Journal of Mathematics Research; Vol. 11, No. 5; October 2019
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Notes on Constraint Qualifications for Second-Order Optimality Conditions

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Received: July 10, 2019 Accepted: September 9, 2019 Online Published: September 15, 2019

doi:10.5539/jmr.v11n5p16 URL: https://doi.org/10.5539/jmr.v11n5p16

Abstract

In the first part of this paper we point out some basic properties of the critical cones used in second-order optimality conditions and give a simple proof of a strong second-order necessary optimality condition by assuming a "modified" first-order Abadie constraint qualification. In the second part we give some insights on second-order constraint qualifications related to second-order local approximations of the feasible set.

Keywords: second-order necessary optimality conditions, second-order constraint qualifications, second-order tangent sets, Abadie-type conditions

Mathematics Subject Classification (2010): 90C29, 90C30, 90C31, 49K27, 49K40

1. Introduction

The study of second-order optimality conditions for a nonlinear programming problem is a classical subject in mathematical programming theory: second-order optimality conditions have a long history that begins with a modern approach in the Karush's Master Thesis (1939). In more recent years several authors have been concerned with second-order (necessary and/or sufficient) optimality conditions, both from a theoretic and from an algorithmic point of view, producing various interesting papers. See, e. g., Andreani, Behling, Haeser and Silva (2017), Andreani, Birgin, Martinez and Schuverdt (2010), Andreani, Echagüe and Schuverdt (2010), Andreani, Martinez and Schuverdt (2007), Arutyunov (1998), Arutyunov and Pereira (2006), Baccari (2004), Baccari and Trad (2005), Bomze (2015), Bonnans and Shapiro (2000), Brinkhuis (2009), Casas and Troltzsch (2002), Cominetti (1990), Di (1996), Gfrerer (2007), Huy and Tuyen (2016), Kawasaki (1988), Minchenko and Leschov (2016), Penot (1998, 2000), Shen, Xue and An (2015).

For a survey on second-order optimality conditions in case of twice-continuously differentiable functions, see Giorgi (2019).

In particular, Andreani, Behling, Haeser and Silva (2017) are concerned with some second-order constraint qualifications of the Abadie-type. In the first part of this paper we shall obtain in a simple and direct way Theorem 3.3 of Andreani and others (2017), theorem previously given without proof by Bazaraa, Sherali and Shetty (2006). We shall exploit some properties of the contingent cone and shall make some remarks on classical necessary second-order optimality conditions for a nonlinear programming problem, in particular on the so-called "critical cones" utilized in these conditions. In the second part of the paper we shall make some considerations on second-order constraint qualifications expressed in terms of second-order local approximations of sets.

The problem considered is

$$(P): \quad \begin{cases} \min f(x) & \\ \text{subject to:} & g_i(x) \leqq 0, \ \forall i \in M; \\ & h_j(x) = 0, \ \forall j \in P, \end{cases}$$

where $M = \{1, ..., m\}$, $P = \{1, ..., p < n\}$, $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$; every $g_i : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., m$; every $h_j : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $j = 1, ..., p$. The functions involved in (P) are assumed to be twice-continuously differentiable on the open set $X \subset \mathbb{R}^n$. With reference to (P) let us denote by K its *feasible set;* given $x^0 \in K$ we define

$$I(x^0) = \left\{ i \in M : g_i(x^0) = 0 \right\}$$

as the set of indices of the *active constraints* at x^0. The *Lagrangian function* associated with (P) is defined as:

$$\mathcal{L}(x, u, w) = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} w_j h_j(x),$$

where $u_i \geqq 0$, $\forall i \in M$.

The *generalized Lagrangian function (or Fritz John-Lagrange function)* associated with (P) is defined as:

$$\mathcal{L}_1(x, u_0, u, w) = u_0 f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} w_j h_j(x)$$

where $u_0 \geqq 0$, $u_i \geqq 0$, $\forall i \in M$ (u_i, $i = 0, 1, ..., m$; $w_j \in \mathbb{R}$, $j = 1, ..., p$, not all zero).

As it is well-known, the two more used first-order necessary optimality conditions for (P) are given by the two following statements.

Theorem 1. (Karush (1939). Kuhn and Tucker (1951)). Let x^0 be a local minimizer of (P) and suppose that an appropriate constraint qualification holds for x^0. Then, the Karush-Kuhn-Tucker (KKT) conditions hold at x^0, i. e. there exist multipliers vectors (u, w) such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^0, u, w) &= 0; \\ u_i g_i(x^0) &= 0, \ i = 1, ..., m; \\ u_i &\geqq 0, \ i = 1, ..., m. \end{aligned}$$

Theorem 2. (F. John (1948)). Let x^0 be a local minimizer of (P). Then, the Fritz John (FJ) conditions hold at x^0, i. e. there exist multipliers (u_0, u, w) such that

$$\begin{aligned} \nabla_x \mathcal{L}_1(x^0, u_0, u, w) &= 0; \\ u_i g_i(x^0) &= 0, \ i = 1, ..., m; \end{aligned}$$

$$u_0 \geqq 0, \ u_i \geqq 0, \ \forall i \in M, \ (u_0, u, w) \neq (0, 0, 0).$$

The set of vectors (u, w) satisfying at $x^0 \in K$ the (KKT) conditions is denoted by $\Lambda(x^0)$, whereas the set (u_0, u, w) satisfying the (FJ) conditions at $x^0 \in K$ is denoted by $\Lambda_0(x^0)$. Obviously $\Lambda(x^0) = \left\{(u, w) : (1, u, w) \in \Lambda_0(x^0)\right\}$. Moreover, we note that $\Lambda_0(x^0) \neq \emptyset$, whereas it may be $\Lambda(x^0) = \emptyset$.

There are many constraint qualifications which suffice for Theorem 1 to hold (see, e. g., for a quite recent survey Giorgi (2018)). We recall here only the following ones.

(a) The *Mangasarian-Fromovitz Constraint Qualification* (MFCQ) holds at $x^0 \in K$ if:

(i) The vectors $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent.

(ii) There exists $z \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_i(x^0) z &< 0, \ i \in I(x^0), \\ \nabla h_j(x^0) z &= 0, \ j = 1, ..., p. \end{aligned}$$

(b) The *Linear Independence Condition* (LI) holds at $x^0 \in K$ if the vectors

$$\left\{\nabla g_i(x^0), \ i \in I(x^0); \ \nabla h_j(x^0), \ j = 1, ..., p\right\}$$

are linearly independent.

(c) The *Strict Mangasarian-Fromovitz Constraint Qualification* (SMFCQ) holds at $x^0 \in K$ if, denoting by $I^+(x^0, u)$ the set of *strictly active* inequality constraints at x^0, i. e.

$$I^+(x^0, u) = \left\{i : i \in I(x^0) \text{ and there is } (u, w) \in \Lambda(x^0) \text{ with } u_i > 0\right\},$$

it holds:

(i) The gradients

$$\left\{\nabla g_i(x^0), \ i \in I^+(x^0, u); \ \nabla h_j(x^0), \ j = 1, ..., p\right\}$$

are linearly independent.

(ii) The system

$$\begin{array}{rcl}\nabla g_i(x^0)z & < & 0,\ i \in I(x^0)\diagdown I^+(x^0,u);\\ \nabla g_i(x^0)z & = & 0,\ i \in I^+(x^0,u);\\ \nabla h_j(x^0)z & = & 0,\ j = 1,...,p,\end{array}$$

has a solution $z \in \mathbb{R}^n$.

It follows that (LI) implies (SMFCQ) and that (SMFCQ) implies (MFCQ). (SMFCQ) was introduced by Kyparisis (1985) who has shown that this condition is both necessary and sufficient to have uniqueness of multipliers vectors u and w in (KKT), i. e. $\Lambda(x^0)$ is a singleton. We note, however, that (SMFCQ) is not properly a constraint qualification, as it involves the multipliers vectors in its definition. Usually, these multipliers vectors depend also from the objective function f; for this reason it is perhaps better to call (SMFCQ) a "regularity condition.

Definition 1. A sequence $\left\{x^k\right\} \subset \mathbb{R}^n \diagdown \left\{x^0\right\}$, with $x^k \longrightarrow x^0$ is called *tangentially convergent* in the direction $y \in \mathbb{R}^n$ to the point x^0 if

$$\lim_{k\longrightarrow+\infty} \frac{x^k - x^0}{\left\| x^k - x^0 \right\|} = y$$

and we write $x^k \overset{y}{\longrightarrow} x^0$.

Obviously, any convergent sequence $x^k \longrightarrow x^0$ (with $x^k \neq x^0$ for all k) contains at least a tangentially convergent subsequence.

The set of all directions y for which there exists a feasible sequence $\left\{x^k\right\} \subset S$, with $S \subset \mathbb{R}^n$, tangentially convergent to $x^0 \in S$, form a cone which is a local cone approximation at x^0 of the set $S \subset \mathbb{R}^n$.

Definition 2. Let $S \subset \mathbb{R}^n$ and $x^0 \in S$; the cone

$$T(S,x^0) = \left\{\lambda y \in \mathbb{R}^n : \exists \left\{x^k\right\} \subset S,\ x^k \overset{y}{\longrightarrow} x^0,\ \lambda \geqq 0\right\}$$

is called *Bouligand tangent cone or contingent cone* to the set S at x^0. If x^0 is an isolated point of S, then we set $T(S,x^0) = \{0\}$.

There are many other equivalent characterizations of $T(S,x^0)$: see. e. g., Bazaraa and Shetty (1976), Aubin and Frankowska (1990), Giorgi and Guerraggio (1992, 2002). For example, we have

$$T(S,x^0) = \left\{y \in \mathbb{R}^n : \exists \left\{x^k\right\} \subset S,\ x^k \longrightarrow x^0,\ \exists \{\lambda_k\} \subset \mathbb{R}_+ \text{ such that } \lambda_k(x^k - x^0) \longrightarrow y\right\}.$$

$$T(S,x^0) = \left\{y \in \mathbb{R}^n : \exists \left\{y^k\right\} \longrightarrow y,\ \exists t_k \longrightarrow 0 \text{ such that } x^0 + t_k y^k \in S\right\}.$$

Note that if $x^0 \in int(S)$, then $T(S,x^0) = \mathbb{R}^n$. $T(S,x^0)$ is a *closed* cone, but not necessarily convex.

Other two local cone approximations widely used in optimization theory are the following ones.

Definition 3. Let $S \subset \mathbb{R}^n$ and $x^0 \in S$; the cone

$$A(S,x^0) = \left\{y \in \mathbb{R}^n : \exists \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}^n,\ \varphi(0) = x^0,\ \varphi'(0) = y,\ \exists \delta > 0,\ \forall \theta \in (0,\delta) : \varphi(\theta) \in S\right\}$$

is called *cone of attainable directions* to S at x^0 or *Kuhn-Tucker tangent cone* to S at x^0.

This cone can be equivalently defined in other ways, e. g. in the following ways.

$$A(S,x^0) = \left\{y \in \mathbb{R}^n : \forall \{t_k\} \subset \mathbb{R}_+,\ t_k \longrightarrow 0,\ \exists \left\{y^k\right\} \longrightarrow y \text{ such that } x^0 + t_k y^k \in S\right\}.$$

$$A(S,x^0) = \left\{y \in \mathbb{R}^n : \exists \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}^n,\ \exists \delta > 0,\ \forall \theta \in (0,\delta) \text{ it holds } \varphi(\theta) = x^0 + \theta y + o(\theta) \in S\right\}.$$

The cone $A(S,x^0)$ is closed, but not necessarily convex and it holds

$$A(S,x^0) \subset T(S,x^0).$$

Definition 4. Let $S \subset \mathbb{R}^n$ and $x^0 \in S$; the cone

$$F(S, x^0) = \left\{ y \in \mathbb{R}^n : \exists \bar{\alpha} > 0 \text{ such that } \forall \alpha \in [0, \bar{\alpha}] \text{ it holds } x^0 + \alpha y \in S \right\}$$

is called *cone of feasible directions* to S at x^0.

Note that $F(S, x^0)$ is a cone containing the origin, however it need not be closed or open or convex. We have the following inclusions:

$$cl(F(S, x^0)) \subset A(S, x^0) \subset T(S, x^0).$$

Theorem 3. Let $x^0 \in S \subset \mathbb{R}^n$ and let S be a convex set; then it holds

$$cl(F(S, x^0)) = A(S, x^0) = T(S, x^0) = cl(cone(S - x^0)).$$

It turns out that in this case the cones $F(S, x^0)$, $A(S, x^0)$ and $T(S, x^0)$ are convex. Here $cone(S)$ is the convex cone generated by S, i. e.

$$cone(S) = \left\{ \sum_{i=1}^{k} \lambda_i x^i : x^i \in S,\ \lambda_i \geqq 0,\ \sum_{i=1}^{k} \lambda_i > 0,\ k \in \mathbb{N} \right\}.$$

This cone turns out to be the intersection of all convex cones that contain S. Moreover, if S is a *polyhedral set, then*

$$F(S, x^0) = A(S, x^0) = T(S, x^0)$$

and $F(S, x^0)$ consists of the vectors of the form $\alpha(y - x^0)$, with $\alpha > 0$ and $y \in S$.

If $S \subset \mathbb{R}^n$ is a nonempty set we denote by S^* its (negative) *polar cone,* given by

$$S^* = \{ y \in \mathbb{R}^n : yx \leqq 0,\ \forall x \in S \}.$$

If S is empty, then S^* is interpreted as the whole space $\mathbb{R}^n$. S^* is a closed convex cone.

The next theorem states the classical second-order necessary conditions for (local) optimality of a point $x^0 \in K$. This result is essentially due to McCormick (1967). See also Fiacco and McCormick (1968) and McCormick (1976, 1983). With reference to problem (P) let us define the so-called *critical cone or cone of critical directions* $Z(x^0)$, where $x^0 \in K$:

$$Z(x^0) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla g_i(x^0) z = 0,\ i \in I^+(x^0, u); \\ \nabla g_i(x^0) z \leqq 0,\ i \in I(x^0) \diagdown I^+(x^0, u); \\ \nabla h_j(x^0) z = 0,\ j = 1, ..., p \end{array} \right\}.$$

Theorem 4. Suppose that $x^0 \in K$ is a local solution of (P) and that the (LI) condition holds at x^0. Then, the (KKT) conditions hold at x^0 with associated unique multipliers vectors u and w; moreover, the following additional second-order necessary conditions hold at x^0 :

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0,\ \forall z \in Z(x^0).$$

The second-order necessary conditions expressed in Theorem 4 are called by some authors "strong second-order necessary optimality conditions" for (P). We draw the reader's attention to the fact that there is not uniformity in the literature on this denomination. The same is true for the "weak second-order necessary optimality conditions" (see further).

Note (see Kyparisis (1985)) that the (LI) condition can be substituted by the weaker (SMFCQ) but *not* by (MFCQ). In other words, (MFCQ) is *not* a *second-order constraint qualification* which assures the strong second-order conditions of Theorem 4. This has been remarked by Anitescu (2000), Arutyunov (1991), Baccari (2004).

2. Abadie Constraint Qualification in Second-Order Optimality

In the previous section we have remarked that (MFCQ) does not assure the validity of Theorem 4, however this constraint qualification assures the validity of second-order necessary optimality conditions with *non fixed* multipliers vectors. In other words, this approach, considered by Ben-Tal (1980) but also by Hettich and Jongen (1977), does not guarantee that the *same* multipliers vectors u and w can be taken such that the corresponding second-order necessary optimality conditions are satisfied for all $z \in Z(x^0)$. We may call "weak second-order necessary optimality conditions" the results contained in the next theorem.

Theorem 5. Assume that $x^0 \in K$ is a local minimum point for (P) that satisfies the (MFCQ) condition. Then, we have that for every $z \in Z(x^0)$ there exist multipliers vectors u and w such that $\Lambda(x^0) \neq \varnothing$ (more precisely $\Lambda(x^0)$ is a convex and compact set) and

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0.$$

For an example of a problem (P) where, at a minimizer x^0, different multipliers are required to get (weak) second-order necessary conditions, see Ben-Tal (1980), Example 2.1.

In the literature another description of the cone $Z(x^0)$ often appears; it is the cone (again called "critical cone"):

$$C(x^0) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla f(x^0) z = 0; \\ \nabla g_i(x^0) z \leqq 0, \ i \in I(x^0); \\ \nabla h_j(x^0) z = 0, \ j = 1, ..., p \end{array} \right\}.$$

Indeed, it can be proved that, under the validity of the (KKT) conditions at $x^0 \in K$, the two cones $C(x_0)$ and $Z(x^0)$ coincide (in Han and Mangasarian (1979) there are some minor inaccuracies).

Theorem 6. Let $x^0 \in K$ verify the (KKT) conditions. Then

$$C(x^0) = Z(x^0).$$

Proof. We first show that $C(x^0) \subset Z(x^0)$. Let $z \in C(x^0)$; clearly we only need to show that for $i \in I^+(x^0, u)$ we have $\nabla g_i(x^0) z = 0$. By (KKT) we have that

$$\nabla f(x^0) z + \sum_{i \in I(x^0)} u_i \nabla g_i(x^0) z + \sum_{j=1}^{p} w_j \nabla h_j(x^0) z = 0.$$

Because $\nabla h_j(x^0) z = 0$, $j = 1, ..., p$, and $u_i = 0$ for $i \in I(x^0) \diagdown I^+(x^0, u)$, we have

$$\nabla f(x^0) z + \sum_{i \in I^+(x^0, u)} u_i \nabla g_i(x^0) z = 0.$$

Because $\nabla f(x^0) z = 0$, and every $u_i > 0$ for all $i \in I^+(x^0, u)$, we have

$$\nabla g_i(x^0) z = 0, \ i \in I^+(x^0, u).$$

Now we prove that $Z(x^0) \subset C(x^0)$. Let z be any point in $Z(x^0)$. It suffices to show that $\nabla f(x^0) z = 0$. As before, we have

$$\nabla f(x^0) z + \sum_{i \in I(x^0)} u_i \nabla g_i(x^0) z + \sum_{j=1}^{p} w_j \nabla h_j(x^0) z = 0.$$

Clearly $\nabla f(x^0) z = 0$, because all the other terms are zero. □

It is possible to obtain also weak necessary second-order optimality conditions of the Fritz John -type, as done by Hettich and Jongen (1977) and by Ben-Tal (1980). Usually, under the said approach, the following cone of critical directions is considered:

$$C_1(x^0) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla f(x^0) z \leqq 0; \\ \nabla g_i(x^0) z \leqq 0, \ i \in I(x^0); \\ \nabla h_j(x^0) z = 0, \ j = 1, ..., p \end{array} \right\}.$$

We may call $C_1(x^0)$ the *extended critical cone* at $x^0 \in K$. Between $C(x^0)$ and $C_1(x^0)$, besides the fact that obviously it holds $C(x^0) \subset C_1(x^0)$, there is the relationship specified in the following result, not often pointed out.

Theorem 7. Let x^0 be a local minimum point for (P). Then the (FJ) conditions are satisfied at x^0 with $u_0 > 0$ if and only if $C(x^0) = C_1(x^0)$.

Proof. If $C(x^0) \neq C_1(x^0)$, it will exist $z \in \mathbb{R}^n$ such that

$$\left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla f(x^0) z < 0; \\ \nabla g_i(x^0) z \leqq 0, \ i \in I(x^0); \\ \nabla h_j(x^0) z = 0, \ j = 1, ..., p. \end{array} \right. \quad (1)$$

We write the first basic (FJ) relation in the form:

$$u_0 \nabla f(x^0) + \sum_{i \in I(x^0)} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0.$$

Multiplying this relation by z, we have that $u_0 = 0$. On the other hand, if $C(x^0) = C_1(x^0)$, we conclude that system (1) has no solution $z \in \mathbb{R}^n$. From the well-known Farkas lemma (see, e. g., Mangasarian (1969)), we deduce the existence of multipliers $u_i \geqq 0$, $i \in I(x^0)$; $w_j \in \mathbb{R}$, $j = 1, ..., p$, such that

$$\nabla f(x^0) + \sum_{i \in I(x^0)} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0. \qquad \square$$

Theorem 5 (weak necessary second-order conditions for (P)) holds also under a constraint qualification weaker than (MFCQ): the second-order Ben-Tal constraint qualification (which is also a first-order constraint qualification). See Ben-Tal (1980), Giorgi (2019), Still and Streng (1996) and Section 3 of the present paper. However, Theorem 5 does not hold under other weak first-order constraint qualifications, such as, for example, the well-known *Abadie constraint qualification* (see, e. g., Bazaraa and Shetty (1976), Giorgi, Guerraggio and Thierfelder (2004), Giorgi (2018)). We recall that the *linearizing cone* at $x^0 \in K$ is given by

$$L(K, x^0) = L(x^0) = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^0) z \leqq 0, \ i \in I(x^0); \ \nabla h_j(x^0) z = 0, \ j = 1, ..., p \right\}.$$

The *Abadie constraint qualification* is said to hold at $x^0 \in K$ if

$$L(x^0) = T(K, x^0). \tag{2}$$

Quite recently, Andreani, Behling, Haeser and Silva (2017) have obtained for (P) strong second-order necessary optimality conditions by means of a "modified" Abadie constraint qualification. In particular, these authors prove the following result (their Theorem 3.3), previously given, without proof, by Bazaraa, Sherali and Shetty (2006, Exercise 5.9).

Theorem 8. Let $x^0 \in K$ be a local minimum point for (P) and suppose that x^0 fulfils the (KKT) conditions. Let us define the set

$$K_1 = \left\{ x \in K : g_i(x) = 0, \ \forall i \in I^+(x^0, u) \right\}.$$

If the following "modified" Abadie constraint qualification holds at x^0 :

$$Z(x^0) = T(K_1, x^0), \tag{3}$$

then we have

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \quad \forall z \in Z(x^0).$$

The previous theorem can be obtained in a simple and direct way as a consequence of more general results which make reference to the Bouligand tangent cone.

Let us consider a minimization problem with an *abstract constraint* (or a *set constraint*):

$$(P_1): \quad \min f(x), \quad x \in S,$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is twice-continuously differentiable on an open set containing the set $S \subset \mathbb{R}^n$, on which no particular topological property is a priori made. We have the following result.

Theorem 9. Let $x^0 \in S$ be a local minimum point for (P_1); then we have:

(i)

$$\nabla f(x^0) y \geqq 0, \quad \forall y \in T(S, x^0),$$

i. e.

$$-\nabla f(x^0) \in (T(S, x^0))^*.$$

(ii) Let $\nabla f(x^0) = 0$; then it holds:

$$y^\top \nabla^2 f(x^0) y \geqq 0, \quad \forall y \in T(S, x^0).$$

Proof.

(i) This result is well-known: see, e. g., Bazaraa and Shetty (1976), Giorgi, Guerraggio and Thierfelder (2004), Gould and Tolle (1971).

(ii) Let $y \neq 0$ be any vector of $T(S, x^0)$. Without loss of generality let $\|y\| = 1$; then there exists a feasible sequence $\left\{x^k\right\} \subset S$, with $x^k \xrightarrow{y} x^0$. Being x^0 a local minimum point of (P_1), the quotients

$$\frac{f(x^k) - f(x^0)}{\left\|x^k - x^0\right\|^2} = \frac{\frac{1}{2}(x^k - x^0)^\top \nabla^2 f(x^0)(x^k - x^0) + o(\left\|x^k - x^0\right\|^2)}{\left\|x^k - x^0\right\|^2}$$

for $k \in \mathbb{N}$ large enough, are nonnegative and convergent to $\frac{1}{2} y^\top \nabla^2 f(x^0) y$. The thesis is therefore proved. □

Remark 1. If $x^0 \in int(S)$, then $T(S, x^0) = \mathbb{R}^n$ and (i) collapses to the classical Fermat necessary condition $\nabla f(x^0) = 0$, whereas condition (ii) recovers the classical second-order necessary optimality condition for an unconstrained optimization problem.

Remark 2. It is possible to weaken the requirement $\nabla f(x^0) = 0$ appearing in condition (ii) of Theorem 9. We have the following result.

Theorem 10. Let $x^0 \in S$ be a local minimum point for (P_1); then we have

$$\left\{y^\top \nabla f(x^0) = 0,\ y \in F(S, x^0)\right\} \Longrightarrow y^\top \nabla^2 f(x^0) y \geqq 0.$$

Proof. By assumption there exists $\bar{\alpha}$ such that $x^0 + \alpha y \in S$, $\forall \alpha \in [0, \bar{\alpha}]$. The Taylor expansion gives

$$\begin{aligned} f(x^0 + \alpha y) - f(x^0) &= \alpha y^\top \nabla f(x^0) + \frac{1}{2}\alpha^2 y^\top \nabla^2 f(x^0) y + o(\|\alpha y\|^2) = \\ &= \frac{1}{2}\alpha^2 y^\top \nabla^2 f(x^0) y + o(\|\alpha y\|^2). \end{aligned}$$

As the first member of the previous equalities is nonnegative, for sufficient small α, it will result $y^\top \nabla^2 f(x^0) y \geqq 0$. □

If S is *polyhedral* it is then possible to substitute in Theorem 10 the cone $F(S, x^0)$ with the cone $T(S, x^0)$, but this substitution is not possible in general, in order to obtain the thesis of Theorem 10.

Let us now consider the problem with both inequality and equality constraints, i. e. problem (P). Let $x^0 \in K$ be a local minimum point for (P) and let some first-order constraint qualification be satisfied at x^0 : for example (MFCQ) or the Abadie constraint qualification (2) or the Guignard-Gould-Tolle constraint qualification:

$$(L(x^0))^* = (T(K, x^0))^*$$

which is the weakest first-order constraint qualification for (P). See, e. g., Gould and Tolle (1971), Giorgi (2018). Therefore at x^0 the (KKT) conditions hold. We recall the definition of K_1 :

$$K_1 = \left\{x \in K : g_i(x) = 0,\ \forall i \in I^+(x^0, u)\right\}.$$

But then, by the complementarity slackness conditions, we have

$$\mathcal{L}(x, u, w) = f(x),\ \forall x \in K_1.$$

As x^0 is a local solution of(P), the same point is also a local solution of the problem

$$\min_{x \in K_1} \mathcal{L}(x, u, w) = f(x).$$

But, thanks to the Karush-Kuhn-Tucker conditions, it holds $\nabla_x \mathcal{L}(x^0, u.w) = 0$. Applying Theorem 9, we have the following result.

Theorem 11. Let $x^0 \in K$ be a local solution of (P) and let (x^0, u, w) be a triple satisfying the related Karush-Kuhn-Tucker conditions. Then we have

$$y^\top \nabla_x^2 \mathcal{L}(x^0, u, w) y \geqq 0, \ \forall y \in T(K_1, x^0).$$

Theorem 11 may be viewed as a relaxed version of Theorem 4, as it holds under *any* first-order constraint qualification. On the other hand, it is not difficult to prove that it holds

$$T(K_1, x^0) \subset Z(x^0).$$

Note that $Z(x^0)$ is just the linearizing cone at x^0 referred to K_1. Obviously, if the "modified" Abadie constraint qualification (3) holds, we get the thesis of Theorem 8. A sufficient condition to have (3) is (LI) but also (SMFCQ).

We point out that similar questions have been treated also by Bomze (2016) who introduces an apparently new "modified" Abadie constraint qualification, called by this author *reflected Abadie constraint qualification:*

$$L(x^0) \subset T(K, x^0) \cup \left[-T(K, x^0)\right]. \tag{4}$$

In the same paper Bomze (2016) remarks that the usual Abadie constraint qualification (2) implies (4) (and it implies also the Guignard-Gould-Tolle constraint qualification, as it is well-known), but in general there are no inclusion relations between the reflected Abadie constraint qualification and the Guignard-Gould-Tolle constraint qualification. Moreover, it must be stressed that the reflected Abadie constraint qualification is *not* a first-order constraint qualification, i. e. it does not assure (KKT) at an optimal point x^0 of (P). Bomze (2016) proves the following result.

Theorem 12. Let x^0 be a local solution of (P), let the Karush-Kuhn-Tucker conditions be verified at x^0, with multipliers vectors (u, w) and let the reflected Abadie constraint qualification (4) be satisfied, with respect to the set K_1. Then we have

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \ \ \forall z \in Z(x^0) = C(x^0).$$

3. Constraint Qualifications via Second-Order Tangent Sets

Local second-order approximations of sets have been used by various authors in obtaining optimality conditions for optimization problems, both for the scalar case and for the vector case. We quote the works of Aghezzaf and Hachimi (1999), Aubin and Frankowska (1990), Bigi and Castellani (2000), Bonnans and Cominetti (1996), Bonnans, Cominetti and Shapiro (1999), Bonnans and Shapiro (2000), Cambini, Martein and Vlach (1999), Castellani and Pappalardo (1996), Cominetti (1990), Jiménez and Novo (2003, 2004), Kawasaki (1988), Penot (1994, 1998, 2000), Ruszczynski (2006). See, for a survey and other bibliographical references, the paper of Giorgi, Jiménez and Novo (2010).

One of the first authors to treat second-order optimality conditions for (P) via a second-order tangent set is Kawasaki (1988) who makes use of the lexicographic order. In the present section we add further results and information to the ones obtained by Kawasaki (1988).

Definition 5. Let $S \subset \mathbb{R}^n$ and $x^0 \in S$. The *second-order contingent set* of S at x^0 in the direction $v \in \mathbb{R}^n$ is the set

$$T^2(S, x^0, v) = \left\{ w \in \mathbb{R}^n : \exists t_n \longrightarrow 0^+, \ \exists w^n \longrightarrow w \text{ such that } x^0 + t_n + \frac{1}{2} t_n^2 w^n \in S, \ \forall n \in \mathbb{N} \right\}.$$

This set can be equivalently described as follows:

$$\begin{aligned} T^2(S, x^0, v) &= \left\{ w \in \mathbb{R}^n : \exists t_n \longrightarrow 0^+, \ \exists x^n \in S \ \text{ such that } \lim_{n \longrightarrow \infty} \frac{x^n - x^0 - t_n v}{\frac{1}{2} t_n^2} = w \right\} = \\ &= \limsup_{t \longrightarrow 0^+} \frac{S - x^0 - tv}{\frac{1}{2} t^2}. \end{aligned}$$

(In the last expression the limit is intended in the Kuratowski sense).

It follows from the above definitions that the points on the parabolic trajectory

$$x(t) = x^0 + tv + \frac{t^2}{2} w, \ \ t \geqq 0,$$

are, for $t = t_k$ and $k \longrightarrow \infty$, very close to the set S :

$$dist(x(t_k), S) = o(t_k^2),$$

where $dist(x, S)$ denotes the Euclidean point-to-set distance from x to S.

Remark 3.

(i) $T^2(S, x^0, v)$ is a closed set contained in $cl\left[cone(cone(S - x^0) - v\right]$. Here $cone(A)$ is the cone generated by the (non necessarily convex) set A.

(ii) If $T^2(S, x^0, v) \neq \emptyset$, then $v \in T(S, x^0)$, but the viceversa does not hold. Therefore, the second-order contingent set may be empty even if the direction v is chosen in the (first-order) contingent cone.

(iii) If $v = 0$, the second-order contingent set collapses into the contingent cone $T(S, x^0)$.

(iv) It holds

$$T^2(S, x^0, \lambda v) = \lambda^2 T^2(S, x^0, v), \ \forall \lambda > 0$$

and

$$T^2(S, x^0, v) + \alpha v \subset T^2(S, x^0, v), \ \forall \alpha \in \mathbb{R}.$$

(v) If S is a *polyhedral set,* then we have, with $v \in T(S, x^0)$,

$$T^2(S, x^0, v) = T(T(S, x^0), v).$$

Therefore in this case $T^2(S, x^0, v)$ is a convex cone (see Ruszczynski (2006)).

(vi) When S is a convex set, $x^0 \in S$ and $v \in T(S, x^0)$, then

$$T^2(S, x^0, v) + T(T(S, x^0), v) \subset T^2(S, x^0, v).$$

See Bonnans, Cominetti and Shapiro (1999).

Definition 6. Let $S \subset \mathbb{R}^n$ and $x^0 \in S$. The *second-order set of attainable directions* of S at x^0 in the direction $v \in \mathbb{R}^n$ is the set

$$A^2(S, x^0, v) = \left\{ \begin{array}{c} w \in \mathbb{R}^n : \exists \delta > 0, \ \exists \gamma : [0, \delta] \longrightarrow \mathbb{R}, \text{ such that} \\ \gamma(0) = x^0, \gamma(t) \in S, \ \forall t \in (0, \delta], \ \lim_{t \longrightarrow 0^+} \frac{\gamma(t) - x^0 - tv}{\frac{1}{2}t^2} = w \end{array} \right\}.$$

This set can be equivalently described as follows:

$$A^2(S, x^0, v) = \left\{ w \in \mathbb{R}^n : \forall t_n \longrightarrow 0^+ \ \exists w^n \longrightarrow w \text{ such that } x^0 + t_n v + \frac{1}{2} t_n^2 w^n \in S, \ \forall n \in \mathbb{N} \right\};$$

$$\begin{aligned} A^2(S, x^0, v) &= \left\{ w \in \mathbb{R}^n : \forall t_n \longrightarrow 0^+, \ \exists x^n \in S \text{ such that } \lim_{n \longrightarrow \infty} \frac{x^n - x^0 - t_n v}{\frac{1}{2} t_n^2} = w \right\} = \\ &= \liminf_{t \longrightarrow 0^+} \frac{S - x^0 - tv}{\frac{1}{2} t^2}. \end{aligned}$$

(The limit operation in the last expression is in the Kuratowski sense).

Remark 4. The set $A^2(S, x^0, v)$ has properties similar to the ones of $T^2(S, x^0, v)$, for example:

(a)

$$A^2(S, x^0, v) \neq \emptyset \Longrightarrow v \in A(S, x^0),$$

where $A(S, x^0)$ is the first-order cone of attainable directions (see Section 2).

(b) If $v = 0$, then $A^2(S, x^0, v) = A(S, x^0)$.

(c)

$$A^2(S, x^0, \lambda v) = \lambda^2 A^2(S, x^0, v), \ \forall \lambda > 0.$$

For other properties of $T^2(S, x^0, v)$ and $A^2(S, x^0, v)$ see, e. g., Aubin and Frankowska (1990), Castellani and Pappalardo (1996), Ward (1993). It is quite easy to prove that for any $S \subset \mathbb{R}^n$ it holds

$$A^2(S, x^0, v) \subset T^2(S, x^0, v), \ \forall v \in \mathbb{R}^n.$$

A first result concerning second-order necessary conditions in terms of $T^2(S, x^0, v)$ takes into consideration the following problem (P_1) :

$$(P_1): \quad \min f(x), \ \ x \in S \subset \mathbb{R}^n.$$

This result is due to Penot (1994); see also Giorgi (2019) and Rusczynski (2006).

Theorem 13. Assume that x^0 is a local solution of (P_1). Then, for every $v \in T(S, x^0)$ such that $\nabla f(x^0)v = 0$, we have

$$\nabla f(x^0)w + v^\top \nabla^2 f(x^0)v \geqq 0, \ \ \forall w \in T^2(S, x^0, v).$$

If we denote by $C(f, x^0)$ and $C^2(f, x^0, v)$, respectively, the sets

$$C(f, x^0) = \left\{v \in \mathbb{R}^n : \nabla f(x^0)v = 0\right\},$$
$$C^2(f, x^0, v) = \left\{w \in \mathbb{R}^n : \nabla f(x^0)w + v^\top \nabla^2 f(x^0)v < 0\right\},$$

Theorem 13 can be expressed as: $\forall v \in T(S, x^0) \cap C(f, x^0)$ it holds

$$T^2(S, x^0, v) \cap C^2(f, x^0, v) = \varnothing.$$

We recall that the *linearizing cone* for (P) at $x^0 \in K$ is given by

$$L(K, x^0) \equiv L(x^0) = \left\{v \in \mathbb{R}^n : \nabla g_i(x^0)v \leqq 0, \ i \in I(x^0); \ \nabla h_j(x^0)v = 0, \ j = 1, ..., p\right\}.$$

Therefore, if the (KKT) conditions hold, the critical cone $Z(x^0) = C(x^0)$ is given by

$$C(x^0) = C(f, x^0) \cap L(x^0).$$

In order to discuss constraint qualifications related to second-order local approximations of sets we need some definitions.

We define the *strict linearizing cone* for (P) at $x^0 \in K$ as

$$L_<(x^0) = \left\{v \in \mathbb{R}^n : \nabla g_i(x^0)v < 0, \ i \in I(x^0); \ \nabla h_j(x^0)v = 0, \ j = 1, ..., p\right\}.$$

We define the *second-order linearizing set* for (P) at $x^0 \in K$ *in the direction* v as

$$L^2(x^0, v) = \left\{ \begin{array}{c} w \in \mathbb{R}^n : \nabla g_i(x^0)w + v^\top \nabla^2 g_i(x^0)v \leqq 0, \ i \in I^*(x^0, v); \\ \nabla h_j(x^0)w + v^\top \nabla^2 h_j(x^0)v = 0, \ j = 1, ..., p \end{array} \right\},$$

where, if $I(x^0) \neq \varnothing$,

$$I^*(x^0, v) = \left\{i \in I(x^0) : \nabla g_i(x^0)v = 0\right\}.$$

Note that $L^2(x^0, v)$ is a closed and convex polyhedral set.

We define the *strict* second-order linearizing set for (P) at $x^0 \in K$ in the direction v as

$$L^2_<(x^0, v) = \left\{ \begin{array}{c} w \in \mathbb{R}^n : \nabla g_i(x^0)w + v^\top \nabla^2 g_i(x^0)v < 0, \ i \in I^*(x^0, v); \\ \nabla h_j(x^0)w + v^\top \nabla^2 h_j(x^0)v = 0, \ j = 1, ..., p \end{array} \right\}.$$

Now we consider some second-order constraint qualifications for (P) in terms of the definitions previously given.

Let be given $K = \left\{x \in \mathbb{R}^n : g_i(x) \leqq 0, \ i = 1, ..., m; \ h_j(x) = 0, \ j = 1, ..., p\right\}$, let $x^0 \in K$ and $v \in L(x^0)$. We say that at (x^0, v) the following second-order constraint qualifications are satisfied if the related properties described below hold.

(1) *Slater constraint qualification,* if each g_i, $i \in I(x^0)$, is a convex function, each h_j, $j = 1, ..., p$, is an affine function and there exists $\bar{x} \in K$ such that

$$g_i(\bar{x}) < 0, \ i \in I(x^0); \ h_j(\bar{x}) = 0, \ j = 1, ..., p.$$

(2) *Second-order linear independence condition,* if the vectors

$$\nabla g_i(x^0),\ i \in I^*(x^0, v);\ \nabla h_j(x^0),\ j = 1, ..., p,$$

are linearly independent.

(3) *Second-order Mangasarian-Fromovitz constraint qualification or second-order Ben-Tal constraint qualification,* if:

a) The vectors $\left\{\nabla h_j(x^0),\ j = 1, ..., p\right\}$ are linearly independent;

b) It holds $L^2_<(x^0, v) \neq \emptyset$.

Still and Streng (1996) call this constraint qualification "second-order Mangasarian-Fromovitz constraint qualification", while Kawasaki (1988) calls the same "second-order Ben-Tal constraint qualification". Indeed, Ben-Tal (1980, page 158) introduced this constraint qualification.

(4) *Second-order Kuhn-Tucker constraint qualification,* if

$$A^2(K, x^0, v) = L^2(x^0, v) \neq \emptyset.$$

(5) *Second-order Abadie constraint qualification,* if

$$T^2(K, x^0, v) = L^2(x^0, v) \neq \emptyset.$$

We note that the second-order versions of the Abadie and of the Kuhn-Tucker constraint qualifications are in fact stronger than their first-order counterparts, while the second order (MFCQ) is weaker than the first-order (MFCQ). More precisely, we have the following results.

Theorem 14.

(a) If the Slater constraint qualification holds, then the first-order (MFCQ) holds.

(b) If the first-order (LI) constraint qualification holds, then the first-order (MFCQ) holds.

(c) If the first-order (LI) constraint qualification holds, then the second-order linear independence condition is satisfied at (x^0, v), $\forall v \in L(x^0)$.

(d) If the first-order (MFCQ) holds, then the second-order Mangasarian-Fromovitz constraint qualification holds at (x^0, v), $\forall v \in L(x^0)$.

(e) If the second-order linear independence condition holds, then the second-order Mangasarian-Fromovitz constraint qualification holds.

(f) If the second-order Mangasarian-Fromovitz constraint qualification holds, then the second-order Kuhn-Tucker constraint qualification holds.

(g) If the second-order Kuhn-Tucker constraint qualification holds, then the second-order Abadie constraint qualification holds.

(h) If $T^2(K, x^0, v) \neq \emptyset$, $\forall v \in L(x^0)$, then $L(x^0) = T(K, x^0)$, i. e. the first-order Abadie constraint qualification holds. In other words, if the second-order Abadie constraint qualification holds, then the first-order Abadie CQ holds.

(i) If $A^2(K, x^0, v) \neq \emptyset$, $\forall v \in L(x^0)$, then $L(x^0) = A(K, x^0)$, i. e. the first-order Kuhn-Tucker constraint qualification holds. In other words, if the second-order Kuhn-Tucker CQ holds, then the first-order kuhn-Tucker CQ holds.

For the proof of the above theorem we need some previous results. In the next lemma some relations between the second order sets previously introduced are given.

Lemma 1. Let $K \subset \mathbb{R}^n$ be the feasible set of (P). Then:

1)

$$T^2(K, x^0, v) \subset L^2(x^0, v), \forall v \in L(x^0).$$

2) If, moreover, the gradients $\left\{\nabla h_j(x^0),\ j = 1, ..., p\right\}$ are linearly independent, then

$$L^2_<(x^0, v) \subset A^2(, x^0, v),\quad \forall v \in L(x^0)$$

and hence

$$L^2_<(x^0, v) \subset A^2(, x^0, v) \subset T^2(K, x^0, v) \subset L^2(x^0, v), \forall v \in L(x^0).$$

Proof. Property 1) is a consequence of the relations

(a)

$$\left\{\lim_{n\longrightarrow\infty}\frac{x^n - x^0 - t_n v}{\frac{1}{2}t_n^2} = w, \text{ with } t_n \longrightarrow 0^+\right\} \Longrightarrow$$

$$\Longrightarrow \lim_{n\longrightarrow\infty}\frac{f(x^n) - f(x^0) - t_n\nabla f(x^0)v}{\frac{1}{2}t_n^2} =$$

$$= \nabla f(x^0)w + v^\top\nabla^2 f(x^0)v.$$

(b) If $\gamma : [0, \varepsilon] \longrightarrow \mathbb{R}^n$ gives

$$\lim_{t\longrightarrow 0^+}\frac{\gamma(t) - x^0 - tv}{\frac{1}{2}t^2} = w,$$

then

$$\lim_{t\longrightarrow 0^+}\frac{f(\gamma(t)) - f(x)^0 - t\nabla f(x^0)v}{\frac{1}{2}t^2} =$$

$$= \nabla f(x^0)w + v^\top\nabla^2 f(x^0)v.$$

For the proof of part 2) let us define the sets

$$G = \{x \in \mathbb{R}^n : g(x) \leqq 0\}, \quad H = \{x \in \mathbb{R}^n : h(x) = 0\}.$$

(Obviously $K = G \cap H$). Applying proposition 4.5 of Ward (1993) we obtain

$$A^2(H, x^0, v) = T^2(H, x^0, v) = K^2(H, x^0, v),$$

where

$$K^2(H, x^0, v) = \left\{w \in \mathbb{R}^n : \nabla h(x^0)w + v^\top\nabla^2 h(x^0)v = 0\right\}.$$

Let $w \in L^2_<(x^0, v)$ and hence $w \in K^2(H, x^0, v)$. Then $w \in A^2(H, x^0, v)$. This means that there exists $\gamma : [0, \delta] \longrightarrow \mathbb{R}^n$ such that $\gamma(0) = x^0$, $\gamma(t) \in H$, $\forall t \in (0, \delta]$,

$$\lim_{t\longrightarrow 0^+}\frac{\gamma(t) - x^0 - tv}{\frac{1}{2}t^2} = w \tag{5}$$

and

$$\lim_{t\longrightarrow 0^+}\frac{\gamma(t) - \gamma(0)}{t} = v. \tag{6}$$

We see that for t sufficiently small $\gamma(t) \in G$. If $i \in M \diagdown I(x^0)$, i. e. $g_i(x^0) < 0$, we have $g_i(\gamma(t)) < 0$, $\forall t \in (0, \varepsilon_i)$. If $i \in I(x^0) \diagdown I^*(x^0, v)$, then $g_i(x^0) = 0$ and $\nabla g_i(x^0)v < 0$ ($v \in L(x^0)$). Hence, by (6)

$$\lim_{t\longrightarrow 0^+}\frac{g_i(\gamma(t)) - g_i(x^0)}{t} = \nabla g_i(x^0)v < 0.$$

Therefore $g_i(\gamma(t)) < 0$, $\forall t \in (0, \varepsilon_i)$. If $i \in I^*(x^0, v)$, then $g_i(x^0) = \nabla g_i(x^0)v = 0$ and by (5) and applying the relations appearing in the proof of part 1) we get

$$\lim_{t\longrightarrow 0^+}\frac{g_i(\gamma(t)) - g_i(x^0) - t\nabla g_i(x^0)v}{\frac{1}{2}t^2} = \lim_{t\longrightarrow 0^+}\frac{g_i(\gamma(t))}{\frac{1}{2}t^2} =$$

$$= \nabla g_i(x^0)w + v^\top\nabla^2 g_i(x^0)v < 0.$$

Hence $g_i(\gamma(t)) < 0$, $\forall t \in (0, \varepsilon_i)$.

For $\varepsilon = \min\{\varepsilon_i,\ i \in M\}$, we have $g(\gamma(t)) < 0$, $\forall t \in (0, \varepsilon)$. Therefore $\gamma(t) \in G \cap H = K$ and hence $w \in A^2(K, x^0, v)$. □

Lemma 2. If $L^2_<(x^0, v) \neq \varnothing$, then $cl(L^2_<(x^0, v)) = L^2(x^0, v)$.

Proof. Let $w \in L^2_<(x^0, v)$, $u \in L^2(x^0, v)$, $a_i = -v^\top\nabla^2 gi(x^0)v$ and $u_\lambda = \lambda w + (1 - \lambda)u$ for $\lambda \in (0, 1)$.

By definition, $\nabla g_i(x^0)w < a_i$ and $\nabla g_i(x^0)u \leqq a_i$, $\forall i \in I^*(x^0, v)$. Therefore, multipliying by λ and by $(1-\lambda)$ respectively and summing, we obtain

$$\nabla g_i(x^0)u_\lambda = \nabla g_i(x^0)(\lambda w + (1-\lambda)u) < \lambda a_i + (1-\lambda)a_i = a_i.$$

Similarly, taking into consideration the gradients $\nabla h_j(x^0)$, we obtain $\nabla h_j(x^0)u_\lambda = 0$, and therefore we deduce that $u_\lambda \in L^2_<(x^0, v)$; being

$$\lim_{\lambda \longrightarrow 0^+} u_\lambda = u,$$

we conclude that $u \in cl(L^2_<(x^0, v))$. □

Proof of Theorem 14.

Propositions (a) and (b) are well-known. Proposition (c) is evident. For proposition (c) see Still and Streng (1996, remark 3.8). For proposition (e), let us consider the set

$$K_* = \left\{x \in \mathbb{R}^n : g_i(x) = 0,\ \forall i \in I^*(x^0, v);\ h_j(x) = 0,\ \forall j = 1, ..., p\right\}.$$

The assumptions imply that the first-order Mangasarian-Fromovitz CQ holds for the set K_* (defined by equalities only at x^0): see Bazaraa and Shetty (1976, theorem 6.2.3 (iii)). Hence, by proposition c), it holds the second-order Mangasarian-Fromovitz CQ at (x^0, v) for all $v \in L_1(K_*, x^0)$, where

$$L_1(K_*, x^0) = \left\{v \in \mathbb{R}^n : \nabla g_i(x^0)v = 0,\ \forall i \in I^*(x^0, v);\ \nabla h_j(x^0)v = 0,\ \forall j = 1, ..., p\right\}.$$

Proposition (f) follows from Lemma 1 and Lemma 2, taking into account that the set $A^2(K, x^0, v)$ is closed.

Proposition (g) folloes from the fact that it holds

$$A^2(K, x^0, v) \subset T^2(K, x^0, v),\ \forall v \in \mathbb{R}^n.$$

Proposition (h) follows from the fact that if $T^2(K, x^0, v) \neq \emptyset$, then $v \in T(K, x^0)$; therefore $L(x^0) \subset T(K, x^0)$. The reverse inclusion always holds, as it is well-known.

The proof of proposition (i) is similar to the proof of proposition (h). □

Remark 5. Still and Streng (1996, example 3.1) have shown that the inclusion of proposition (d) of the previous theorem is strict. If the second-order Mangasarian-Fromovitz CQ holds, then we have the following rule to calculate the second-order tangent sets $T^2(K, x^0, v)$ and $A^2(K, x^0, v)$:

$$A^2(K, x^0, v) = T^2(K, x^0, v) = L^2(K, x^0, v).$$

The following basic result has been obtained by Kawasaki (1988) who makes reference to the lexicographic order and, for vector (Pareto) optimization problems, by Aghezzaf and Hachimi (1999) and by Bigi and Castellani (2000). For the reader's convenience we give a proof.

Theorem 15. Let $x^0 \in K$ be a local minimum point for (P) and let the second-order Abadie constraint qualification be verified at (x^0, v) for all $v \in C(^0) = Z(x^0)$. Then for each $v \in C(x^0)$ there exist multipliers $\mu \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ such that the (KKT) conditions hold at x^0 and, moreover,

$$v^\top(\nabla^2 f(x^0) + \sum_{i=1}^m \mu_i \nabla^2 g_i(x^0) + \sum_{j=1}^p \nabla^2 h_j(x^0))v \geqq 0.$$

In particular, it will be $\mu_i = 0$ for all $i \notin I^*(x^0, v)$.

Proof. As the second-order Abadie constraint qualification holds at (x^0, v) for all $v \in C(x^0)$, then the first-order Abadie constraint qualification holds at x^0 and the (KKT) conditions will hold at x^0. Then, by Theorem 13 we have the implication

$$v \in T(K, x^0) \cap C(f, x^0) \Longrightarrow T^2(K, x^0, v) \cap C^2(f, x^0, v) = \emptyset,$$

where

$$C^2(f, x^0, v) = \left\{w \in \mathbb{R}^n : \nabla f(x^0)w + v^\top \nabla^2 f(x^0)v < 0\right\}.$$

In other words, we have, being $T(K, x^0) = L(x^0)$ and $C(x^0) = L(x^0) \cap C(f, x^0)$,

$$L^2(x^0, v) \cap C^2(f, x^0, v) = \varnothing, \quad \forall v \in C(x^0).$$

This means that the following system, with $v \in C(x^0)$,

$$\begin{cases} \nabla f(x^0)w + v^\top \nabla^2 f(x^0)v < 0 \\ \nabla g_i(x^0)w + v^\top \nabla^2 g_i(x^0)v \leqq 0, \ \forall i \in I^*(x^0, v) \\ \nabla h_j(x^0)w + v^\top \nabla^2 h_j(x^0)v = 0, \ \forall j = 1, ..., p, \end{cases}$$

has no solution $w \in \mathbb{R}^n$.

By the nonhomogeneous Farkas lemma of Still and Streng (1996, lemma 4.2), it will exist therefore scalars α, λ and vectors μ, ν, with $(\alpha, \lambda, \mu) \geqq 0$, $(\alpha, \lambda) \neq 0$, such that, with $v \in C(x^0)$,

$$\lambda \nabla f(x^0) + \sum_{i \in I^*(x^0,v)} \mu_i \nabla g_i(x^0) + \sum_{j=1}^{p} \nu_j \nabla h_j(x^0) = 0 \tag{7}$$

$$v^\top \left(\lambda \nabla^2 f(x^0) + \sum_{i \in I^*(x^0,v)} \mu_i \nabla^2 g_i(x^0) + \sum_{j=1}^{p} \nu_j \nabla^2 h_j(x^0) \right) v = \alpha \geqq 0 \tag{8}$$

If $\alpha = 0$, then $\lambda \neq 0$ and the theorem is proved.

Then, let $\alpha > 0$ and suppose $\lambda = 0$. Let us multiply relation (7) by $w \in \mathbb{R}^n$; by summing to (8), we obtain, for each $w \in \mathbb{R}^n$

$$\sum_{i \in I^*(x^0,v)} \mu_i (\nabla g_i(x^0)w + v^\top \nabla^2 g_i(x^0)v) + \sum_{j=1}^{p} \nu_j (\nabla h_j(x^0)w + v^\top \nabla^2 h_j(x^0)v) = \alpha > 0.$$

But, being the second-order Abadie constraint qualification be verified, we can choose $w \in L^2(x^0, v)$ and thus obtaining a contradiction with the last relation, as in this case the first member of the same is $\leqq 0$ for this choice of w. □

Remark 6. Bigi and Castellani (2000) consider a Pareto optimization problem and define the second-order Abadie constraint qualification without requiring the condition $T^2(K, x^0, v) \neq \varnothing$, but this is necessary, as a counterexample, for a Pareto problem, given by Jiménez and Novo (2003) shows. We note, moreover, that in fact the assumption of the Abadie second-order constraint qualification of Theorem 15 can be replaced by the weaker *second-order Guignard-Gould-Tolle constraint qualification,* i. e., with $v \in L(x^0)$ and $T^2(K, x^0, v) \neq \varnothing$,

$$(T^2(K, x^0, v))^* = (L^2(x^0, v))^*$$

for every $v \in C(x^0)$, or equivalently,

$$L^2(x^0, v) = cl(conv(T^2(K, x^0, v))),$$

for every $v \in C(x^0)$.

Finally, as in Bigi and Castellani (2000), it is possible to weaken further the second-order Abadie or Guignard-Gould-Tolle constraint qualification, by requiring that these conditions hold only for some vectors $v \in C(x^0)$. Obviously, in this case, the related second-order necessary optimality conditions will hold only for those $v \in C(x^0)$.

References

Aghezzaf, B., & Hachimi, M. (1999). Second-order optimality conditions in multiobjective optimization problems. *Journal of Optimization Theory and Applications, 102*(1), 37-50. https://doi.org/10.1023/A:1021834210437

Andreani, R., Behling, R., Haeser, G., & Silva, P. J. (2017). On second-order optimality conditions in nonlinear optimization. *Optimization Methods and Software, 32*(1), 22-38. https://doi.org/10.1080/10556788.2016.1188926

Andreani, R., Birgin, E. G., Martłnez, J. M., & Schuverdt, M. L. (2010). Second-order negative-curvature methods for box-constrained and general constrained optimization. *Computational Optimization and Applications, 45*(2), 209-236.

Andreani, R., Echagüe, C. E., & Schuverdt, M. L. (2010). Constant-rank condition and second-order constraint qualifications. *J. Optim. Theory Appl., 146*, 255-266. https://doi.org/10.1007/s10957-010-9671-8

Andreani, R., Martinez, J. M., & Schuverdt, M. L. (2007). On second-order optimality conditions for nonlinear programming. *Optimization, 56,* 529-542. https://doi.org/10.1080/02331930701618617

Anitescu, M. (2000). Degenerate nonlinear programming with a quadratic growth condition. *SIAM Journal on Optimization, 10*(4), 1116-1135. https://doi.org/10.1137/S1052623499359178

Arutyunov, A. V. (1991). Perturbations of extremal problems with constraints and necessary optimality conditions. *Journal of Soviet Mathematics, 54*(6), 1342-1400. https://doi.org/10.1007/BF01373649

Arutyunov, A. (1998). Second-order conditions in extremal problems. The abnormal points. *Transactions of the American Mathematical Society, 350*(11), 4341-4365.

Arutyunov, A., & Pereira, F. L. (2006). Second-order necessary optimality conditions for problems without a priori normality assumptions. *Mathematics of Operations Research, 31*(1), 1-12. https://doi.org/10.1287/moor.1050.0172

Aubin, J. P., & Frankowska, H. (1990), Set-Valued Analysis, Birkh äuser, Boston.

Baccari, A. (2004). On the classical necessary second-order optimality conditions. *Journal of Optimization Theory and Applications, 123*(1), 213-221. https://doi.org/10.1023/B:JOTA.0000043998.04008.e6

Baccari, A., & Trad, A. (2005). On the classical necessary second-order optimality conditions in the presence of equality and inequality constraints. *SIAM Journal on Optimization, 15*(2), 394-408. https://doi.org/10.1137/S105262340342122X

Bazaraa, M. S., & Shetty, C. M. (1976). Foundations of Optimization, Springer Verlag, New York. https://doi.org/10.1007/978-3-642-48294-6

Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (2006). Nonlinear Programming. Theory and Algorithms. Third Edition, Wiley-Interscience, New York. https://doi.org/10.1002/0471787779

Ben-Tal, A. (1980). Second-order and related extremality conditions in nonlinear programming. *Journal of Optimization Theory and Applications, 31*(2), 143-165. https://doi.org/10.1007/BF00934107

Bigi, G., & Castellani, M. (2000). Second order optimality conditions for differentiable multiobjective problems. *RAIRO-Operations Research, 34*(4), 411-426. https://doi.org/10.1051/ro:2000122

Bomze, I. M. (2016). Copositivity for second-order optimality conditions in smooth optimization problems. *Optimization, 65*, 779-795. https://doi.org/10.1080/02331934.2015.1083019

Bonnans, J. F., & Cominetti, R. (1996). Perturbed optimization in Banach spaces III: Semi-infinite optimization. *SIAM journal on control and optimization, 34*(5), 1555-1567. https://doi.org/10.1137/S0363012994276883

Bonnans, J. F., Cominetti, R., & Shapiro, A. (1999). Second order optimality conditions based on parabolic second order tangent sets. *SIAM Journal on Optimization, 9*(2), 466-492. https://doi.org/10.1137/S1052623496306760

Bonnans, J. F., & Shapiro, A. (2000). Perturbation Analysis of Optimization Problems, Springer Verlag, Berlin. https://doi.org/10.1007/978-1-4612-1394-9

Brinkhuis, J. (2009). A linear programming proof of second-order conditions of non-linear programming. *European J. of Operational Research, 192*, 1001-1007. https://doi.org/10.1016/j.ejor.2007.10.062

Cambini, A., Martein, L., & Vlach, M. (1999). Second order tangent sets and optimality conditions. *Mathematica japonicae, 49*(3), 451-461.

Casas, E., & Tröltzsch, F. (2002). Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory. *SIAM Journal on Optimization, 13*(2), 406-431. https://doi.org/10.1137/S1052623400367698

Castellani, M., & Pappalardo, M. A. S. S. I. M. O. (1996). Local second-order approximations and applications in optimization. *Optimization, 37*(4), 305-321. https://doi.org/10.1080/02331939608844224

Cominetti, R. (1990). Metric regularity, tangent sets and second-order optimality conditions. *Appl. Math. Optim., 21,* 265-287. https://doi.org/10.1007/BF01445166

Deepmala. (2014). *A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications,* Ph. D. Thesis, Pt. Ravishnankar Shukla University, Raipur 492 010, Chatisgarth, India. https://doi.org/10.1186/1687-1812-2014-216

Di, S. (1996). Classical optimality conditions under weaker assumptions. *SIAM Journal on Optimization, 6*(1), 178-197. https://doi.org/10.1137/0806010

Dubey, R., Deepmala, & Mishra, V. N. (2019). *Higher-order symmetric duality in nondifferentiable multiobjective fractional programming problem over cone constraints,* Statistics, Optimization and Information Theory, in press. ISSN: 2310-5070.

Dubey, R., Ishra, V. N., & Tomar, P. (2018). *Duality relations for second-order programming problem under* (G, α_f)-*bonvexity assumptions,* Asian-European Journal of Mathematics. https://doi.org/10.1142/S1793557120500448

Dubey, R., Mishra, L. N., & Mishra, V. N. (2018). Duality relations for a class of a multiobjective fractional programming problem involving support functions. *American Journal of Operations Research, 8*(04), 294.

Dubey, R., & Mishra, V. N. (2019). Symmetric duality results for second-order nondifferentiable multiobjective programming problem. *RAIRO-Operations Research, 53*(2), 539-558. https://doi.org/10.1051/ro/2019044

Dubey, R., Vandana, M. V., & Mishra, V. N. (2018). Second-order multiobjective symmetric programming problem and duality relations under (F, G f) convexity. *Glob J Eng Sci Res, 5*(8), 187-199.

Fiacco, A. V., & McCormick, G. P. (1990). Nonlinear programming: sequential unconstrained minimization techniques (Vol. 4). Siam. https://doi.org/10.1137/1.9781611971316

Gfrerer, H. (2007). Second-order necessary conditions for nonlinear optimization problems with abstract constraints: the degenerate case. *SIAM Journal on Optimization, 18*(2), 589-612. https://doi.org/10.1137/050641387

Giorgi, G. (2018). A Guided Tour in Constraint Qualifications for Nonlinear Programming under Differentiability Assumptions (No. 160). University of Pavia, Department of Economics and Management.

Giorgi, G. (2019). On Second-Order Optimality Conditions in Smooth Nonlinear Programming Problems (No. 171). University of Pavia, Department of Economics and Management.

Giorgi, G., Guerraggio, A., & Guerraggio, A. (1992). On the notion of tangent cone in mathematical programming. *Optimization, 25*(1), 11-23. https://doi.org/10.1080/02331939208843804

Giorgi, G., & Guerraggio, A. (2002). Characterizations, comparisons, algebraic and topological properties of tangent cones. *Journal of Statistics and Management Systems, 5*(1-3), 275-294. https://doi.org/10.1080/09720510.2002.10701060

Giorgi, G., Guerraggio, A., & Thierfelder, J. (2004). Mathematics of optimization: smooth and nonsmooth case. Elsevier.

Giorgi, G., Jimnez, B., & Novo, V. (2010). An overview of second order tangent sets and their application to vector optimization. *SeMA Journal, 52*(1), 73-96. https://doi.org/10.1007/BF03322575

Giorgi, G., & Kjeldsen, T. H. (2014). A historical view of nonlinear programming: Traces and emergence. In Traces and Emergence of Nonlinear Programming (pp. 1-43). Birkhäuser, Basel. https://doi.org/10.1007/978-3-0348-0439-4_1

Gould, F. J., & Tolle, J. W. (1971). A necessary and sufficient qualification for constrained optimization. *SIAM Journal on Applied Mathematics, 20*(2), 164-172. https://doi.org/10.1137/0120021

Han, S. P., & Mangasarian, O. L. (1979). Exact penalty functions in nonlinear programming. *Mathematical programming, 17*(1), 251-269. https://doi.org/10.1007/BF01588250

Hettich, R., & Jongen, H. T. (1977). On first and second order conditions for local optima for optimization problems in finite dimensions. *Methods Oper. Res, 23*, 82-97.

Huy, N. Q., & Van Tuyen, N. (2016). New second-order optimality conditions for a class of differentiable optimization problems. *Journal of Optimization Theory and Applications, 171*(1), 27-44. https://doi.org/10.1007/s10957-016-0980-4

Jimnez, B., & Novo, V. (2003). Second order necessary conditions in set constrained differentiable vector optimization. *Mathematical Methods of Operations Research, 58*(2), 299-317. https://doi.org/10.1007/s001860300283

Jimnez, B., & Novo, V. (2004). Optimality conditions in differentiable vector optimization via second-order tangent sets. *Applied Mathematics and Optimization, 49*(2), 123-144. https://doi.org/10.1007/s00245-003-0782-6

John, F. (1948). Extremum problems with inequalities as subsidiary conditions, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, 187C204.

Karush, W. (2014). Minima of functions of several variables with inequalities as side conditions. In Traces and Emergence of Nonlinear Programming (pp. 217-245). Birkhauser, Basel. https://doi.org/10.1007/978-3-0348-0439-4_10

Kawasaki, H. (1988). Second-order necessary conditions of the Kuhn-Tucker type under new constraint qualifications. *Journal of Optimization Theory and Applications, 57*(2), 253-264. https://doi.org/10.1007/BF00938539

Kuhn, H. W., & Tucker, A. W. (1951). Nonlinear programming, in (J. Neyman, ed.) Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability.

Kyparisis, J. (1985). On uniqueness of Kuhn-Tucker multipliers in nonlinear programming. *Mathematical Programming, 32*(2), 242-246. https://doi.org/10.1007/BF01586095

Mangasarian, O. L. (1969). Nonlinear Programming, McGraw-Hill Book Co., New York.

McCormick, G. P. (2014). Second order conditions for constrained minima. In Traces and Emergence of Nonlinear Programming (pp. 259-270). Birkhauser, Basel. https://doi.org/10.1007/978-3-0348-0439-4_12

McCormick, G. P. (1976). *Optimality criteria in nonlinear programming;* in R. W. Cottle and C. E. Lemke (Eds.), SIAM-AMS Proceedings, Vol. IX, American Math. Society, Providence, RI, 27-38.

McCormick, G. P. (1983). Nonlinear programming; theory, algorithms, and applications (No. 04; T57. 8, M2.).

Minchenko, L., & Leschov, A. (2016). On strong and weak second-order necessary optimality conditions for nonlinear programming. *Optimization, 65*(9), 1693-1702. https://doi.org/10.1080/02331934.2016.1179300

Penot, J. P. (1994). Optimality conditions in mathematical programming and composite optimization. *Mathematical Programming, 67*(1-3), 225-245. https://doi.org/10.1007/BF01582222

Penot, J. P. (1998). Second-order conditions for optimization problems with constraints. *SIAM Journal on Control and Optimization, 37*(1), 303-318. https://doi.org/10.1137/S0363012996311095

Penot, J. P. (2000). Recent advances on second-order optimality conditions. In Optimization (pp. 357-380). Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-57014-8_25

Ruszczyski, A. P., & Ruszczynski, A. (2006). Nonlinear optimization (Vol. 13). Princeton university press.

Shen, C., & Xue, W. (2015). A new result on second-order necessary conditions for nonlinear programming. *Operations Research Letters, 43*(1), 117-122. https://doi.org/10.1016/j.orl.2014.12.013

Still, G., & Streng, M. (1996). Optimality conditions in smooth nonlinear programming. *Journal of optimization theory and applications, 90*(3), 483-515. https://doi.org/10.1007/BF02189792

Ward, D. (1993). Calculus for parabolic second-order derivatives. *Set-valued analysis, 1*(3), 213-246. https://doi.org/10.1007/BF01027635