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Blow-up for Semidiscretization of Semilinear Parabolic Equation With Nonlinear Boundary Condition

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Abstract

This paper deals with the study of the numerical approximation for the following semilinear equation with a nonlinear absorption term $u_t = u_{xx} - \lambda u^p,\ 0 < x < 1,\ t > 0$, and a nonlinear flux boundary condition $u_x(0,t) = 0,\ u_x(1,t) = u^q(1,t),\ t > 0$. We give conditions under which the positive semidiscrete solution blows up in a finite time. Convergence of the numerical blow-up time to the theoretical one when the mesh size goes to zero is established. Finally, we use an efficient algorithm to estimate the blow-up time.

Keywords: semilinear equation, numerical blow-up, nonlinear boundary, finite differences, arc length transformation, Aitken Δ^2 method

1. Introduction

Consider the following semilinear parabolic problem

$$\begin{cases} u_t = u_{xx} - \lambda u^p, \quad 0 < x < 1,\ t > 0 \\ u_x(0,t) = 0,\ u_x(1,t) = u^q(1,t), \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \end{cases} \tag{1}$$

where $p, q > 1$, $\lambda > 0$ are given constants, and u_0 is a positive smooth function defined on $[0,1]$ such that $u_0'(0) = 0$ and $u_0'(1) = u_0^q(1)$. It is proved that regularity solutions exist for this problem see (Gómez, Márquez, & Wolanski, 1993). For differential equations, solutions can become unbounded in finite time, we say that they blow up, or they can be defined for all time and we call them global solutions. We call a blow-up point, the point of the space where the solution become unbounded in finite time.

For problem (1), Gómez J. L. et al. (Gómez et al., 1993) prove that under certain conditions on $u_0,\ \lambda,\ p$ and q, blow-up occurs in finite time at the boundary $x = 1$; in particular,

- when $p < q$ and $u_0 > u_\lambda$ where u_λ is the unique positive stationary solution,
- when $p = q,\ u_0 > 0$ and $\lambda < 1$.

Rossi J. D. (Rossi, 1998) investigated the blow-up rate for positive solutions of problem (1). He also characterize the blow-up profile in similarity variables. Problem (1) can be considered as a heat conduction problem. In this case, u represents the temperature, see (Assalé, Boni, & Diabate, 2008).

we focus in this paper on the numerical approximations of (1). Since the solution u blows up in finite time, it is worth asking what can be stated about numerical approximations of this type of problems. For previous work on numerical approximations of blowing up solutions we refer to (Abia, López-Marcos, & Martínez, 1996; Adou, Touré, & Coulibaly, 2019; Assalé et al., 2008; Dratman, 2010; Edja, Touré, & Koua, 2018; Ganon, Taha, & Touré, 2019; N'dri, Touré, & Yoro, 2018; Touré, N'Guessan, & Diabate, 2015; Taha, Touré, & Mensah, 2012 and references therein).

This paper is structured as follows : in section 2, we introduce a semidiscrete scheme of the problem (1). In Section 3, we give some properties of this semidiscrete scheme. In Section 4, under suitable conditions, we show that the semidiscrete solution blows up in a finite time and this numerical blow-up time converges to the theoretical one when the mesh size goes to zero. Finally, in section 5, we illustrate our analysis by giving some numerical results.

2. Semidiscrete Problem

Let I be a positive integer and define the grid $x_i = ih,\ i = 0, \ldots, I$, where $h = \frac{1}{I}$ is the mesh parameter. We approximate the solution u of the problem (1) by the solution $U_h(t) = (U_0(t), \ldots, U_I(t))^T$ of the following semidiscrete scheme

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - \lambda U_i^p(t), \quad i = 0, \ldots, I-1, \quad t \in (0, T_h), \tag{2}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) + \frac{2}{h} U_I^q(t) - \lambda U_I^p(t), \quad t \in (0, T_h), \tag{3}$$

$$U_i(0) = \varphi_i > 0, \quad i = 0, \ldots, I, \tag{4}$$

where for $t \in (0, T_h)$,

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad i = 1, \ldots, I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2},$$

$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}$$

and $[0, T_h)$, the maximal time interval on which $\|U_h(t)\|_\infty = \max_{0 \le i \le I} |U_i(t)|$ is finite. We say that $U_h(t)$ blows up in a finite time if the time T_h is finite, and T_h is called the blow-up time of $U_h(t)$.
Denote

$$\delta_*^2 U_i(t) = \begin{cases} \delta^2 U_i(t) & \text{if } i = 0, \ldots, I-1, \\ \delta^2 U_I(t) + \frac{2}{h} U_I^q(t) & \text{if } i = I. \end{cases}$$

3. Properties of the Semidiscrete Problem

The below comparison lemma is another form of the maximum principle for the semidiscrete equations.

Lemma 1 *Let $f \in C^0(\mathbb{R}, \mathbb{R})$, if $V_h, W_h \in C^1([0, T), \mathbb{R}^{I+1})$ are such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) - f(V_i(t)) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) - f(W_i(t)), \quad i = 0, \ldots, I, \quad t \in (0, T), \tag{5}$$

$$V_i(0) < W_i(0), \quad i = 0, \ldots, I, \tag{6}$$

then we have $V_i(t) < W_i(t), \quad 0 \le i \le I, \quad t \in (0, T)$.

Proof. Let us define the functions $Z_i(t) = W_i(t) - V_i(t),\ 0 \le i \le I,\ t \in [0, T)$. Let t_0 be the first $t \in (0, T)$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. It is not difficult to see that

$$\frac{d}{dt} Z_{i_0}(t_0) = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if } 1 \le i_0 \le I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0 \quad \text{if } i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0 \quad \text{if } i_0 = I,$$

which implies that

$$\frac{d}{dt} Z_{i_0}(t_0) - \delta^2 Z_{i_0}(t_0) - f(W_{i_0}(t_0)) + f(V_{i_0}(t_0)) \le 0,$$

but this inequality contradicts (5) and the proof is complete.

Lemma 2 *Let U_h be a solution of* (2)-(4). *Then,*

$$U_i(t) > 0, \quad i = 0, \ldots, I,\ t \in [0, T_h).$$

Proof. Let t_0 be the first $t \in (0, T_h)$ such that $U_i(t) > 0$ for $t \in [0, t_0)$, $0 \le i \le I$, but $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. One can easily check that

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{\epsilon\to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0-\epsilon)}{\epsilon} \le 0,$$
$$\delta^2 U_{i_0}(t_0) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} > 0 \quad \text{if } 1 \le i_0 \le I-1,$$
$$\delta^2 U_{i_0}(t_0) = \frac{2U_1(t_0) - 2U_0(t_0)}{h^2} > 0 \quad \text{if } i_0 = 0,$$
$$\delta^2 U_{i_0}(t_0) = \frac{2U_{I-1}(t_0) - 2U_I(t_0)}{h^2} > 0 \quad \text{if } i_0 = I,$$

which implies that

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) + \lambda U_{i_0}^p(t_0) < 0, \quad \text{if } 0 \le i_0 \le I-1,$$
$$\frac{dU_I(t_0)}{dt} - \delta^2 U_I(t_0) - \frac{2}{h} U_I^q(t_0) + \lambda U_I^p(t_0) < 0, \quad \text{if } i_0 = I.$$

But these inequalities contradict (2)-(4) and we get the expected result.

Lemma 3 *Let U_h be a solution of* (2)-(4) *and the initial condition at* (4) *verifies*

$$\delta_*^2 \varphi_i - \lambda \varphi_i^p > 0, \quad 0 \le i \le I.$$

Then, $\dfrac{dU_i(t)}{dt} > 0$ *for* $0 \le i \le I,\ t \in [0, T_h)$.

Proof. Consider the functions $W_i(t) = \dfrac{dU_i(t)}{dt},\ 0 \le i \le I,\ t \in [0, T_h)$. Let t_0 be the first $t \in (0, T_h)$ such that $W_i(t) > 0$ for $t \in [0, t_0)$, but $W_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. We may assume without lost of generality that i_0 is the smallest integer which satisfies the above equality. Then we have

$$\frac{dW_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{W_{i_0}(t_0) - W_{i_0}(t_0-k)}{k} \le 0, \quad 0 \le i_0 \le I,$$
$$\delta^2 W_{i_0}(t_0) = \frac{W_{i_0+1}(t_0) - 2W_{i_0}(t_0) + W_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \le i_0 \le I-1,$$
$$\delta^2 W_{i_0}(t_0) = \frac{2W_1(t_0) - 2W_0(t_0)}{h^2} > 0, \quad i_0 = 0,$$
$$\delta^2 W_{i_0}(t_0) = \frac{2W_{I-1}(t_0) - 2W_I(t_0)}{h^2} > 0, \quad i_0 = I,$$

which implies by a simple computation that

$$\frac{dW_{i_0}(t_0)}{dt} - \delta^2 W_{i_0}(t_0) + \lambda p U_{i_0}^{p-1}(t_0) W_{i_0}(t_0) < 0 \quad \text{if } 0 \le i_0 \le I-1, \tag{7}$$
$$\frac{dW_I(t_0)}{dt} - \delta^2 W_I(t_0) + \left(\lambda p U_I^{p-1}(t_0) - \frac{2q}{h} U_I^{p-1}(t_0)\right) W_I(t_0) < 0 \quad \text{if } i_0 = I. \tag{8}$$

But inequalities (7)-(8) contradict (2)-(3), and the lemma is proved.

Lemma 4 *Let U_h be a solution of* (2)-(4) *and the initial condition at* (4) *verifies*

$$\varphi_i < \varphi_{i+1}, \quad 0 \le i \le I.$$

Then, $U_i(t) < U_{i+1}(t), \quad 0 \le i \le I-1,\ t \in [0, T_h)$.

Proof. Let t_0 be the first $t > 0$ such that $U_{i+1}(t) - U_i(t) > 0$ for $t \in [0, t_0),\ 0 \le i \le I-1$, but $U_{i_0+1}(t) - U_{i_0}(t) = 0$ for a certain $i_0 \in \{0, \ldots, I-1\}$. We may suppose without lost of generality that i_0 is the smallest integer which verifies the above equality. Let us now consider the functions $Z_i(t) = U_{i+1}(t) - U_i(t)$ for $0 \le i \le I-1,\ t \in [0, T_h)$. We have

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{\epsilon\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-\epsilon)}{\epsilon} \le 0, \quad 0 \le i_0 \le I-1,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \le i_0 \le I-2,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_1(t_0) - 3Z_0(t_0)}{h^2} > 0, \quad i_0 = 0,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{I-2}(t_0) - 3Z_{I-1}(t_0)}{h^2} > 0, \quad i_0 = I-1,
\end{aligned}$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + \lambda(U^p_{i_0+1}(t_0) - U^p_{i_0}(t_0)) < 0 \quad \text{if } 0 \le i_0 \le I-2, \tag{9}$$

$$\frac{dZ_{I-1}(t_0)}{dt} - \delta^2 Z_{I-1}(t_0) - \frac{2}{h}U^q_I(t_0) + \lambda(U^p_I(t_0) - U^p_{I-1}(t_0)) < 0 \quad \text{if } i_0 = I-1. \tag{10}$$

The inequalities (9)-(10) contradict (2)-(3) and the desired result follows.

The following theorem gives conditions under which the solution U_h of (2)-(4) converges to the corresponding one of (1).

Theorem 1 *Assume that the problem* (1) *has a solution* $u \in C^{4,1}([0,1]\times[0,T_d])$ *and the initial condition at* (4) *verifies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \text{ as } h \to 0, \tag{11}$$

where $u_h(t) = (u(x_0,t), \ldots, u(x_I,t))^T$*. Then, for h small enough, the semidiscrete problem* (2)-(4) *has a unique solution* $U_h \in C^1([0,T_d], \mathbb{R}^{I+1})$ *such that*

$$\max_{0\le t\le T_d} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \text{ as } h \to 0. \tag{12}$$

Proof. Let $\gamma > 0$ be such that

$$\|u(\cdot,t)\|_\infty \le \gamma \ \text{ for } \ t \in [0, T_d]. \tag{13}$$

Then the problem (2)-(4) has for each h, a unique solution $U_h \in C^1([0,T_h), \mathbb{R}^{I+1})$.
Let $t(h) \le \min\{T_d, T_h\}$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1. \tag{14}$$

The relation (11) implies $t(h) > 0$ for h small enough. Using the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \le \|u(\cdot,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \ \text{ for } \ t \in (0, t(h)),$$

which leads to

$$\|U_h(t)\|_\infty < 1 + \gamma \ \text{ for } \ t \in (0, t(h)). \tag{15}$$

Let $e_h(t) = U_h(t) - u_h(t)$, $t \in (0, t(h))$ be the discretization error and consider the function

$$Z(x,t) = (\|\varphi_h - u_h(0)\|_\infty + Mh^2)e^{(L+1)t + Cx^2}$$

where $M,\ L,\ C$ are non-negative constants. We denote by $Z(x_i,t)$ the discretisation in space of $Z(x,t)$.
For suitable non-negative constants $M,\ L,\ C$, we prove, using Lemma 1 that

$$|e_i(t)| < Z(x_i,t), \quad 0 \le i \le I,\ t \in (0, t(h)), \quad \text{see (Taha et al., 2012) for more details.}$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + Mh^2)e^{(L+1)t + C}, \quad t \in (0, t(h)). \tag{16}$$

Now, we prove that $t(h) = T_d$. Suppose that $t(h) < T_d$. From (14) and (16), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + Mh^2)e^{(L+1)T_d + C}.$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $1 \leq 0$, which is impossible.

4. Numerical Blow-up

From now on, we suppose $u_0(1) \geq 1$. Under suitable assumptions, we prove that the solution U_h of the semidiscrete problem (2)-(4) blows up in finite time and that its semidiscrete blow-up time converges to the real one when the mesh size goes to zero.

We set

$$(H_1): \quad u_0 > 0,\ u_0' \geq 0 \text{ and } u_0'' - \lambda u_0^p > 0 \text{ in } [0,1],$$
$$(H_2): \quad p \leq q \text{ and } 0 < \lambda < \frac{(q-1)(p+1)}{(q+1)(p-1)}.$$

Theorem 2 *Let $q > 2$. Assume that the problem* (1) *has a solution u which blows up in finite time T such that $u \in C^{4,1}([0,1] \times [0,T))$ and the initial condition at* (4) *verifies $\|\varphi_h - u_h(0)\|_\infty = o(1)$ as $h \to 0$. Under the assumptions (H_1) and (H_2), the unique solution U_h of* (2)-(4) *blows up in finite time T_h for sufficiently small h, and we have :*

$$\lim_{h\to 0} T_h = T.$$

Proof. For the proof, we use the Theorem 1.4 given in (Ushijima, 2000). We have to check conditions A0, A1''' and A2' of this theorem before applying it.

Step 1 (Condition A0) The solution u of (1) blows up in finite time T see (Gómez et al., 1993).

Step 2 (Condition A1''') We define the energy I of problem (1) by

$$I[u](t) = \frac{1}{2}\int_0^1 u_x^2(x,t)dx - \frac{1}{q+1}u^{q+1}(1,t) + \frac{\lambda}{p+1}\int_0^1 u^{p+1}(x,t)dx, \quad t \in [0,T). \tag{17}$$

For any solution u, this energy I is monotone non-increasing function of t. In fact,

$$\frac{d}{dt}I[u](t) = -\int_0^1 u_t^2(x,t)dx \leq 0.$$

Because assumption (H_1) holds, we know from (Gómez et al., 1993) that $u > 0,\ u_t \geq 0$ and from (Chipot, Fila, & Quittner, 1991) that $u_x \geq 0$.

Introduce the functional J as follows :

$$J[u](t) = \int_0^1 u^2(x,t)dx, \quad t \in [0,T). \tag{18}$$

We have

$$\begin{aligned}
\frac{d}{dt}J(t) &= 2\int_0^1 u(x,t)u_t(x,t)dx \\
&= 2\left(-2I[u](t) + \frac{q-1}{q+1}u^{q+1}(1,t) - \lambda\frac{p-1}{p+1}\int_0^1 u^{p+1}(x,t)dx\right) \\
&\geq -4I[u](t) + 2\frac{(q-1)(p+1) - \lambda(q+1)(p-1)}{(q+1)(p+1)}u^{q+1}(1,t) \text{ because } u_0(1) \geq 1 \\
&\geq -4I[u_0] + 2\frac{(q-1)(p+1) - \lambda(q+1)(p-1)}{(q+1)(p+1)}u^{q+1}(1,t) \text{ because } I \text{ is non-increasing.}
\end{aligned} \tag{19}$$

Set $\alpha = 2\frac{(q-1)(p+1) - \lambda(q+1)(p-1)}{(q+1)(p+1)} > 0$ because of H_2.

Then we have

$$\begin{aligned}
\frac{d}{dt}J(t) &\geq -4I[u_0] + \alpha u^{q+1}(1,t) \\
&= -4I[u_0] + \alpha\left(u^{q+1}(0,t) + (q+1)\int_0^1 u_x(x,t)u^q(x,t)dx\right).
\end{aligned} \tag{20}$$

Since u_t is nonnegative, we have $u_{xx} \geq 0$. Which implies that u_x is a continuous and non-decreasing fonction with respect to x. Then, there exists $\xi(t) \in (0,1)$ such that $\int_0^1 u_x(x,t)u^q(x,t)dx = u_x(\xi(t),t)\int_0^1 u^q(x,t)dx$.

Denote $\Gamma_1 = -4I[u_0] + \alpha u_0^{q+1}(0)$ and $\Gamma_2 = \inf\Big\{\alpha(q+1)u_x(\xi(t),t),\ t \in [0,T]\Big\}$.
Thus, we have

$$\begin{aligned} \frac{d}{dt}J(t) &\geq \Gamma_1 + \Gamma_2 \int_0^1 u^q(x,t)dx \\ &\geq \Gamma_1 + \Gamma_2 (J)^{q/2}. \end{aligned} \tag{21}$$

We obtain relation (21) by using Jensen's inequality.

Define $H(t) = -4I[u_0] + \alpha u^{q+1}(1,t),\ t \in [0,T)$.
From (19), we have $\frac{d}{dt}J(t) \geq H(t),\ t \in [0,T)$ and $\lim_{t\to T} H(t) = \infty$ since the blow-up point of u is $x = 1$.

Now, for $t \in [0,T_h)$, we denote by

$$I_h[U_h](t) = \frac{1}{2h}\sum_{i=0}^{I-1}\Big(U_{i+1}(t) - U_i(t)\Big)^2 - \frac{1}{q+1}U_I^{q+1}(t) + \frac{\lambda h}{p+1}\sum_{i=0}^{I} U_i^{p+1}(t), \tag{22}$$

$$J_h(t) = h\sum_{i=0}^{I} U_i^2(t), \tag{23}$$

$$H_h(t) = -4I[\varphi_h] + \alpha U_I^{q+1}(t), \tag{24}$$

the numerical approximations of I, J and H, respectively.

By a simple computation, we obtain for $t \in [0,T_h)$,

$$\frac{dJ_h(t)}{dt} \geq H_h(t) \ \text{ and } \ \frac{dH_h(t)}{dt} = \alpha(q+1)U_I^q(t)\frac{dU_I(t)}{dt} \geq 0.$$

A straightforward calculation yields the following inequality

$$\frac{dJ_h(t)}{dt} \geq \Gamma_1' + \Gamma_2'(J_h)^{q/2},$$

where $\Gamma_1' = -4I[\varphi_h] + \alpha\varphi_0^{q+1}$ and
$\Gamma_2' = \inf\Big\{\alpha(q+1)\frac{U_{k_0+1}(t)-U_{k_0}(t)}{h},\ t \in [0,T_h]\Big\} > 0$, with $k_0 \in \{1,\ldots,I-2\}$ fixed.

Putting $G(s) = \Gamma_1' + \Gamma_2'(s)^{q/2}$, it is clear that

$$\frac{dJ_h(t)}{dt} \geq G(J_h),$$

and there exists $s_0 > 0$ such that

$$\begin{cases} G(s) > 0 \quad \text{for } s > s_0, \\ \displaystyle\int_{s_0}^{\infty} \frac{1}{G(s)}ds < \infty \quad \text{since } \ q > 2. \end{cases}$$

Condition (A"') of theorem 1.4 in (Ushijima, 2000) is satisfied.

Step 3 (Condition A2') By virtue of theorem 1, we show that for any $\epsilon > 0$,

$$\lim_{h\to 0}\sup_{t\in[0,T-\epsilon]} |J[u](t) - J_h[U_h](t)| = 0 \ \text{ and } \ \lim_{h\to 0}\sup_{t\in[0,T-\epsilon]} |H(t) - H_h(t)| = 0.$$

Finally, conditions A0, A1"' and A2' are satisfied. According to theorem 1.4 of (Ushijima, 2000), we obtain the desired results.

5. Numerical Experiments

In this section, we estimate the blow-up time of (2)-(4) by using the algorithm proposed by C. Hirota and K. Ozawa (Hirota & Ozawa, 2006). This algorithm deals with the numerical blow-up time of ODEs (here, the semidiscrete scheme (2)-(4)). We first use the arc length transformation technique to transform the semidiscrete scheme (2)-(4) like this :

$$\begin{cases} \dfrac{d}{d\ell}\begin{pmatrix} t \\ U_0 \\ \vdots \\ U_I \end{pmatrix} = \dfrac{1}{\sqrt{1+\sum_{i=1}^{I} f_i^2}}\begin{pmatrix} 1 \\ f_0 \\ \vdots \\ f_I \end{pmatrix}, \quad 0 < \ell < \infty, \\ t(0) = 0, \quad U_i(0) = \varphi_i, \quad 0 \le i \le I, \end{cases} \tag{25}$$

where

$$\begin{aligned} f_0 &= \frac{2}{h^2}\Big(U_1 - U_0\Big) - \lambda U_0^p(t), \\ f_i &= \frac{1}{h^2}\Big(U_{i+1} - 2U_i + U_{i-1}\Big) - \lambda U_i^p(t), \quad 1 \le i \le I-1, \\ f_I &= \frac{2}{h^2}\Big(U_{I-1} - U_I\Big) + \frac{2}{h}U_I^q(t) - \lambda U_I^p(t), \end{aligned}$$

"ℓ" is such that $\ell^2 = dt^2 + \sum_{i=1}^{I} U_i^2$ and is called the arc length.

The variables t and U_i are fonctions of ℓ, and it is proved in (Hirota & Ozawa, 2006) that

$$\lim_{\ell\to\infty} t(\ell) = T_h \quad \text{and} \quad \lim_{\ell\to\infty} \|U_h(\ell)\|_\infty = \infty.$$

Then we introduce $\{\ell_j\}$, a sequence of the arc length and we apply an ODE solver to (25) for each value of $\{\ell_j\}$ in order to generate a sequence that converges linearly to the blow-up time. This sequence is finally accelerated by the Aitken Δ^2 method see (Hirota & Ozawa, 2006). As ODE solver, we have chosen DOP54. This code is MATLAB version of the well-known FORTRAN code DOPRI5 which has been written by Hairer and Wanner (Hairer, Nørsett, & Wanner, 1993). We find in DOP54, three tolerances parameters, RelTol, AbsTol and InitialStep. RelTol and AbsTol parameters indicate the tolerances of relative and absolute errors respectively, and we use InitialStep to choose how errors are controlled, see (Hirota & Ozawa, 2006) for more details. For our experiments we set RelTol= AbsTol = 1.d–15, InitialStep = 0, $\ell_j = 2^{10}\cdot 2^j$ ($j = 0,\ldots,12$) and the initial data

$$\varphi_i = e^{i*h-\ln(e^1-1)} - \frac{i*h}{e^1-1}, \quad 0 \le i \le I.$$

This initial data guarantees that if q increases, the flow on the boundary also increases since $U_I(t) \ge \varphi_I = 1,\ t > 0$. But it can not ensure the growth of the absorption term in the equation by that of p because $0 < \varphi_i \le 1,\ i = 0,\ldots,I$. Obviously, if λ increases, the absorption term in the equation also becomes large.

In the following tables, T_h is the approximate blow-up time corresponding to meshes of $I = 16, 32, 64, 128, 256, 512, 1024$, n is the number of iterations and the order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}$$

Table 1. For $p = 2.5,\ q = 2.5,\ \lambda = 0.5$

I	T_h	n	s
16	0.216 071 202	4154	–
32	0.210 906 554	6952	–
64	0.209 320 480	12404	1.70
128	0.208 851 002	23101	1.76
256	0.208 715 516	44733	1.79
512	0.208 677 132	97408	1.82
1024	0.208 666 410	293764	1.84

Table 2. For $p = 2.5,\ q = 3,\ \lambda = 0.5$

I	T_h	n	s
16	0.146 361 159	3086	–
32	0.142 014 795	5194	–
64	0.140 656 932	9289	1.68
128	0.140 250 111	17290	1.74
256	0.140 131 638	33341	1.78
512	0.140 097 837	70399	1.81
1024	0.140 088 342	202200	1.83

Table 3. For $p = 2.5,\ q = 4,\ \lambda = 0.5$

I	T_h	n	s
16	0.084 607 888	2020	–
32	0.081 038 633	3425	–
64	0.079 895 889	6155	1.64
128	0.079 548 019	11473	1.72
256	0.079 445 569	22048	1.76
512	0.079 416 093	45078	1.80
1024	0.079 407 758	120059	1.82

Table 4. For $p = 2.5,\ q = 3,\ \lambda = 1$

I	T_h	n	s
16	0.184 326 256	3342	–
32	0.178 963 408	5666	–
64	0.177 345 272	10193	1.73
128	0.176 872 688	19073	1.78
256	0.176 737 698	37068	1.81
512	0.176 699 759	81769	1.83
1024	0.176 689 229	248435	1.85

Table 5. For $p = 2.5,\ q = 3,\ \lambda = 1.2$

I	T_h	n	s
16	0.207 781 947	3481	–
32	0.201 749 362	5923	–
64	0.199 961 413	10684	1.75
128	0.199 446 098	20047	1.79
256	0.199 300 396	39162	1.82
512	0.199 259 776	88792	1.84
1024	0.199 248 575	276972	1.86

Remark 1

- From the tables above, we can ensure the convergence of T_h to the blow-up time of the solution of (1), since the order (s) of the method goes to 2, which is just the accuracy of the difference approximation in space.
- It comes from these tables that there is a relationship between the blow-up time, the flow on the boundary and the absorption term in the equation. In fact, when the absorption term in the equation is constant ($p = 2.5,\ \lambda = 0.5$) and that the flow on the boundary increases (by $q = 2.5$ to $q = 4$), then the blow-up is accelerated (from 0.208 to 0.079) see Tables 1-3. Whereas when the flow on the boundary is a constant ($q = 3$) and that the absorption term in the equation becomes large (by $\lambda = 0.5$ to 1.2 with $p = 2.5$), then the blow-up is delayed (from 0.140 to 0.199) see Tables 2, 4, 5.
- The numerical method used in this paper has some advantages. Firstly, to obtain the numerical blow-up, we are not obliged to put an additional condition on the initial data, which is not the case of some numerical methods (see Theorem 4.3, Theorem 8.6 in (Assalé et al., 2008) and Theorem 6 in (Adou et al., 2019)). Secondly, when the mesh size goes to zero, that is I becomes large, with the method of this paper, the number of iterations n increases slowly. This fact allow us to obtain easily results for high values of I ($I = 512,\ 1024$).

Others illustrations are given by some plots in the below figures.

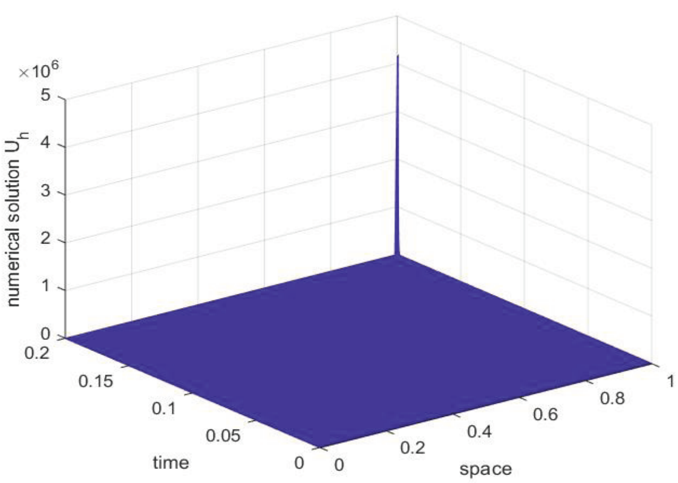

Figure 1. Evolution of the numerical solution for $I = 128,\ p = 2.5,\ q = 3$ and $\lambda = 1.2$

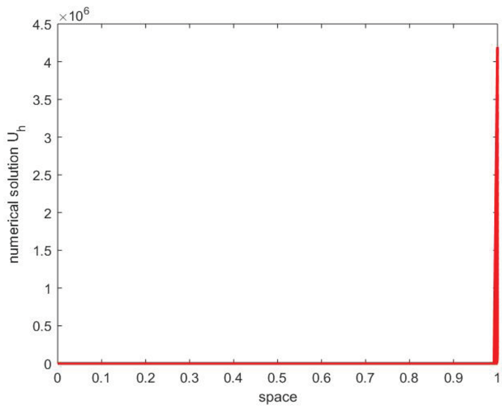

Figure 2. Evolution of U_h according to the node for $I = 128, p = 2.5, q = 3$and$\lambda = 1.2$

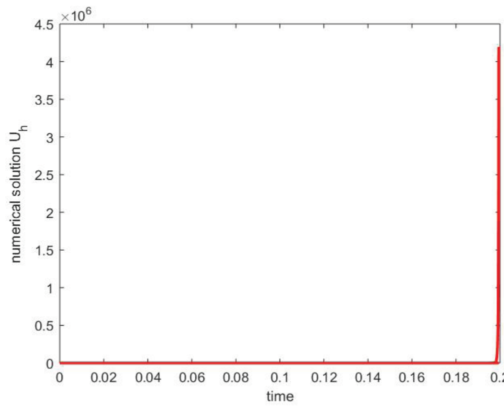

Figure 3. Evolution of U_h according to the time for $I = 128,\ p = 2.5,\ q = 3$ and $\lambda = 1.2$

Remark 2

From the Figures 1, 2 and 3, we can observe that the numerical solution blows up in finite time at the last node, which is in agreement with the result established theoretically (Gómez et al., 1993 and Chipot et al., 1991).

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