right){L_t}^{-1}\left(\frac{s}{s^2+1}\right) \tag{4.19}$$

Thus we obtain the solution of (4.13) as

$$u(x,t) = \sin x \cos t \tag{4.20}$$

which is the required solution.

Example 4.3. Consider the wave equation:

$$u_{tt}(x,t) = u_{xx}(x,t) - 12x^2 \;;\;\; 0 < x < \pi,\;\; t > 0 \tag{4.21}$$

with initial conditions

$$u(x,0) = \; x^4 + \sin x \;; \qquad\qquad u_t(x,0) = 0\,, \tag{4.22}$$

and boundary conditions

$$u(0,t) = 0 \;; \qquad\qquad u(\pi,t) = \pi^4,\;\; t \geq 0 \tag{4.23}$$

applying the double Laplace transform on both sides of (4.21), we have

$$s^2\bar{u}(p,s) - s\bar{u}(p,0) - \bar{u}_t(p,0) - p^2\bar{u}(p,s) + p\bar{u}(0,s) + \bar{u}_x(0,s) = -L_xL_t(12x^2) \tag{4.24}$$

further application of single Laplace transform to conditions (4.22) and (4.23) gives

$$\bar{u}(p,0) = \frac{4!}{p^5} + \frac{1}{p^2+1};\;\; \bar{u}_t(p,0) = 0;\;\; \bar{u}(0,s) = 0 \; and \;\; \bar{u}_x(0,s) = \frac{s}{s^2+1} \tag{4.25}$$

Substitute (4.25) in (4.24) and simplify to obtain

$$\bar{u}(p,s) = \; \frac{s}{(p^2+1)(s^2+1)} + \frac{4!}{sp^5} \tag{4.26}$$

applying inverse double Laplace transform on (4.26) , we obtain

$$u(x,t) = {L_x}^{-1}{L_t}^{-1}\left[\frac{s}{(p^2+1)(s^2+1)}\right] + {L_x}^{-1}{L_t}^{-1}\left[\frac{4!}{sp^5}\right] \tag{4.27}$$

Thus we obtain the solution of (4.21) as

$$u(x,t) = \sin x \cos t + x^4 \tag{4.28}$$

which is the required solution.

Example 4.4. Consider the wave equation:

$$u_{tt}(x,t) = u_{xx}(x,t) + \sin x \;;\;\; 0 < x < \pi,\;\; t > 0 \tag{4.29}$$

with initial conditions

$$u(x,0) = \; \sin x \;; \qquad\qquad u_t(x,0) = \sin x\,, \tag{4.30}$$

and boundary conditions

$$u(0,t) = 0 \;; \qquad\qquad u(\pi,t) = 0\,, t \geq 0 \tag{4.31}$$

applying the double Laplace transform on both sides of (4.29), we have

$$s^2\bar{u}(p,s) - s\bar{u}(p,0) - \bar{u}_t(p,0) - p^2\bar{u}(p,s) + p\bar{u}(0,s) + \bar{u}_x(0,s) = L_xL_t[\sin x] \tag{4.32}$$

further application of single Laplace transform to conditions (4.30) and (4.31) gives

$$\bar{u}(p,0)=\frac{1}{p^2+1};\ \ \bar{u}_t(p,0)=\frac{1}{p^2+1};\ \ \bar{u}(0,s)=0\ and\ \ \bar{u}_x(0,s)=\frac{1}{s}+\frac{1}{s^2+1} \tag{4.33}$$

Substitute (4.33) in (4.32) and simplify to obtain

$$\bar{u}(p,s)=\frac{1}{s(p^2+1)}+\frac{1}{(p^2+1)(s^2+1)} \tag{4.34}$$

applying inverse double Laplace transform on (4.34) , we obtain

$$u(x,t)=L_x{}^{-1}L_t{}^{-1}\ \frac{1}{s(p^2+1)}+\ L_x{}^{-1}L_t{}^{-1}\ \frac{1}{(p^2+1)(s^2+1)} \tag{4.35}$$

Thus we obtain the solution of (4.29) as

$$u(x,t)=\sin x+\sin x\sin t \tag{4.36}$$

which is the required solution.

Example 4.5. Consider the homogenous wave equation:

$$u_{tt}(x,t)=u_{xx}(x,t)-3u(x,t)\,;\ \ 0<x<\pi,\ \ t>0 \tag{4.37}$$

with initial conditions

$$u(x,0)=0\ \ ; \qquad\qquad u_t(x,0)=2\cos x\,, \tag{4.38}$$

and boundary conditions

$$u(0,t)=\sin 2t\,; \qquad u(\pi,t)=-\sin 2t\,, \tag{4.39}$$

applying the double Laplace transform on both sides of (4.37), we have

$$s^2\bar{u}(p,s)-s\bar{u}(p,0)-\bar{u}_t(p,0)-p^2\bar{u}(p,s)+p\bar{u}(0,s)+\bar{u}_x(0,s)= \\ -L_xL_t(3u) \tag{4.40}$$

further application of single Laplace transform to conditions (4.38) and (4.39) gives

$$\bar{u}(p,0)=0;\ \ \bar{u}_t(p,0)=\frac{2p}{p^2+1};\ \ \bar{u}(0,s)=\frac{2}{s^2+4}\ and\ \ \bar{u}_x(0,s)=0 \tag{4.41}$$

Substitute (4.41) in (4.40) and simplify to obtain

$$\bar{u}(p,s)=\frac{2p}{(p^2+1)(s^2+4)}+\frac{6p}{(s^2-p^2)(p^2+1)(s^2+4)}-\frac{1}{(s-p)(s+p)}L_xL_t[3u] \tag{4.42}$$

applying inverse double Laplace transform on (4.42) , we obtain

$$u(x,t)=\cos x\sin 2t+L_x{}^{-1}L_t{}^{-1}\left\{\frac{1}{(s-p)(s+p)}\left[\frac{6p}{(p^2+1)(s^2+4)}-L_xL_t(3u)\right\}\right. \tag{4.43}$$

To apply the Iterative method, we substitute (3.8) in (4.43) and apply (3.12) – (3.14). Thus the components of the solution are:

$$u_0(x,t)=\cos x\sin 2t\,, \tag{4.44}$$

$$u_1(x,t)=\ L_x{}^{-1}L_t{}^{-1}\left\{\frac{1}{s^2-p^2}\left[\frac{6p}{(p^2+1)(s^2+4)}-3L_xL_t(u_0(x,t))\right\}\right.=0 \tag{4.45}$$

$$u_{n+1}(x,t)=\ L_x{}^{-1}L_t{}^{-1}\left\{\frac{1}{s^2-p^2}\left[\frac{6p}{(p^2+1)(s^2+4)}-3L_xL_t\left(\sum_{k=0}^{n}u_k(x,t)\right)\right\}\right.\ ;\ \ n\geq 1 \tag{4.46}$$

$$u_2(x,t)=\ L_x{}^{-1}L_t{}^{-1}\left\{\frac{1}{s^2-p^2}\left[\frac{6p}{(p^2+1)(s^2+4)}-3L_xL_t(u_0(x,t))\right\}\right.\ =0 \tag{4.47}$$

Similarly, $u_3(x,t)=u_4(x,t)=u_5(x,t)=0\ \ and\ so\ on.$

Thus we obtain the solution of (4.1) as

$$u(x,t)=\cos x\sin 2t \tag{4.48}$$

which is the required solution.

5. Discussion of Result

The aim of the paper is to apply a coupling of double Laplace transform and iterative method to solve linear and non-linear wave equations. This has been demonstrated successfully to get exact solutions to the equations. The iteration process leads to disappearance of noise terms and exact solutions of equations are obtained at first iteration. The simplicity of the method was demonstrated and it doesn't need linearization, perturbation, small parameters etc. for nonlinear operators necessary for some existing techniques.

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