On Liouvillian Solutions of Third Order Homogeneous Linear Differential Equations

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Abstract

In this article we will consider third order homogeneous differential equations: $L(y) = y''' + a_1 y' + a_0 y$ $(a_0, a_1 \in k)$ whose Galois group $\mathfrak{G}(L)$ is imprimitive. This case is characterised by the fact that the third symmetric power equation $L^{\circledS 3}(y) = 0$ has an exponential solution whose square is rational (Singer & Ulmer 1993). If $L(y) = 0$ has a Liouvillian solution z whose logarithmic derivative $u = z'/z$ is algebraic over a differential field $(k, ')$, we will give an algorithm to find the relation between a_0, a_1, the semi-invariant $S = Y_1 Y_2 Y_3$ which is unique up to multiplication by a constant, the coefficients c_0, c_1 of the minimal polynomial $P(U)$ of u and their derivatives. The aim of this work is to diminutize the number of constants C_m stated in the algorithm of Singer & Ulmer (Singer & Ulmer 1993 Algorithm p. 31) whose determination is not easy to do, and we will achieve this by using Groebner Basis.

Keywords: differential Galois Theory, Liouvillian solutions, Groebner Basis, invariant theory, representation theory

1. Introduction

This section is a recall of some definitions in ***Differential Galois theory*** that we will depend on in our study. In the second section we will give an algorithm to find the relation between a_0, a_1, the semi-invariant $S = Y_1 Y_2 Y_3$ which is unique up to multiplication by a constant, the coefficients c_0, c_1 of the minimal polynomial $P(U)$ of u and their derivatives. The aim of this work is to diminutize the number of constants C_m stated in the algorithm of Singer & Ulmer (Singer & Ulmer 1993 Algorithm p. 31) whose determination is not easy to do, and we will achieve this by using Groebner Basis, and in the third section we will expose the codes Maple and Magma we have used for more knowledge about Invariant Theory refer to (Benson 1993, Cox 1992, Sturmfels 1993)

Differential Galois theory

Definition 1. (Singer & Ulmer 1998 p.3) *A differential field* $(k, ')$ is a field k together with a derivation $'$ in k. The set of all constants $C = \{a \in k, a' = 0\}$ is a subfield of $(k, ')$.

Definition 2. (Singer & Ulmer 1998 Defination1.1.) A differential field extension (K, Δ) of (k, δ) is a *Liouvillian extension* if there is a tower of fields :

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K$$

Where K_{i+1} is a simple field extension $K_i(\eta_i)$ of K_i, such that one of the following holds:

➤ η_i is algebraic over K_i,

➤ $\delta(\eta_i) = f_i \in K_i$ (extension by an integral $\eta_i = \int f_i$),

➤ $\delta(\eta_i)/\eta_i = f_i \in K_i$ (extension by the exponential of an integral $\eta_i = e^{\int f_i}$).

A function contained in a Liouvillian extension of k is called a *Liouvillian function* over k.

Definition 3. (Fakler 1997 p. 29) Let $(k, ')$ be a differential field of characteristic 0 and C is its field of constants algebraically closed. Consider the following ordinary homogeneous linear differential equation:

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \quad (a_i \in k) \tag{1}$$

over $(k, ')$ with a system $\{y_1, \ldots, y_n\}$ of fundamental solutions. By extending the derivation $'$ to a system of fundamental solutions and by adjunction of these solutions and their derivatives to $(k, ')$ in a way the field of constants does not change, one gets $K = k\langle y_1, \ldots, y_n \rangle$, the so-called ***Picard-Vessiot extension*** (PVE) of $L(y) = 0$. With these assumptions, the PVE of $L(y) = 0$ always exists and is unique up to differential isomorphism. This extension plays the

same role for a differential equation as a splitting field for a polynomial equation.

Definition 4. (Fakler 1997 p. 29) The set of all automorphisms of K, which fix $(k,')$ elementwise and commute with the derivation in K is a group, *it is the differential Galois group* $\mathfrak{S}(K/k)=\mathfrak{S}(L)$ of $L(y)=0$.

Let $L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0y=0$ $(a_i \in k)$ be an $n-th$ order linear diferential equation with Galois group $\mathfrak{S}(L)\in GL(n,C)$. To a basis $\{y_1,\ldots,y_n\}$ of the solution space of $L(y)=0$ we associate the *evaluation morphism*:

$$\Phi:\left[Y_1,\ldots,Y_n\right]\to K$$
$$Y_i \mapsto y_i$$

The *value of a semi-invariant* of $\mathfrak{S}(L)\in GL(n,C)$ is its image in K under the above evaluation morphism.

The morphism Φ has the following properties:

a. Φ restricts to a bijection between linear forms and solutions of $L(y)=0$

b.   Semi-invariants of finite order n are sent to the $n-th$ roots of elements of k. In Particular invariants are fixed under the action of $\mathfrak{S}(L)$ and are sent to elements of k.

Definition 5. (Singer & Ulmer 1997 Definition 1) Let $L(y)=0$ be a homogeneous linear diferential equation of order n and let $\{y_1,\ldots,y_n\}$ be a fundamental system of solutions. The differential equation $L^{\circledS m}(y)=0$ whose solution space, denoted S_m is spanned by monomials of degree m in $y_1,\ldots,y_n$ is called the $m-th$ *symmetric power* of $L(y)=0$.

An algorithm to construct $L^{\circledS m}(y)=0$ is given in (Singer 1980). The value of a semi-invariant of degree m and order j is an $j-th$ root of a solution of $L^{\circledS m}(y)=0$ (Singer & Ulmer 1993 Lemma 1.6.).

Definition 6. (Singer & Ulmer 1997 p.2) Let $L(y)=y^{(n)}+a_{n-2}y^{(n-2)}+\cdots+a_1y'+a_0y=0$ $(a_i \in k)$ be a linear differential equation of order n over k. A solution of (1) in k is a *rational solution*, a solution in an algebraic extension of k is an *algebraic solution*, , a solution whose logarithmic derivative is in k is an *exponential solution*, , a solution belonging to a field obtained from k by successive adjunction of exponentials, integrals and algebraic functions is a *Liouvillian solution*.

Definition 7. (Ulmer 1994. P.3) Let $G \subset GL(V)$ be a linear group acting irreducibly on the vector space V of dimension n over C. Then G is said to be imprimitive if there exist subspaces $\{V_1,\ldots,V_d\}$ with $d \succ 1$ such that $V=V_1\oplus\cdots\oplus V_d$ and, for each $\sigma \in G$, the mapping $V_i \to \sigma(V_i)$ is a permutation of the set $A=\{V_1,\ldots,V_d\}$ The set A is called a system of imprimitivity of G. If all the subspaces V_i are of one dimension, then G is called monomial. An irreducible group $G \subseteq GL(V)$ which is not imprimitive is called primitive.

for more knowledge about *Differential Galois theory* refer to Magid 1994, Kolchin 1948.

2. Algorithm to Diminutize the Number of Constants C_m Stated in the Algorithm of Singer & Ulmer 1993

In this section we will consider linear ordinary differential equations of third order the form: $L_{SL}(y)=y'''+a_1y'+a_0y$ $(a_0,a_1 \in k)$ without losing the Liouvillian character of the solutions (Kolchin 1948, p.184, OKKO 2018 Remark. P. 51), whose Galois group $\mathfrak{S}(L)$ is imprimitive. This case is characterised by the fact that the third symmetric power $L^{\circledS 3}(y)=0$ has an exponential solution whose square is rational or equivalently it has a semi-invariant of degree 3 and of order ≤ 2 exists (cf. Singer & Ulmer 1993 Theorem 4.6). If $L(y)=0$ has a Liouvillian solution z whose logarithmic derivative $u=z'/z$ is algebraic over *a differential field* $(k,')$, we will give an algorithm to find the coefficients of the minimal polynomial $P(U)$ of u in terms of a_0, a_1, the semi-invariant $S=Y_1Y_2Y_3$ which is unique up to multiplication by a constant, and their derivatives by using Groebner Basis. Befor this we will introduce a lemma (Ulmer 2003, Lemma 3.1).) and recall it.

Let $R_m=k\left[Z_{1,0},\ldots,Z_{m,n-1}\right]$ be the polynomial ring in nm variables. We will take the same notations of Felix Ulmer in

(Ulmer 2003) : $W_{i_1,\ldots,i_m}$ denotes for the monic polynomial corresponding to $\sum_{\sigma \in S_m} Z_{\sigma(1)}^{(i_1)}Z_{\sigma(2)}^{(i_2)}\cdots Z_{\sigma(m)}^{(i_m)} \in R_m$

for integers $n \succ i_1 \geq i_2 \geq \cdots \geq i_m \geq 0$. Let $S=Z_1Z_2\cdots Z_m \in R_m$ and $S',\ldots,S^{(j)}$ the first j derivatives of

S, then S, its derivatives and the $N=\binom{m+n-1}{n-1}$ polynomials $W_{i_1,\ldots,i_m}(\in R_m)$ are related by a matrix

$A_j \in \left(k, (j+1) \times N \right)$:

$$\begin{pmatrix} S \\ S' \\ \cdots \\ S^{(j)} \end{pmatrix} = A_j \begin{pmatrix} W_{0,\ldots,0} \\ W_{1,0,\ldots,0} \\ \cdots \\ W_{n-1,\ldots,n-1} \end{pmatrix}$$

The theorem 3 and its proof in (Singer & Ulmer 1997.) show that if $g \in K$ is an exponential solution of $L^{\circledS m}(y) = 0$ and

$g = \Psi_{z_1,\ldots,z_m}(S)$, for $m \in \mathbb{N}$ and $z_1, \cdots, z_m$ are m solutions, different from zero, of the equation $L_{SL}(y) = 0$.

Then for all polynomial $P(U) \in K[U]$:

$$P(U) = \left(\prod_{i=1}^{m} z_i \right) U^m - \left(z_1' z_2 \ldots z_m + \cdots + z_1 z_2 \ldots z_m' \right) U^{m-1} + \cdots + (-1)^m \prod_{i=1}^{m} z_i'$$

$$= \Psi_{z_1,\ldots,z_m}\left(W_{0,\ldots,0} \right) U^m - \Psi_{z_1,\ldots,z_m}\left(W_{1,0,\ldots,0} \right) U^{m-1} + \cdots + (-1)^m \Psi_{z_1,\ldots,z_m}\left(W_{1,\ldots,1} \right)$$

the polynomial :

$$Q(U) = \frac{P(U)}{\prod_{i=1}^{m} z_i}$$

is in the ring $K[U]$. And in this case all roots u_i of $Q(U)$ give Liouvillian solutions $e^{\int u_i}$ of $L(y) = y''' + a_2 y'' + a_1 y' + a_0 y \quad (a_0, a_1, a_2 \in k)$ (cf. Remark. OKKO 2018).

Lemma 3.1. (Ulmer F. 2003, Lemma 3:1). *For a third order linear differential equation* $L(y) = y''' + a_1 y' + a_0 y$ *over*

k *the order of* $L^{\circledS 3}(y) = 0$ *is 7, 9 or 10. The order of* $L^{\circledS 3}(y) = 0$ *is 7 if and only if* $2a_0 = a_1'$. *In this case*

$L(\mathrm{y}) = \tilde{L}^{\circledS 2}(y)$ *where* $\tilde{L}(\mathrm{y}) = y'' + \frac{a_1}{4} y$.

Proof: Let Φ_m be the $\mathfrak{S}-$morphism obtained by restriction of the evaluation morphism Φ to the vector space of homogeneous forms of degree m of $C[Y_1, Y_2, Y_3]$ (Singer & Ulmer 1993 Lemma 3.5). Because Φ_1 is a bijection, (Lemma 3.5 (4) in Singer & Ulmer 1993) implies that the kernel of Φ_2 is at most of dimension 1, i.e. $L^{\circledS 2}(y) = 0$ is of order at least 5. If Φ_2 is a bijection, then (Lemma 3.5 (4) in Singer & Ulmer 1993) implies that $L^{\circledS 3}(y) = 0$ is of order 9 or 10. Suppose now that the kernel of Φ_2 is of dimension 1 and generated by $F_2 \in C[Y_1, Y_2, Y_3]$. Note that F_2 must be irreducible since otherwise Φ_1 is not a bijection. The homogeneous polynomials of degree 3 divisible by F_2 correspond to the $C-$Span of $Y_1 F_2$, $Y_2 F_2$ and $Y_3 F_2$ are in the kernel of Φ_3, showing $L^{\circledS 3}(y) = 0$ is of order ≤ 7. If $L^{\circledS 3}(y) = 0$ is of order ≤ 6, it must exist a polynomial $F_3 \in C[Y_1, Y_2, Y_3]$ in the kernel of Φ_3 which is not a multiple of F_2. If F_2 or F_3 is of degree zero in Y_3, then we get a non constant homogeneous polynomial $F \in C[Y_1, Y_2]$ whose evaluation via Φ is zero. Since F is homogeneous and C is algebraically closed, F factors as a product of linear forms $\prod_i (\alpha_i Y_1 - \beta_i Y_2)$. The evaluation via Φ gives a linear relation between the basis elements y_1, y_2 and thus a contradiction. Denote $F_{2,3} \in C[Y_1, Y_2]$ the resultant of F_2 and F_3 with respect to Y_3. Since F_2 and F_3 are relatively prime and are both of degree ≥ 1 in Y_3, we get that $F_{2,3} \in C[Y_1, Y_2]$ is homogeneous of degree ≥ 1. Arguing as above with F, we get a contradiction. Thus $L^{\circledS 3}(y) = 0$ is of order 7 in this case. This proves the first assertion. We want to characterise the fact That $L^{\circledS 3}(y) = 0$ is of order 7 or equivalently that $L^{\circledS 2}(y) = 0$ is of order 5. This has been done (Singer 1985) Lemma 3.4, but we will give a proof using the introduced formalism. According to section 2.1 we have that $L^{\circledS 2}(y) = 0$ is of order ≤ 5 if and only if the matrix A_5 is not invertible (i.e. if and only if $(z_1 z_2 z_3), \ldots, (z_1 z_2 z_3)^5$ are linearly dependent). Performing a fraction free Gaussian elimination on the matrix $(A_5)^t$ we get:

$$\begin{bmatrix} 1 & 0 & 0 & -2a_0 & -2a_0' & 2a_1a_0 - 2a_0'' \\ 0 & 1 & 0 & -a_1 & -5a_0 - a_1' & -7a_0' + a_1^2 - a_1'' \\ 0 & 0 & 2 & 0 & -8a_1 & -10a_1' - 10a_0 \\ 0 & 0 & 0 & 6 & 0 & -30a_1 \\ 0 & 0 & 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 108(a_1' - 2a_0) \end{bmatrix}$$

Therefore $L^{\circledS 3}(y) = 0$ is of order 7 if and only if $a_1' = 2a_0$. A direct computation shows that in this case $L(y) = \tilde{L}^{\circledS 2}(y)$.

Now we will give an algorithm to find the coefficient of $P(U)$ in terms of a_0, a_1, the semi-invariant $S = Y_1 Y_2 Y_3$ and their derivatives :

The Algorithm:

➢ Write S and its derivatives in terms of the polynomials $w_{i,j,k}$, the coefficients a_1, a_0 and their derivatives (see section 3. , NormalForm1 and NormalForm2 procedures).

➢ Calculate Groebner Basis GR of the ideal $I = \langle S - w_{0,0,0}, \ldots, S^{(9)} - \ldots, c_0 - w_{1,1,1}, c_1 - w_{1,1,0} \rangle$

$\subset R = k \langle w_{0,0,0}, w_{1,0,0}, w_{2,0,0}, w_{1,1,0}, w_{2,1,0}, w_{1,1,1}, w_{2,2,0}, w_{2,1,1}, w_{2,2,1}, w_{2,2,2}, a_{0,0}$

$a_{1,0}, a_{0,1}, \ldots, a_{0,6}, a_{1,6}, S, S', S'', \ldots, S^{(9)}, c_0, c_1 \rangle$

such that the order on the ring R is the elimination order on the first 10 variables, so we can find relations between s, a_1, a_0, their derivatives and the coefficients $c_1 = z_1' z_2' z_3 + z_1' z_2 z_3' + z_1 z_2' z_3'$,

$c_0 = -z_1' z_2' z_3'$ of $P(U)$ in the ring $k[S, S', S'', \ldots, S^{(9)}]$ (see section 3. , s, a_1, a_0 and their derivatives procedure). After calculating Groebner Basis GR we find that its latest six elements have only the variables s, a_0, a_1, their derivatives and the coefficients c_0, c_1.

➢ Eliminate one more time the coefficient c_0 considred as a variable in the ring $k[c_0, c_1, a_0, a_1, a_1', a_1', \ldots, a_0^{(6)}, a_1^{(6)}, s, s', s'', \ldots, s^{(9)}]$ by calculating Groebner Basis Gr of the ideal $id = \langle GR_{16}, \ldots, GR_{21} \rangle$

(see section 3, eliminate c_0). So we obtain a relation between the variables $S, S', \ldots, S^{(9)}, a_0, a_0', \ldots, a_0^{(6)}$, $a_1, a_1', \ldots, a_1^{(6)}$ and c_1.

➢ We determin the value of c_1 with Maple. We substitute this value in the element of Groebner Basis

Gr which contains S, a_1, a_0, their derivatives and the coefficients c_0, c_1 in order to express c_0 in terms of S, a_1, a_0 and their derivatives. In reality, the element Gr we speak about is Gr_4, whatever the order of the symmetric equation $L^{\circledS 3}(y) = 0$ is 7, 9 or 10.

$Gr_4 = c_0 a_0^2 a_1 - \frac{7}{9} c_0 a_0'^2 + \frac{2}{3} c_0 a_0 a_0'' + \frac{43}{20} a_0^3 a_1 S + \frac{1}{4} a_0 a_1^2 a_0' S - \frac{1}{30} a_0^2 a_1^2 a_0' S' + \frac{1}{3} a_1^3 a_0' S'$

$\qquad - \frac{1}{4} a_0 a_1^3 S'' + \frac{63}{20} c_1 a_0^3 + \frac{1}{4} c_1 a_0 a_1 a_0' - \frac{23}{144} a_0 a_0'^2 S + \frac{11}{240} a_0^2 a_0'' S - \frac{13}{48} a_0 a_1 a_0''' S - \frac{17}{80}$

$\qquad a_0^2 a_0' S' - \frac{67}{54} a_1 a_0'^2 S + \frac{169}{144} a_0 a_1 a_0'' S' - \frac{111}{80} a_0^3 S'' + \frac{5}{12} a_0 a_1 a_0' S' + \frac{17}{24} a_0^2 a_1 S'''$

$\qquad - \frac{49}{108} a_1^2 a_0' S''' + - \frac{49}{144} a_0 a_1^2 S^{(4)} - \frac{1}{9} c_1 a_0' a_0'' + \frac{1}{12} c_1 a_0 a_0''' + \frac{1}{36} a_0' a_0^{(4)} S - \frac{1}{48} a_0 a_0^{(5)} S$

$\qquad + \frac{1}{6} a_0' a_0'' S' - \frac{49}{108} a_1^2 a_0' S''' + - \frac{49}{144} a_0 a_1^2 S^{(4)} - \frac{1}{9} c_1 a_0' a_0''$

$\qquad - \frac{7}{48} a_0 a_0^{(4)} S' + \frac{4}{9} a_0' a_0'' S'' - \frac{11}{24} a_0 a_0''' S'' + \frac{14}{27} a_0''^2 S'''$

$\qquad - \frac{25}{36} a_0 a_0' S''' - \frac{7}{24} a_0 a_0'' S^{(4)} - \frac{7}{40} a_0^2 a_0''' S + \frac{7}{54} a_0'^3$

$$-\frac{7}{72}a_0 a_1 S^{(6)} - \frac{1}{108}a_0' S^{(7)} + \frac{1}{144}a_0 S^{(8)}$$

Examples.

➤ $L^{\circledS 3}(y) = 0$ is of order 7 .

Example1. Take the equation :

$$L(y) = \frac{d^3 y}{dx^3} + \frac{32x^2 - 27x + 27}{36x^2(x-1)^2}\frac{dy}{dx} - \frac{64x^3 - 81x^2 + 135x - 54}{72x^3(x-1)^3}y$$

whose symmetric power $L^{\circledS 3}(y) = 0$ is of order 7 . The $\mathbb{Q}-$ vector space of Liouvillian solutions of $L^{\circledS 3}(y) = 0$ is generated by only one solution $s = x^4 - 2x^3 + x^2$. By using our algorithm, we obtain the coefficients of the minimal polynomial $P(U)$:

$c_3 = s = x^4 - 2x^3 + x^2$

$c_2 = s' = 4x^3 - 6x^2 + 2x$

$c_1 = 25313961574400 x^{11} - 122936959713280 x^{10} + 369414924755712 x^9$

$\quad\quad -759762796466484 x^8 + 1160223530219379 x^7 - 1352965374477072 x^6$

$\quad\quad +1190804183867628 x^5 - 779912777349666 x^4 + 353889936963693 x^3$

$\quad\quad -95718851436546 x^2 + 11393791055748 x - \frac{41822280936}{108}\big(21010579456 x^9$

$\quad\quad -79774543872 x^8 + 234889910208 x^7 - 435358558917 x^6 + 634598874729 x^5$

$\quad\quad -673422918633 x^4 + 534201757263 x^3 - 298530368262 x^2 + 96095161620 x$

$\quad\quad -12699314136\big)$

$c_0 = 40936268627968 x^{12} - 221509596151808 x^{11} + 669402865431552 x^{10}$

$\quad\quad -1391212752866640 x^9 + 2084599015102737 x^8 - 2335386769620696 x^7$

$\quad\quad +1894909901929902 x^6 - 1033536296671596 x^5 + 272124855869121 x^4$

$\quad\quad +84555330577452 x^3 - 101906800574184 x^2 + 33293137560192 x$

$\quad\quad -\frac{3816809262000}{216} x(x-1)(64x^3 - 81x^2 + 135x - 54)(328290304 x^6$

$\quad\quad -830984832 x^5 + 1925952309 x^4 - 2335090518 x^3 + 2196559377 x^2$

$\quad\quad -1191608820 x + 235172484\big)$

➤ $L^{\circledS 3}(y) = 0$ is of order 9 **.**

Example2. Take the equation:

$$L(y) = \frac{d^3 y}{dx^3} + \frac{8x-3}{x(x-1)}\frac{d^2 y}{dx^2} + \frac{3264x^2 - 2452x + 729}{108x^2(x-1)^2}\frac{dy}{dx}$$

$$-\frac{71280x^3 - 80410x^2 + 47466x - 10935}{1458x^3(x-1)^3}y$$

(example 4.5 in Ulmer 2003.) that can be transferred to the equation:

$$L_{SL}(y) = \frac{d^3 y}{dx^3} + \frac{96x^2 - 76x + 81}{108x^2(x-1)^2}\frac{dy}{dx} - \frac{2592x^3 - 3068x^2 + 5400x - 2187}{2916x^3(x-1)^3}y$$

by using procedure transfereL with Maple program (Okko 2018). The equation $L_{SL}(y)$ is irreducible, its differential Galois group is G_{162} and its third symmetric power $L^{\circledS 3}(y) = 0$ is of order 9 . The value of the semi-invariant is $s = \sqrt{(x-1)^5}\sqrt{x^3}$ whose square is rational and it generates the vector space of Liouvillian solutins of $L^{\circledS 3}(y) = 0$. By

using our algorithm we obtain the coefficients of the minimal polynomial $P(U)$:

$$c_3 = s = \sqrt{(x-1)^5}\sqrt{x^3}$$

$$c_2 = s' = \frac{5}{2}\sqrt{x^3}\sqrt{(x-1)^3} + \frac{3}{2}\sqrt{x}\sqrt{(x-1)^5}$$

$$c_1 = \frac{1}{432}\Big(27680816981606400x^{12} - 146369305291456512x^{11} + 460721269652889600x^{10}$$

$$-1009013808719533056x^9 + 1645470628884383488x^8 - 2078192403916191552x^7$$

$$+2032861092998975552x^6 - 1522715003967278160x^5 + 839232768599394624x^4$$

$$-307532640935458044x^3 + 62289431318500884x^2 - 4200887273556699x$$

$$-231958332276525\Big)\Big/\sqrt{x}\sqrt{x-1}\Big(5743767158784x^9 - 20393922244608x^8$$

$$+ 60302794885632x^7 - 109841060253248x^6 + 161225407720576x^5$$

$$-171833666743152x^4 + 138087079432992x^3 - 79030988627868x^2 + 25962912325800x - 3471675001929\Big)$$

$$c_0 = \frac{1}{23328}\Big(1208622863106441216x^{12} - 5620819173370232832x^{11} + 15699354875538112512x^{10}$$

$$-30138292242681495552x^9 + 41327897457410536960x^8 - 42973740430560205568x^7$$

$$+32763357563840003520x^6 - 18873014663604337920x^5 + 8966085528248791344x^4$$

$$-3863141918254277136x^3 + 1563648342203575476x^2 - 500362208768814480x$$

$$+79072079758122051\Big)\Big/\sqrt{x^3}\sqrt{x-1}\Big(5743767158784x^9 - 20393922244608x^8$$

$$+60302794885632x^7 - 109841060253248x^6$$

$$+161225407720576x^5 - 171833666743152x^4 + 138087079432992x^3 - 79030988627868x^2$$

$$+25962912325800x - 3471675001929\Big)$$

➤ $L^{\circledS 3}(y)=0$ is of order 10.

Example3. Let's take the equation (example 4.9 in Ulmer 2003):

$$L(y) = \frac{d^3y}{dx^3} + \frac{7x-3}{x(x-1)}\frac{d^2y}{dx^2} + \frac{2(138x^2-115x+7)}{27x^2(x-1)^2}\frac{dy}{dx}$$

$$+ \frac{10(162x^2-28x-7)}{729x^2(x-1)^2}y$$

It becomes as follow after using the transformation transfereL with Maple program (see transfereL Okko 2018):

$$L_{SL}(y) = \frac{d^3y}{dx^3} + \frac{2}{27}\frac{12x^2-7x+7}{x^2(x-1)^2}\frac{dy}{dx} - \frac{2}{729}\frac{(324x^3-301x^2+525x-224)}{x^3(x-1)^3}y$$

whose third symmetric power $L^{\circledS 3}(y)=0$ is of order 10, and its differential Galois group $\mathfrak{S}(L)=G_{81}$ (example 4.5 in Ulmer 2003). If $\{Y_1, Y_2, Y_3\}$ is a basis of the solution space of $L(y)=0$, then the invariants subspace of degree 3 of G_{81} is generated by the polynomial $I_1 = Y_1 Y_2 Y_3$ and the 3 semi-invariants spaces are generated by the polynomials $S_1 = Y_1^3 + Y_2^3 + Y_3^3$, $S_2 = Y_3^3 + w^3 Y_1^3 + (-w^3-1)Y_2^3$, and $S_3 = Y_3^3 + (-w^3-1)Y_1^3 + w^3 Y_2^3$, (cf. Ulmer 2003 section 4.3), W is the Wronskian. $s = x^4 - 2x^3 + x^2$ is an unique solution whose square is rational. By using our algorithm, we obtain the coefficients of the minimal polynomial $P(U)$:

$c_3 = s = x^4 - 2x^3 + x^2$

$c_2 = s' = 4x^3 - 6x^2 + 2x$

$c_1 = 25313961574400x^{11} - 26409495661920x^{10} + 369414924755712x^9$

$\qquad - 124939714582914x^8 + 172153006123784x^7 - 187209130557757x^6 + 156749233965863x^5$

$\qquad - 100541914181821x^4 + 47033417633853x^3 - 13872317475296x^2 + 1989962456888x$

$\qquad - 63264083456\big/54\big(11218295232x^9 - 31314188304x^8 + 84221681796x^7 - 135184221739x^6$

$\qquad + 181586038333x^5 - 181243862121x^4 + 137006478371x^3 - 78056142304x^2 + 26777872800x$

$\qquad - 3793656832\big)$

$c_0 = \big(73768485297024x^{12} - 333419069906784x^{11} + 855218148700536x^{10} - 369414924755712x^9$

$\qquad + 2139932814242671x^8 - 2227665744927448x^7 + 1741090046828426x^6 - 963045565711138x^5$

$\qquad + 324050153614243x^4 - 1905494677324x^3 - 56321850003132x^2 + 22301556579456x$

$\qquad - 2762481114400\big)/729\, x\big(34624368x^6 - 64482264x^5 - 143934343x^4 - 155095486x^3$

$\qquad + 138557839x^2 - 79850400x + 16935968\big)\big(324x^3 - 301x^2 + 525x - 224\big)$

3.　　In this section we find the procedures which we have spoken about in the Algorithm (in section 2.) proposed in this paper.

Code Maple

1.

```
#NormalForm1
with(diffalg):
with(Groebner):
with(PolynomialIdeals):
NormalForm1:=proc(y1,y2,y3,a0,a1)
local x,i,s,R,R1,b0,b1;
global y10,y20,y30,y11,y21,y31,y12,y22,y32,a00,a10,a01,
a02,a03,a04,a05,a06;
s:=y1(x)*y2(x)*y3(x);
R[0]:=s;
R1[0]:=subs([y1(x)=y10,y2(x)=y20,y3(x)=y30],R[0]);
b0:=a0(x);b1:=a1(x);
for i from 1 to 9 do
if i>=3 then R[i]:=diff(R[i-1],x);
R[i]:=subs(diff(y1(x),x$3)=b1*diff(y1(x),x)+
b0*y1(x),diff(y2(x),x$3)=b1*diff(y2(x),x)+b0*
y2(x),diff(y3(x),x$3)=b1*diff(y3(x),x)+b0*y3(x),
R[i]);
else
R[i]:=diff(R[i-1],x);R[i]:=subs(diff(b1,x)=2*b0,R[i]);
fi;
R1[i]:=subs([y1(x)=y10,y2(x)=y20,y3(x)=y30,diff(y1(x),x)=y11,
diff(y2(x),x)=y21,diff(y3(x),x)=y31,diff(y1(x),x$2)=y12,diff (y2(x),x$2)=y22,diff(y3(x),x$2)=y32,b0=a00,b1=a10,diff
(b0,x)=a01,diff(b0,x$2)=a02,diff(b0,x$3)=a03,diff(b0,x$
4)=a04,diff(b0,x$5)=a05,diff(b0,x$6)=a06],R[i]);
od;
```

```
return([seq(R1[i], i= 0..9)]);
end:
```

2.

```
#NormalForm2
NormalForm2:=proc(l1,l2,order)
local j,NF ;global x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,
x14,x15,x16,x17,x18,x19;
for j from 1 to nops(l1) do
NF[j-1]:=NormalForm(l1[j],l2,T);
od;
return([seq([j-1,NF[j-1]], j=1..10)]);
end:
T:=lexdeg([y10,y20,y30,y11,y21,y31,y12,y22,y32],[w000,
w100,w200,w110,w111,w222,w210,w220,w211,w221]):
B:=[w000-y10*y20*y30,w100-(y11*y20*y30+y10*y21*y30+y10*y20*
y31),w110-(y11*y21*y30+y11*y20*y31+y10*y21*y31),w200-(y12*
y20*y30+y10*y22*y30+y10*y20*y32),w210-(y11*y22*y30+y12*y21*
y30+y12*y20*y31+y11*y20*y32+y10*y22*y31+y10*y21*y32),w111-
(y11*y21*y31),w220-(y12*y22*y30+y12*y20*y32+y10*y22*y32),w211
-(y12*y21*y31+y11*y22*y31+y11*y21*y32),w221-(y12*y22*y31+y12*
y21*y32+y11*y22*y32),w222-(y12*y22*y32)]:
N2:=NormalForm2(N1,B,T);
```

Code Magma

1.

```
//s,a0,a1 and their derivatives
Q:=RationalField();
p<w000,w100,w200,w110,w210,w111,w220,w211,w221,w222,a00,
a10,a01,a02,a03,a04,a05,a06,s0,s1,s2,s3,s4,s5,s6,s7,s8,
s9,c0,c1>:=PolynomialRing(Q,30,"elim",10);
N2:=[w000, w100, w200+2*w110,6*w111+3*w210+3*a00*w000+
a10*w100, ...];
//N2 is the result of le resultat NormalForm2 procedure realized by Maple
id:=ideal<p|s0-N2[1],s1-N2[2],s2-N2[3],s3-N2[4],s4-N2[5],s5-
N2[6],s6-N2[7],s7-N2[8],s8-N2[9],s9-N2[10],c0-w111,c1-w110>;
GR:=GroebnerBasis(id);
```

2 .

```
//eliminate c0
p1<c0,c1,a00,a10,a01,a02,a03,a04,a05,a06,s0,s1,s2,s3,
s4,s5,s6,s7,s8,s9>:=PolynomialRing(Q,20,"elim",1);
sd:=[la suite des 6 derniers elements de la base de
Groebner GR];
id:=ideal<p1|sd>;
```

Gr:=GroebnerBasis(id);

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