Journal of Mathematics Research; Vol. 11, No. 5; October 2019
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Study of Fuzzy Ideals in BCK Algebra

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Received: August 7, 2019 Accepted: September 2, 2019 Online Published: September 9, 2019

doi:10.5539/jmr.v11n5p11 URL: https://doi.org/10.5539/jmr.v11n5p11

Abstract

In this paper, some fuzzy ideals in BCK-Algebra has been discussed and some theorems relating with fuzzy ideals and anti-fuzzy ideals in BCK-Algebra have been proved.

Keywords: BCK Algebra, ideal, fuzzy ideal, anti-Fuzzy ideal, anti-fuzzy p-ideal

1. Introduction

The study of BCK-algebra was initiated by Imai and Is ́eki (IMAI& ISKI, 1966) in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. The concept of fuzzy sets was introduced by L. A. Zadeh (ZADEH, 1965). And Rosenfeld (AZRIEL ROSENFELD, 1971) The first to apply the concept of fuzzy sets to algebraic systems in 1971. And O.Xi applied the concept of fuzzy sets on BCK-algebras.

2. Preliminaries

Definition 2.1. (Meng & Guo, 2005) Let $(X,*,0)$ be a groupoid with a distinguished element 0 and a binary operation$*$. Then $(X,*,0)$ is a BCI-algebra if: for all $x, y, z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0 \; and \; y * x = 0 \; imply \; x = y$

In a BCI-algebra X, a partially ordered relation $\leq$ can be deIned by

$x \leq y$ If and only if $x * y = 0$

A BCI-algebra is said to be a BCK-algebra if it satisfies:

(IIV) $0 * x = 0$ for all$x \in X$.

A BCK/BCI-algebra X has the following properties: (Jun& Lee, 2011)

(b1) $(\forall x \in X)\,(x * 0 = x)$.

(b2) $(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)$.

(b3) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

(b4) $(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y)$.

In particular, if X is a BCK-algebra then the following property hold:

(b5) $(\forall x, y \in X)\,((x * y) * x = 0)$.

Taking the set $A = \{0, a, b\}$ combined with the operator$*$ defined in the table below

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

It is clear that BCI-algebra axioms are satisfied, so A is BCI-algebra, but is not BCK-algebra, notice that $0 * a \neq 0$ from the table means that Axiom (IIV) is not satisfied from BCK-algebra.

Definition 2.2. (Zahedi& Bozorgee, 1999) A BCK-algebra X is said to be bounded if there exists an element $1 \in X$ such that $x \leq 1$ for all $x \in X$.

We will denote $1 * x$ by Nx for brief. We note that $N1 = 0$ and $N0 = 1$ in a bounded BCK-Algebra.

Definition 2.3. (Zahedi& Bozorgee, 1999) A BCK-algebra X is called commutative if

$$x * (x * y) = y * (y * x) \text{ For all } x, y \in X$$

Definition 2.4. (Ahsan & Deeba, 1989) A BCK algebra X is called implicative if

$$x * (y * x) = x \text{ for all } x, y \in X$$

Definition 2.5. (Jun, & Öztürk, 2018) A BCK-algebra X is positive implicative if it satisfies

the following condition:

$$(\forall x, y, z \in X)((x * z) * (y * z) = (x * y) * z).$$

Definition 2.6. (Meng & Guo, 2005) A nonempty subset I of a BCK/BCI-algebra X is said to be an ideal

if it satisfies:

(I1) $0 \in I$, and (I2) for all $x, y \in X$; $x * y \in I$ and $y \in I$ imply $x \in I$.

Any ideal I has the property: (I3) $x \in I$ and $y \leq x$ imply y $\in$ I

Definition 2.7. (Kordi& Moussavi, 2007) Let X be a set. A fuzzy set μ in X is a function

$$\mu : \mathrm{X} \rightarrow [0, 1].$$

Definition 2.8. (Kordi& Moussavi, 2007) A fuzzy set μ in a BCI-algebra X is said to be a fuzzy

ideal in X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$,

(F2) $\mu(x) \geq min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

If μ is fuzzy ideal in X then, for $x, y, z \in X$, such that $(x * y) * z = 0$, then

$$\mu(x) \geq min\{\mu(y), \mu(z)\}$$

Definition 2.9. (Hong & Jun, 1998)A fuzzy subset μ of a BCK-algebra X is called an anti-fuzzy ideal of X if

(i) $\mu(0) \leq \mu(x)$,

(ii) $\mu(x) \leq max\{\mu(x * y), \mu(y)\}$,

For all $x, y \in X$.

Definition 2.10 (Kordi& Moussavi, 2007) A nonempty subset A in a BCI-algebra X is

called a p-ideal of X if it satisfies

$(i1)$ $0 \in A$,

(i2) If for all $x, y, z \in X$, $(x * z) * (y * z) \in A$ and $y \in A$, imply that $x \in A$.

If we put $z = 0$, then it follows that A is an ideal. Thus, every p-ideal is an ideal.

Definition 2.11. (Kordi& Moussavi, 2007) Let X be a BCI-algebra. A fuzzy subset μ in X is called a fuzzy p-ideal if it satisfies

(F1) $\mu(0) \geq \mu(x)$,

(F2) $\mu(x) \geq min\{\mu((x * z) * (y * z)), \mu(y)\}$, for all $x, y, z \in X$.

Definition 2.12. (Mostafa, Omar & Ahmed, 2011) A fuzzy set μ in X is called anti fuzzy p-ideal

of X if it satisfies:

(AF1) $\mu(0) \leq \mu(x)$,

(AF5) $\mu(x) \leq max\left\{\mu\big((x * z) * (y * z)\big), \mu(y)\right\}$ for all $x, y, z \in X$

Definition 2.13. (Solairaju & Ragavan , 2011) A fuzzy ideal μ in a BCI-algebra X is said to be closed if for all $x \in X, \mu(0 * x) \geq \mu(x)$.

Theorem 2.14. (Meng & Guo, 2005) Let μ be a fuzzy ideal in a BCI-algebra X. Then μ is closed if and only if for any $x, y \in X, \mu(x * y) \geq \mu(x) \wedge \mu(y)$, that is, μ is a fuzzy subalgebra in X.

3. Results

In this paper, we defined ideals generated by a set in finite and implicative BCK-Algebra, and main ideal generated by element in implicative BCK Algebra.

We found also equivalence definition for fuzzy ideals and anti-fuzzy ideals in BCK Algebra. We changed fuzzy ideals into fuzzy sub-Algebra of BCK-Algebra, and we proved in implicative BCI Algebra that the fuzzy set is anti- fuzzy p-ideal in BCI-Algebra if and only if the fuzzy set is anti-fuzzy ideal within certain conditions.

Theorem 3.1. Let I be a ideal in a BCK-algebra X, $x \in I$ then

$$x * y \in I \quad ; \forall y \in I$$

Proof. Since $x * y \leq x$ for all $x, y \in X$,then $(x * y) * x = 0$, And since I is an ideal, so $0 \in I$, hence $(x * y) * x \in I$, but $x \in I$, so by the definition of ideal we get $x * y \in I$.

Theorem 3.2. Let X be implicative and finite BCK algebra, and $\emptyset \neq S \subseteq X$ then, the set:

$$A = \{ x \in X \, ; \; x \leq x_1 \vee x_2 \vee \ldots \vee x_n \; ; \; x_i \in S \quad , i = 1,2, \ldots, n \, ; n \in Z^+\}$$

is ideal in X contains S, and it is the smallest ideal which contains S.

Proof. Since $0 * x = 0$ for all $x \in X$, then $0 * (x_1 \vee x_2 \vee \ldots \vee x_n) = 0$

$\Longrightarrow \; 0 \leq x_1 \vee x_2 \vee \ldots \vee x_n \qquad \Longrightarrow \; 0 \in A$

If $x * y \in A$ & $y \in A$ then $x * y \leq x_1 \vee x_2 \vee \ldots \vee x_n$& $y \leq x_1 \vee x_2 \vee \ldots \vee x_n$

$\Longrightarrow \; (x * y) * (x_1 \vee x_2 \vee \ldots \vee x_n) = 0$ & $y * (x_1 \vee x_2 \vee \ldots \vee x_n) = 0$

And by the definition 2.5. We get

$(x * (x_1 \vee x_2 \vee \ldots \vee x_n)) * (y * (x_1 \vee x_2 \vee \ldots \vee x_n)) = 0$

$\Longrightarrow \; \big(x * (x_1 \vee x_2 \vee \ldots \vee x_n)\big) * 0 = 0$

$\Longrightarrow \; \big(x * (x_1 \vee x_2 \vee \ldots \vee x_n)\big) * 0 = 0$

$\Longrightarrow \; x * (x_1 \vee x_2 \vee \ldots \vee x_n) = 0$

$\Longrightarrow \; x \leq x_1 \vee x_2 \vee \ldots \vee x_n$

$\Longrightarrow \; x \in A$

So, we conclude the A is ideal in X.

If B is another ideal in X contains S, then

$\forall x \in A \;\Longrightarrow\; x \in X \, ; \, x \leq x_1 \vee x_2 \vee \ldots \vee x_n \, ; \; x_i \in S \subseteq B \quad , i = 1,2, \ldots, n$

$\Longrightarrow \; x_1 \vee x_2 \vee \ldots \vee x_n \in B \qquad \Longrightarrow \; x \in B \;\Longrightarrow\; A \subseteq B$

We conclude that A is ideal in X contains S and it's the smallest ideal in X contain S.

Theorem 3.3. Let X be implicative and finite BCK algebra and let $a \in X$, then the set $A = \{ x \in X \; ; \; x \leq a \}$ is an ideal in X contains a and it's the smallest ideal in X contains a.

Proof. $0 \leq a \Longrightarrow 0 \in A$ If $x * y \in A$ & $y \in A$ then $x * y \leq a$ & $y \leq a$

$\Longrightarrow \; x * y \leq a \Longrightarrow (x * y) * a = 0$ In addition, by the definition 2.5. We get

$\Longrightarrow y * a = 0 \quad (x * a) * (y * a) = 0,$ But $y \leq a$

$\Longrightarrow (x * a) * 0 = 0 \Longrightarrow \; x * a = 0$

$\Longrightarrow \; x \leq a \;\Longrightarrow\; x \in A$

Therefore, we conclude the A is ideal in X.

If B is another ideal in X contains a, then $\forall x \in A \Longrightarrow x \leq a \;\Longrightarrow x \in B$

So $A \subseteq B$. We conclude that A is ideal in X contains a and it is the smallest ideal in X contain a.

Definition 3.4. Let X be implicative and finite BCK algebra and let $a \in X$, then we call the ideal exists in theorem (3.2.) with main ideal generated by element a, and we denote it by $< a >$, so

$$< \mathrm{a} >= \{ x \in X \;\; ; \;\; x \leq a \}$$

Notice 3.5. The two theorems (3.1.) and (3.2.) are not satisfied in BCI algebra.

Theorem 3.6. Let X be BCK algebra and let μ be fuzzy set in X, then μ is anti-fuzzy ideal in X if and only if for any element $a \in [0,1]$ then the next set is ideal in X.

$$A_a = \{x \in X \ ; \ \mu(x) \leq a\} \quad ; A_a \neq \emptyset$$

Proof. Suppose that μ is anti-fuzzy ideal in X, and since for $x \in A_a$ Take t $\in$ [0, 1] such that $A_a \neq \emptyset$ and let $x, y \in X$ such that $x \in A_a$. Then $\mu(x) \leq a$, and also we have μ is anti-fuzzy ideal in X, so $\mu(0) \leq \mu(x)$

$\Longrightarrow \mu(0) \leq \mu(x) \leq a \quad \Longrightarrow \mu(0) \leq a \quad \Longrightarrow 0 \in A_a$

Let $x * y \in A_a$ & $y \in A_a$

We have $x * y \in A_a \Longrightarrow \mu(x * y) \leq a$

And $y \in A_a \Longrightarrow \mu(y) \leq a$, we have μ is anti-fuzzy ideal in X, so

$\mu(x) \leq \mu(x * y) \wedge \mu(y) \leq a \wedge a = a$

$\Longrightarrow \mu(x) \leq a \quad \Longrightarrow \quad x \in A_a$ Therefore, A_a is an ideal in X.

Now suppose for contradiction that μ is not an anti-fuzzy ideal of X.

If (i) from 2.9. is not true, that is $\mu(0) > \mu(x)$, for some $x \in X$ then we take

$a_1 = \frac{1}{2}(\mu(0) + \mu(x))$, so $\mu(0) > a_1$ & $\mu(x) < a_1$, hence

$0 \leq \mu(x) < a_1 \leq 1$, therefore $x \in A_a$, which means that $A_{a_1} \neq \emptyset$.

As A_a is an ideal we have $0 \in A_{a_1}$ implies $\mu(0) \leq a_1$, which is a contradiction because $\mu(0) > a_1$, therefore $\mu(0) \leq \mu(x)$ for all $x \in X$.

If $\mu(x) > \mu(x * y) \vee \mu(y)$, we take $a_2 = \frac{1}{2}(\mu(x) + (\mu(x * y) \vee \mu(y))$, then

$(\mu(x * y) \vee \mu(y)) < a_2 < \mu(x)$, so $x \notin A_{a_2}$ which gives a contradiction by (I2) from 2.6. Therefore

$\mu(x) \leq \mu(x * y) \vee \mu(y)$.

Theorem 3.7. Let X be BCK algebra and let μ be fuzzy set in X, if

$(x * a) * (y * b) = (x * y) * (a * b) \quad \forall \ x, y, a, b \in X$, then

1. $\mu(x * y) = \mu(y * x) \quad , \forall \ x, y \in X$

2. μ is fuzzy sub-algebra of X.

Proof. 1- we have $\mu\,(x * y) = \mu((x * y) * 0)$, and by (III) from 2.1.

$\mu\,(x * y) = \mu\big((x * y) * (x * x)\big)$, and by the assumption

$\mu\,(x * y) = \mu\big((x * x) * (y * x)\big)$

$\mu\,(x * y) = \mu\big(0 * (y * x)\big) = \mu(0) \geq \mu(y * x)$, therefore

$\mu(x * y) \geq \mu(y * x)$

Also $\mu\,(y * x) = \mu((y * x) * 0)$, and by (III) from 2.1.

$\mu\,(y * x) = \mu\big((y * x) * (y * y)\big)$, and by the assumption

$\mu\,(y * x) = \mu\big((y * y) * (x * y)\big) = \mu\big(0 * (x * y)\big)$

$\mu\,(y * x) = \mu(0) \geq \mu(x * y)$, therefore $\mu(y * x) \geq \mu(x * y)$

Hence $\mu(x * y) = \mu(y * x)$.

2- we have $\mu\,(x * y) = \mu((x * y) * 0)$, and by (III) from 2.1.

$\mu\,(x * y) = \mu\big((x * y) * (x * x)\big)$, , and by the assumption

$\mu\,(x * y) = \mu\big((x * x) * (y * x)\big)$

$\mu\,(x * y) = \mu\big(0 * (y * x)\big)$

$\mu\,(x * y) = \mu(0) \geq \mu(y) \wedge \mu(z)$, because

$0 \leq z \quad , \forall \ z \in X \Longrightarrow 0 * y \leq z$, and according to the definition 2.8.

$\mu(0) \geq \mu(y) \wedge \mu(z)$, hence $\mu(x * y) \geq \mu(y) \wedge \mu(z)$, that means μ is fuzzy sub-algebra of X according by 2.14.

Theorem 3.8. Let X be BCI algebra and let μ be fuzzy set in X, then μ is anti-fuzzy p-ideal in X if and only if μ is anti-fuzzy ideal and satisfied the condition $x \leq x * y \quad \forall x, y \in X$.

Proof. Proof of the necessity of the condition exists in the reference [7]

Let us prove the opposite, let μ is anti-fuzzy ideal and satisfied the condition

$x \leq x * y \quad \forall x, y \in X$. According to definition 2.9. $\mu(0) \leq \mu(x)$.

We have $x \leq x * y \implies x * y \leq (x * y) * z$

$\implies x * y \leq (x * z) * (y * z)$

$\implies (x * y) * \big((x * z) * (y * z)\big) = 0$. We have μ is anti-fuzzy ideal, therefore

$\mu(x * y) \leq \mu((x * y) * ((x * z) * (y * z)) \vee \mu((x * z) * (y * z))$

$\mu(x * y) \leq \mu(0) \vee \mu((x * z) * (y * z))$, but $\mu(0) \leq \mu((x * z) * (y * z))$, so

$\mu(x * y) \leq \mu((x * z) * (y * z))$, hence

$\mu(x * y) \vee \mu(y) \leq \mu((x * z) * (y * z)) \vee \mu(y)$, but $\mu(x) \leq \mu(x * y) \vee \mu(y)$, therefore $\mu(x) \leq \mu((x * z) * (y * z)) \vee \mu(y)$, hence μ is anti-fuzzy p-ideal.

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