A New Eighth Order Runge-Kutta Family Method

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Abstract

In this article, a new family of Runge-Kutta methods of 8^{th} order for solving ordinary differential equations is discovered and depends on the parameters b_8 and $a_{10,5}$. For $b_8 = 49/180$ and $a_{10,5} = 1/9$, we find the Cooper-Verner method [1]. We show that the stability region depends only on coefficient $a_{10,5}$. We compare the stability regions according to the values of $a_{10,5}$ with respect to the stability region of Cooper-Verner.

Keywords: Runge-Kutta, ordinary differential equations, Cooper-Verner, region of stability

1. Introduction

Since the time of Newton, one of the main problems of mathematicians is the resolution of various differential equations. Practically, the immense quantity of these equations were not resolvable in the analytical aspect. This has led to the development of numerical methods for their resolution. The method of Runge-Kutta, named RK is used to find a good numerical result. Currently, a large number of high-order RK methods are known (5..10) [1, 2, 4, 8, 9, 10]. However, they are not all found. For example, methods of 8^{th} order depending of several parameters are not presented in the literature. The field of stability has not been fully investigated. In this article we discover new family of order 8 depending on several parameters.This family summarizes the Cooper-Verner method [1]. It depends on the coefficients b_8 and $a_{10,5}$. For $b_8 = 49/180$ and $a_{10,5} = 1/9$, we find the Cooper-Verner method [1]. Subsequently, we show that the region of stability of this family depends only on the coefficient $a_{10,5}$ and not on b_8. The study will consist in comparing the regions of stability according to the values of $a_{10,5}$ with respect to the stability region from Cooper-Verner.

The study will be led by respecting the following plan: in section 2 Presentation of the new family RK8 method, section 3 The stability region, section 4 Comparison of some stability regions, section 5 Conclusion.

2. Presentation of the New Family RK8 Method

Consider a general form of the first-order ODE given below:

$$y' = f(x, y(x)), \tag{1}$$

with the initial condition $y(x_0) = y_0$ for the interval $x_0 \le x \le x_n$. Here, x is the independent variable, y is the dependent variable, n is the number of point values, and f is the function of the derivation. The goal is to determine the unknown function $y(x)$ whose derivative satisfies (1) and the corresponding initial values. In doing so, let us discretize the interval $x_0 \le x \le x_n$ to be

$$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, ..., x_n = x_0 + nh, \tag{2}$$

where h is the fixed step size. With the initial condition $y(x_0) = y_0$, the unknown function $y_1, y_2, y_3, \cdots, y_n$ can be solved by using the RK8 method.

The family of 8^{th} order method is thus obtained by the resolution of the 200 equations with 11 stages (see Appendix A) on Maple. However, the 200 equations are obtained from the successive derivatives of the exact solution up to 8 (see Appendix B).

Lets consider the Butcher tableau of 8 order 11 steps RK method(see Fig. 1):

$$
\begin{array}{c|ccccccccccc}
0 & & & & & & & & & & & \\
c_2 & a_{2,1} & & & & & & & & & & \\
c_3 & a_{3,1} & a_{3,2} & & & & & & & & & \\
c_4 & a_{4,1} & a_{4,2} & a_{4,3} & & & & & & & & \\
c_5 & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & & & & & & & \\
c_6 & a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & & & & & & \\
c_7 & a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & & & & & \\
c_8 & a_{8,1} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & a_{8,7} & & & & \\
c_9 & a_{9,1} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & a_{9,6} & a_{9,7} & a_{9,8} & & & \\
c_{10} & a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & & \\
c_{11} & a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} & \\
\hline
 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11}
\end{array}
$$

Figure 1. Butcher tableau of RK8 family

with free parameters b_8 and $a_{10,5}$.

Some of these coefficients have fixed values, not depending on b_8 and $a_{10,5}$, these coefficients are:

$$b_1 = \frac{1}{20};\ b_2 = 0;\ b_3 = 0; b_4 = 0;\ b_5 = 0; b_6 = 0;\ b_9 = \frac{16}{45};\ b_{10} = \frac{49}{180};\ b_{11} = \frac{1}{20}; \tag{3}$$

$$c_2 = \frac{1}{2};\ c_3 = \frac{1}{2};\ c_4 = \frac{7+\sqrt{21}}{14}; c_5 = \frac{7+\sqrt{21}}{14};\ c_6 = \frac{1}{2}; \tag{4}$$

$$c_7 = \frac{7-\sqrt{21}}{14}; c_8 = \frac{7-\sqrt{21}}{14};\ c_9 = \frac{1}{2};\ c_{10} = \frac{7+\sqrt{21}}{14};\ c_{11} = 1 \tag{5}$$

$$a_{2,1} = \frac{1}{2}; \tag{6}$$

$$a_{3,1} = \frac{1}{4};\ a_{3,2} = \frac{1}{4}; \tag{7}$$

$$a_{4,1} = \frac{1}{7};\ a_{4,2} = \frac{-7-3\sqrt{21}}{98};\ a_{4,3} = \frac{21+5\sqrt{21}}{49}; \tag{8}$$

$$a_{5,1} = \frac{11+\sqrt{21}}{84};\ a_{5,2} = 0;\ a_{5,3} = \frac{4\sqrt{21}}{63} + \frac{2}{7};\ a_{5,4} = \frac{21-\sqrt{21}}{252}; \tag{9}$$

$$a_{6,1} = \frac{5+\sqrt{21}}{48};\ a_{6,2} = 0;\ a_{6,3} = \frac{9+\sqrt{21}}{36};\ a_{6,4} = \frac{-231+14\sqrt{21}}{360};\ a_{6,5} = \frac{63-7\sqrt{21}}{80}; \tag{10}$$

$$a_{7,1} = \frac{10-\sqrt{21}}{42};\ a_{7,2} = 0; \tag{11}$$

$$a_{9,1} = \frac{1}{32};\ a_{9,2} = 0; \tag{12}$$

$$a_{10,1} = \frac{1}{14};\ a_{10,2} = 0; a_{10,9} = \frac{4\sqrt{21}}{35} + \frac{132}{245}; \tag{13}$$

$$a_{11,1} = 0;\ a_{11,2} = 0; a_{11,9} = \frac{28-28\sqrt{21}}{45};\ a_{11,10} = \frac{49-7\sqrt{21}}{18}; \tag{14}$$

And the others are expressed in terms of b_8 and $a_{10,5}$:

$$b_7 = -b_8 + \frac{49}{180}; \tag{15}$$

$$a_{7,3} = -(24/35)a_{10,5} - 136/105 - (12/245)a_{10,5}\sqrt{21} + (656/2205)\sqrt{21}; \tag{16}$$

$$a_{7,4} = 7 - (3/10)a_{10,5}\sqrt{21} - (71/45)\sqrt{21} + (3/10)a_{10,5}; \tag{17}$$

$$a_{7,5} = -(3/10)a_{10,5} + (3/10)a_{10,5}\sqrt{21} - 43/6 + (169/105)\sqrt{21}; \tag{18}$$

$$a_{7,6} = -(277/735)\sqrt{21} + 181/105 + (12/245)a_{10,5}\sqrt{21} + (24/35)a_{10,5}; \tag{19}$$

$$a_{8,1} = -\frac{180b_8\sqrt{21} - 49\sqrt{21} - 1800b_8 + 343}{7560b_8};\ a_{8,2} = 0; \tag{20}$$

$$a_{8,5} = -\frac{441a_{10,5}\sqrt{21} - 3240a_{7,5}b_8 - 28\sqrt{21} + 882a_{7,5} - 2205a_{10,5} + 147}{3240b_8}; \tag{21}$$

$$a_{8,6} = \frac{72a_{10,5}\sqrt{21} + 1620a_{7,6}b_8 - 29\sqrt{21} - 441a_{7,6} - 252a_{10,5} + 119}{1620b_8}; \tag{22}$$

$$a_{8,3} = -\frac{900b_8\sqrt{21} + 11340a_{7,2}b_8 + 11340a_{8,6}b_8 - 98\sqrt{21} - 3087a_{7,2} - 4860b_8 + 686}{11340b_8}; \tag{23}$$

$$a_{8,7} = \frac{49}{1620b_8}; \tag{24}$$

$$a_{8,4} = \frac{(c_8^2/2) - a_{8,2}c_2 - a_{8,3}c_3 - a_{8,5}c_5 - a_{8,6}c_6 - a_{8,7}c_7}{c_4}; \tag{25}$$

$$a_{9,3} = (1/8)a_{10,5}\sqrt{21} - (1/8)a_{10,5} - (1/72)\sqrt{21} + 1/72; \tag{26}$$

$$a_{9,4} = -49/288 - (7/32)a_{10,5}\sqrt{21} + (7/288)\sqrt{21} + (49/32)a_{10,5}; \tag{27}$$

$$a_{9,5} = (7/32)a_{10,5}\sqrt{21} - (35/576)\sqrt{21} - (49/32)a_{10,5} + 21/64; \tag{28}$$

$$a_{9,6} = -(1/8)a_{10,5}\sqrt{21} + (1/8)a_{10,5} + (1/72)\sqrt{21} + 5/36; \tag{29}$$

$$a_{9,7} = 91/576 + (7/192)\sqrt{21} - (585/1568)b_8\sqrt{21} - (405/224)b_8; \tag{30}$$

$$a_{9,8} = (585/1568)b_8\sqrt{21} + (405/224)b_8; \tag{31}$$

$$a_{10,3} = -(6/49)a_{10,5}\sqrt{21} - (2/7)a_{10,5} + (2/147)\sqrt{21} + 2/63; \tag{32}$$

$$a_{10,4} = 1/9 - a_{10,5}; \tag{33}$$

$$a_{10,6} = (2/7)a_{10,5} - 803/2205 + (6/49)a_{10,5}\sqrt{21} - (59/735)\sqrt{21}; \tag{34}$$

$$a_{10,7} = 1/9 + (1/42)\sqrt{21} + (2295/686)b_8 + (495/686)b_8\sqrt{21}; \tag{35}$$

$$a_{10,8} = -(2295/686)b_8 - (495/686)b_8\sqrt{21}; \tag{36}$$

$$a_{11,3} = (2/3)a_{10,5}\sqrt{21} - (2/3)a_{10,5} - (2/27)\sqrt{21} + 2/27; \tag{37}$$

$$a_{11,4} = -(7/6)a_{10,5}\sqrt{21} + (7/54)\sqrt{21} + (49/6)a_{10,5} - 49/54; \tag{38}$$

$$a_{11,5} = (7/27)\sqrt{21} - 77/54 - (49/6)a_{10,5} + (7/6)a_{10,5}\sqrt{21}; \tag{39}$$

$$a_{11,6} = (2/3)a_{10,5} - 64/135 - (2/3)a_{10,5}\sqrt{21} + (94/135)\sqrt{21}; \tag{40}$$

$$a_{11,7} = 7/18 - (265/98)b_8\sqrt{21} - (215/14)b_8; \tag{41}$$

$$a_{11,8} = (265/98)b_8\sqrt{21} + (215/14)b_8; \tag{42}$$

The numerical solution is given by the formula

$$y_{i+1} = y_i + h\left(\sum_{s=1}^{11} b_s k_s\right) \tag{43}$$

with

$$k_s = f\left(x_i + c_s h, y_i + h\sum_{j=1}^{s-1} a_{s,j}k_j\right),\quad x_{i+1} = x_i + h \tag{44}$$

We can notice that if $b_8 = 49/180$ and $a_{10,5} = 1/9$, then we find the method of Cooper-Verner [1].

3. The Stability Regions

The concept of stability is based on the discrete solution. It gives back account of the actual behavior of the approximate solution for a given practical value, therefore non-zero, of the h step. In real calculation, the errors accumulate. This is particularly evident in the process of solving a differential equation where one progresses step by step from an initial value. There are various stability conditions. First of all the numerical solution must remain limited. This requirement of minimum stability may be insufficient in practice, the bound obtained being often an exponential of the duration which therefore grows infinitely when it increases. We then introduce more stability criteria, demanding that the digital solution reproduce the behavior physical of the exact solution. The concept of absolute stability, in its simplest form, is based on the analysis of the behavior, according to the values of the step h, of the numerical solutions of the equation model $[1, 3, 5, 6]$:

$$u'(t) = \lambda u(t) \tag{45}$$

Using (44) and (45), we obtain:

$$k_1 = \lambda y_i; \tag{46}$$

$$\text{for } \; s > 1, \; k_s = \lambda\left(y_i + h\sum_{j=1}^{s-1} a_{s,j}k_j\right); \tag{47}$$

Which give:

$$y_{i+1} = \zeta\,(h\lambda)\,y_i \tag{48}$$

Let's put $z = h\lambda$. We obtain by Maple:

$$\begin{gathered}\zeta(z) = 1 - z + (1/2)z^2 - (1/6)z^3 + (1/24)z^4 - (1/120)z^5 + (1/720)z^6 - (1/5040)z^7 + (1/40320)z^8 + \\ (797/50803200)z^9 + (499/152409600)\sqrt{21}z^9 - (1/25200)z^9 a_{10,5} - (37/4233600)\sqrt{21}z^9 a_{10,5} + \\ (1/470400)z^{10} + (1/2083725)\sqrt{21}z^{10} - (31/940800)z^{10}a_{10,5} - (61/8467200)\sqrt{21}z^{10}a_{10,5} + \\ (13/4267468800)\sqrt{21}z^{11} - (11/1612800)z^{11}a_{10,5} - (353/237081600)\sqrt{21}z^{11}a_{10,5} + (1/29030400)z^{11}\end{gathered} \tag{49}$$

The absolute stability region is the set

$$\{z \in \mathbb{C} | \;\; |\zeta(z)| \leq 1\} \tag{50}$$

4. Comparison of Some Stability Regions

The stability region depends on coefficient $a_{10,5}$. We obtain by Maple different regions according to values of $a_{10,5}$.

For $a_{10,5} = 1/9$, we obtain the stability region of the Cooper-Verner method (see Fig. 2). From the point of view of the values of x, if we choose $a_{10,5} = \frac{1}{10}$(see Fig. 3) or $a_{10,5} = \frac{1}{10.9}$(see Fig. 4) or $a_{10,5} = \frac{1}{11.9}$(see Fig. 5), then we see that the stability regions are bigger than the region of Cooper-Verner.

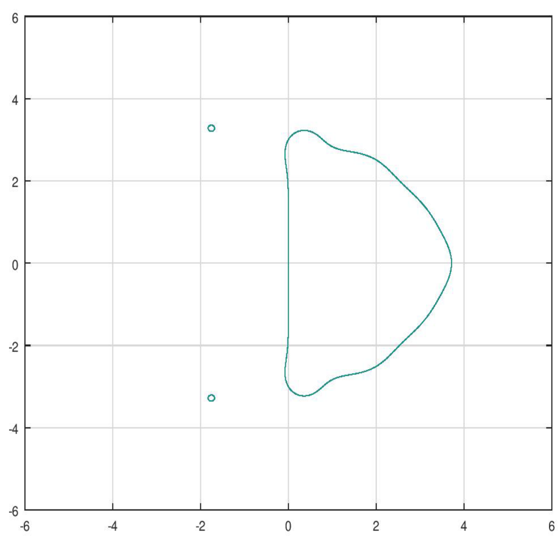

Figure 2. Stability region for a[10,5]=1/9

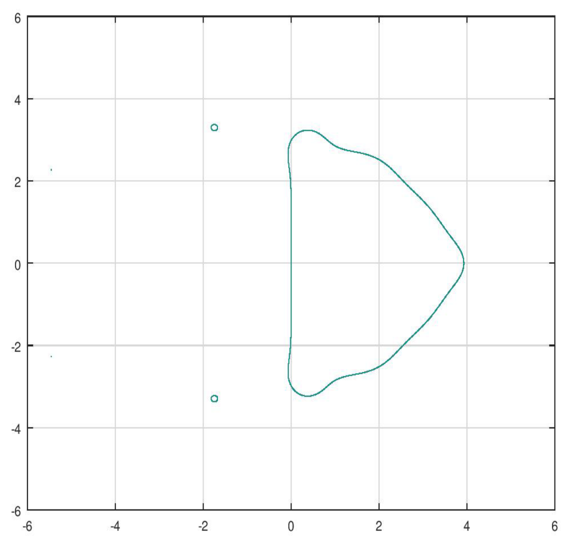

Figure 3. Stability region for a[10,5]=1/10

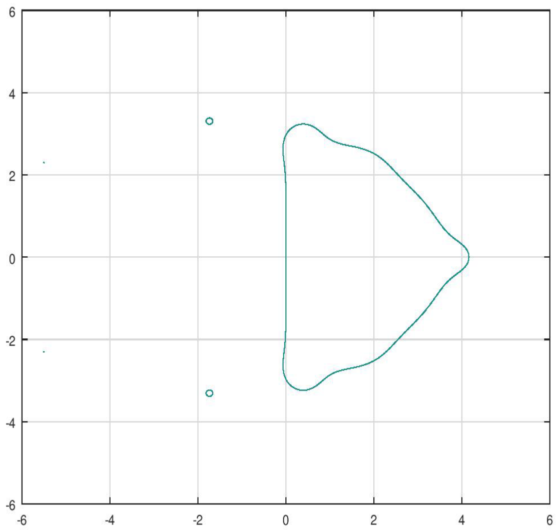

Figure 4. Stability region for a[10,5]=1/10.9

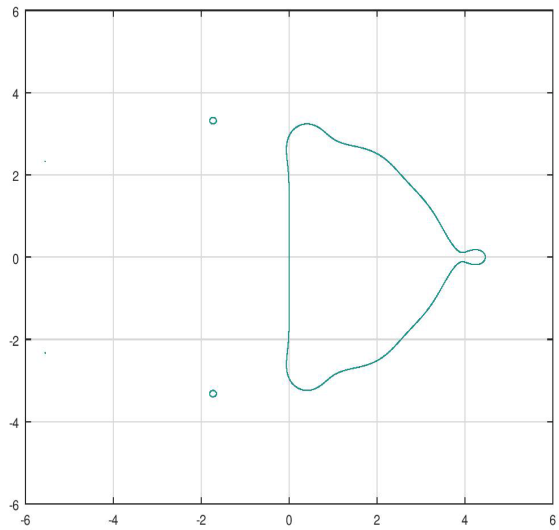

Figure 5. Stability region for a[10,5]=1/11.9

Let's see some stability region of $a_{10,5} = \{1, 10, 100, 1000\}$. (see Fig. 6, Fig. 7, Fig. 8, Fig. 9). We see that the more $a_{10,5} > 1$, the more the stability region becomes smaller. We can notice that all stability regions are smaller than of the Cooper-Verner one.

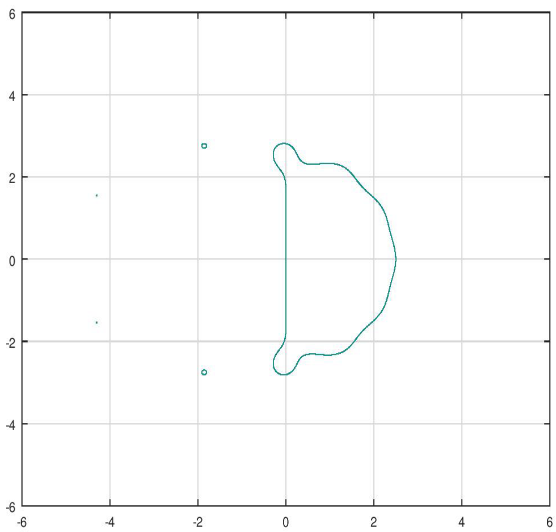

Figure 6. Stability region for a[10,5]=1

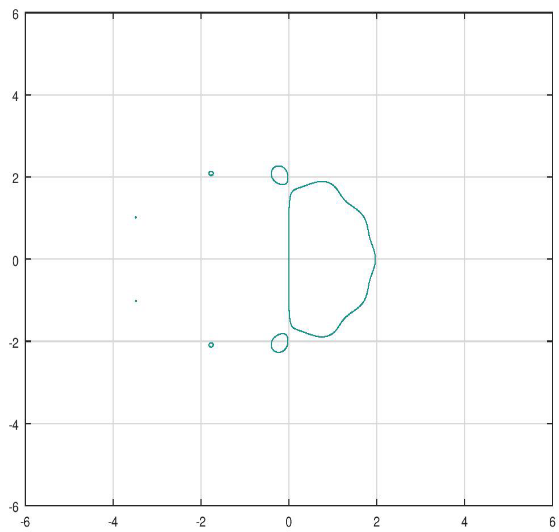

Figure 7. Stability region for a[10,5]=10

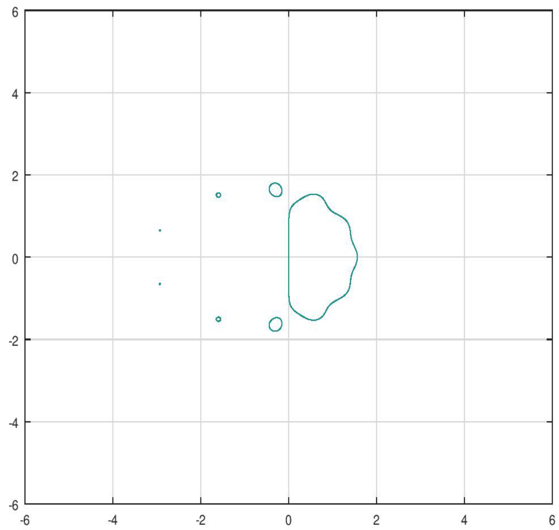

Figure 8. Stability region for a[10,5]=100

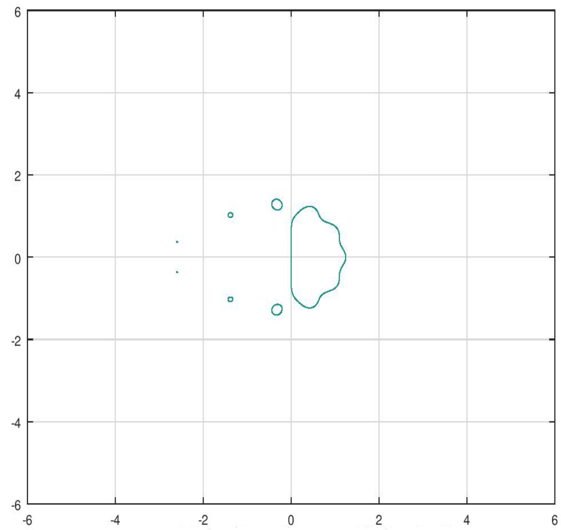

Figure 9. Stability region for a[10,5]=1000

Let's see some stability region of $a_{10,5} = \{0, -1, -10, -100\}$. (see Fig. 10, Fig. 11, Fig. 12, Fig. 13). We can see that the more $a_{10,5} < 0$, the more the stability region is smaller. We also notice that all the stability regions are smaller than that of the Cooper-Verner one.

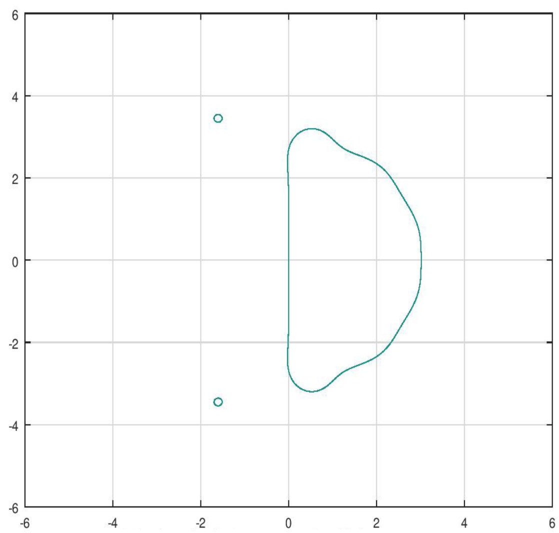

Figure 10. Stability region of a[10,5]=0

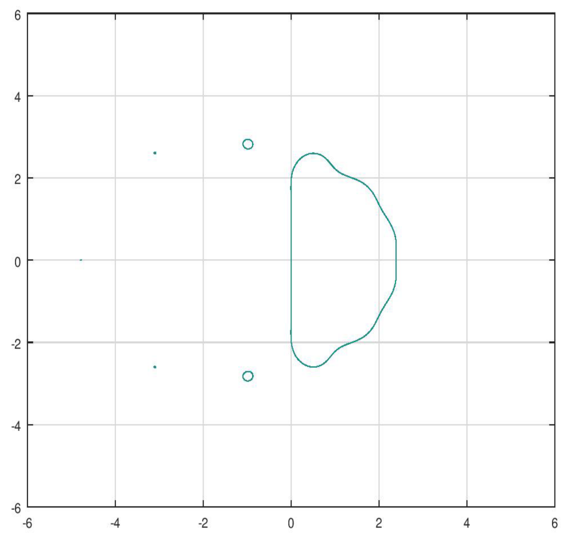

Figure 11. Stability region for a[10,5]=-1

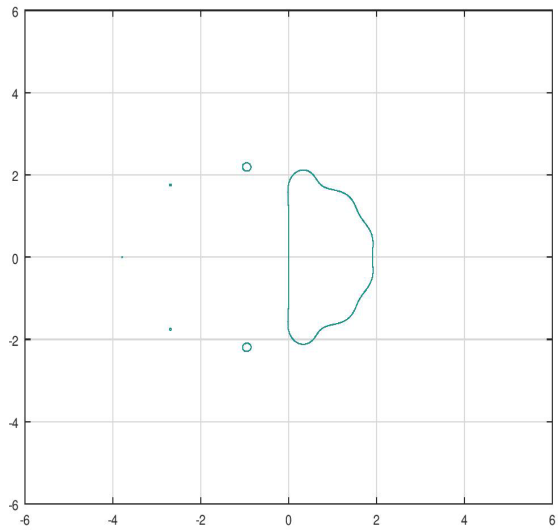
Figure 12. Stability region for a[10,5]=-10

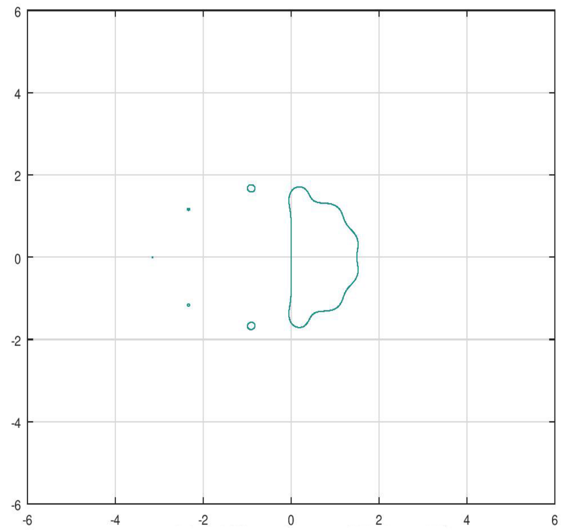
Figure 13. Stability region for a[10,5]=-100

5. Conclusion

A new family of Runge-Kutta method 8^{th} order is discovered. This family depends one the parameters $a_{10,5}$ and b_8. For $a_{10,5} = 1/9$ and $b_8 = 49/180$, we find the method of Cooper-Verner. The stability region depends on the value of $a_{10,5}$ but not of the b_8 coefficient. If we want to obtain a region of stability greater than that of the Cooper-Verner method, then it is not better to choose negative values of $a_{10,5}$ or values greater than 1. You have to choose wisely $a_{10,5}$ between 0 and 1.

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Appendix A

The 200 equations of order 8 with s stages

$\sum_{i}^{s} b_i = 1$
$\sum_{i}^{s} b_i c_i = \frac{1}{2}$
$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}$
$\sum_{i,j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$
$\sum_{i=1}^{s} b_i c_i^3 = \frac{1}{4}$
$\sum_{i,j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$
$\sum_{i,j=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{12}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$

$\sum_{i=1}^{s} b_i c_i^4 = \frac{1}{5}$
$\sum_{i,j=1}^{s} b_i c_i^2 a_{ij} c_j = \frac{1}{10}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j a_{ik} c_k = \frac{1}{20}$
$\sum_{i,j=1}^{s} b_i c_i a_{ij} c_j^2 = \frac{1}{15}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} a_{jk} c_k = \frac{1}{30}$
$\sum_{i,j=1}^{s} b_i a_{ij} c_j^3 = \frac{1}{20}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j a_{jk} c_k = \frac{1}{40}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} c_k^2 = \frac{1}{60}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l = \frac{1}{120}$

$\sum_{i=1}^{s} b_i c_i^5 = \frac{1}{6}$
$\sum_{i,j=1}^{s} b_i c_i^3 a_{ij} c_j = \frac{1}{12}$
$\sum_{i,j=1}^{s} b_i c_i^2 a_{ij} c_j^2 = \frac{1}{18}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} a_{jk} c_j = \frac{1}{36}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j a_{ik} c_k = \frac{1}{24}$
$\sum_{i,j=1}^{s} b_i c_i a_{ij} c_j^3 = \frac{1}{24}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j a_{jk} c_k = \frac{1}{48}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} a_{jk} c_k^2 = \frac{1}{72}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} c_l = \frac{1}{144}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^2 a_{ik} c_k = \frac{1}{36}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{ik} a_{kl} c_l = \frac{1}{72}$
$\sum_{i,j=1}^{s} b_i a_{ij} c_j^4 = \frac{1}{30}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^2 a_{jk} c_k = \frac{1}{60}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j a_{jk} c_k^2 = \frac{1}{90}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} c_l = \frac{1}{180}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k a_{jl} c_l = \frac{1}{120}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} c_k^3 = \frac{1}{120}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} c_l = \frac{1}{240}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l^2 = \frac{1}{360}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} c_m = \frac{1}{720}$

$\sum_{i=1}^{s} b_i c_i^6 = \frac{1}{7}$
$\sum_{i,j=1}^{s} b_i c_i^4 a_{ij} c_j = \frac{1}{14}$
$\sum_{i,j=1}^{s} b_i c_i^3 a_{ij} c_j^2 = \frac{1}{21}$
$\sum_{i,j,k=1}^{s} b_i c_i^3 a_{ij} a_{jk} c_k = \frac{1}{42}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} c_j a_{ik} c_k = \frac{1}{28}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j a_{ik} a_{kl} c_l = \frac{1}{84}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j a_{ik} c_k^2 = \frac{1}{42}$
$\sum_{i,j=1}^{s} b_i c_i^2 a_{ij} c_j^3 = \frac{1}{28}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} c_j a_{jk} c_k = \frac{1}{56}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} a_{jk} c_k^2 = \frac{1}{84}$
$\sum_{i,j,k,l=1}^{s} b_i c_i^2 a_{ij} a_{jk} a_{kl} c_l = \frac{1}{168}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{ik} a_{il} c_j c_k c_l = \frac{1}{56}$
$\sum_{i,j=1}^{s} b_i c_i a_{ij} c_j^4 = \frac{1}{35}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j^2 a_{jk} c_k = \frac{1}{70}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j a_{jk} c_k^2 = \frac{1}{105}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j a_{jk} a_{kl} c_l = \frac{1}{210}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} a_{jk} c_k^3 = \frac{1}{140}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} c_k a_{kl} c_l = \frac{1}{280}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} c_l^2 = \frac{1}{420}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} a_{lm} c_m = \frac{1}{840}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^3 a_{ik} c_k = \frac{1}{56}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} c_k a_{il} c_l = \frac{1}{112}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k^2 a_{il} c_l = \frac{1}{168}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l a_{im} c_m = \frac{1}{336}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{ik} c_j^2 c_k^2 = \frac{1}{63}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{ik} a_{kl} c_l = \frac{1}{126}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{il} a_{lm} c_m = \frac{1}{252}$
$\sum_{i,j=1}^{s} b_i a_{ij} c_j^5 = \frac{1}{42}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^3 a_{jk} c_k = \frac{1}{84}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^2 a_{jk} c_k^2 = \frac{1}{126}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{jk} a_{kl} c_l = \frac{1}{252}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} a_{jl} c_k c_l = \frac{1}{168}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j a_{jk} c_k^3 = \frac{1}{168}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} c_k a_{kl} c_l = \frac{1}{336}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} c_l^2 = \frac{1}{504}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} a_{lm} c_m = \frac{1}{1008}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k a_{jl} c_l^2 = \frac{1}{252}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{jl} a_{lm} c_m = \frac{1}{504}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} c_k^4 = \frac{1}{210}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k^2 a_{kl} c_l = \frac{1}{420}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} c_l^2 = \frac{1}{630}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} a_{lm} c_m = \frac{1}{1260}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{km} c_l c_m = \frac{1}{840}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l^3 = \frac{1}{840}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l a_{lm} c_m = \frac{1}{1680}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} c_m^2 = \frac{1}{2520}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} a_{mn} c_n = \frac{1}{5040}$

$\sum_{i=1}^{s} b_i c_i^7 = \frac{1}{8}$
$\sum_{i,j=1}^{s} b_i c_i^5 a_{ij} c_j = \frac{1}{16}$
$\sum_{i,j=1}^{s} b_i c_i^4 a_{ij} c_j^2 = \frac{1}{24}$
$\sum_{i,j,k=1}^{s} b_i c_i^4 a_{ij} a_{jk} c_k = \frac{1}{48}$
$\sum_{i,j,k=1}^{s} b_i c_i^3 a_{ij} a_{ik} c_j c_k = \frac{1}{32}$
$\sum_{i,j=1}^{s} b_i c_i^3 a_{ij} c_j^3 = \frac{1}{32}$
$\sum_{i,j,k=1}^{s} b_i c_i^3 a_{ij} c_j a_{jk} c_k = \frac{1}{64}$
$\sum_{i,j,k=1}^{s} b_i c_i^3 a_{ij} a_{jk} c_k^2 = \frac{1}{96}$
$\sum_{i,j,k,l=1}^{s} b_i c_i^3 a_{ij} a_{jk} a_{kl} c_l = \frac{1}{192}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} c_j a_{ik} c_k^2 = \frac{1}{48}$
$\sum_{i,j,k,l=1}^{s} b_i c_i^2 a_{ij} c_j a_{ik} a_{kl} c_l = \frac{1}{96}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{ik} a_{il} c_j c_k c_l = \frac{1}{64}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{ik} a_{il} c_k c_l = \frac{1}{96}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{ik} c_j c_k a_{il} a_{lm} c_m = \frac{1}{192}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j a_{ik} c_k^3 = \frac{1}{64}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j a_{ik} c_k a_{kl} c_l = \frac{1}{128}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j a_{ik} a_{kl} c_l^2 = \frac{1}{192}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} c_j a_{ik} a_{kl} a_{lm} c_m = \frac{1}{384}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j^2 a_{ik} a_{kl} c_l = \frac{1}{144}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j^2 a_{ik} c_k^2 = \frac{1}{72}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} a_{jk} c_k a_{il} a_{lm} c_m = \frac{1}{288}$
$\sum_{i,j=1}^{s} b_i c_i^2 a_{ij} c_j^4 = \frac{1}{40}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} c_j^2 a_{jk} c_k = \frac{1}{80}$
$\sum_{i,j,k,l=1}^{s} b_i c_i^2 a_{ij} a_{jk} a_{jl} c_k c_l = \frac{1}{160}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} c_j a_{jk} c_k^2 = \frac{1}{120}$
$\sum_{i,j,k,l=1}^{s} b_i c_i^2 a_{ij} c_j a_{jk} a_{kl} c_l = \frac{1}{240}$
$\sum_{i,j,k=1}^{s} b_i c_i^2 a_{ij} a_{jk} c_k^3 = \frac{1}{160}$
$\sum_{i,j,k,l=1}^{s} b_i c_i^2 a_{ij} a_{jk} c_k a_{kl} c_l = \frac{1}{320}$
$\sum_{i,j,k,l=1}^{s} b_i c_i^2 a_{ij} a_{jk} a_{kl} c_l^2 = \frac{1}{480}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i^2 a_{ij} a_{jk} a_{kl} a_{lm} c_m = \frac{1}{960}$
$\sum_{i,j=1}^{s} b_i c_i a_{ij} c_j^5 = \frac{1}{48}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j^3 a_{jk} c_k = \frac{1}{96}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j^2 a_{jk} c_k^2 = \frac{1}{144}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j^2 a_{jk} a_{kl} c_l = \frac{1}{288}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j a_{jk} a_{jl} c_k c_l = \frac{1}{192}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} c_j a_{jk} c_k^3 = \frac{1}{192}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j a_{jk} c_k a_{kl} c_l = \frac{1}{384}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} c_j a_{jk} a_{kl} c_l^2 = \frac{1}{576}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} c_j a_{jk} a_{kl} a_{lm} c_m = \frac{1}{1152}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} c_k^2 a_{jl} c_l = \frac{1}{288}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} a_{jk} c_k a_{jl} a_{lm} c_m = \frac{1}{576}$
$\sum_{i,j,k=1}^{s} b_i c_i a_{ij} a_{jk} c_k^4 = \frac{1}{240}$

$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} c_k^2 a_{kl} c_l = \frac{1}{480}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} c_k a_{kl} c_l^2 = \frac{1}{720}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} c_k a_{kl} a_{lm} c_m = \frac{1}{1440}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} a_{km} c_l c_m = \frac{1}{960}$
$\sum_{i,j,k,l=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} c_l^3 = \frac{1}{960}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} c_l a_{lm} c_m = \frac{1}{1920}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} a_{lm} c_m^2 = \frac{1}{2880}$
$\sum_{i,j,k,l,m=1}^{s} b_i c_i a_{ij} a_{jk} a_{kl} a_{lm} a_{mn} c_n = \frac{1}{5760}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j a_{ik} c_k^4 = \frac{1}{80}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{ik} c_k^2 a_{kl} c_l = \frac{1}{160}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{ik} a_{kl} a_{km} c_l c_m = \frac{1}{320}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{ik} c_k a_{kl} c_l^2 = \frac{1}{240}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{ik} c_k a_{kl} a_{lm} c_m = \frac{1}{480}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{ik} a_{kl} c_l^3 = \frac{1}{320}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{ik} a_{kl} c_l a_{lm} c_m = \frac{1}{640}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{ik} a_{kl} a_{lm} c_m^2 = \frac{1}{960}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} c_j a_{ik} a_{kl} a_{lm} a_{mn} c_n = \frac{1}{1920}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^3 a_{ik} c_k^2 = \frac{1}{96}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^3 a_{ik} a_{kl} c_l = \frac{1}{192}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{ik} c_k a_{kl} c_l = \frac{1}{192}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{il} c_l a_{lm} c_m = \frac{1}{384}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{ik} a_{kl} c_l^2 = \frac{1}{288}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{il} a_{lm} c_m^2 = \frac{1}{576}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j^2 a_{ik} a_{kl} a_{lm} c_m = \frac{1}{576}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} c_k a_{il} a_{lm} a_{mn} c_n = \frac{1}{1152}$
$\sum_{i,j=1}^{s} b_i a_{ij} c_i^6 = \frac{1}{56}$

$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^4 a_{jk} c_k = \frac{1}{112}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^3 a_{jk} c_k^2 = \frac{1}{168}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^3 a_{jk} a_{kl} c_l = \frac{1}{336}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{jk} c_l a_{jl} c_k = \frac{1}{224}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{jk} c_k a_{jl} a_{lm} c_m = \frac{1}{672}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} c_k a_{jl} c_l^2 = \frac{1}{336}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^2 a_{jk} c_k^3 = \frac{1}{224}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{jk} c_k a_{kl} c_l = \frac{1}{448}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j^2 a_{jk} a_{kl} c_l^2 = \frac{1}{672}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j^2 a_{jk} a_{kl} a_{lm} c_m = \frac{1}{1344}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{jl} a_{jm} c_k c_l c_m = \frac{1}{448}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j a_{jk} c_k^4 = \frac{1}{280}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} c_k^2 a_{kl} c_l = \frac{1}{560}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} a_{km} c_m c_l = \frac{1}{1120}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} c_k a_{kl} c_l^2 = \frac{1}{840}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{jk} c_k a_{kl} a_{lm} c_m = \frac{1}{1680}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} c_l^3 = \frac{1}{1120}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} c_l a_{lm} c_m = \frac{1}{2240}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} a_{lm} c_m^2 = \frac{1}{3360}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} c_j a_{jk} a_{kl} a_{lm} a_{mn} c_n = \frac{1}{6720}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k a_{jl} c_l^3 = \frac{1}{448}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{jl} c_l a_{lm} c_m = \frac{1}{896}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{jl} a_{lm} c_m^2 = \frac{1}{1344}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} c_k a_{jl} a_{lm} a_{mn} c_n = \frac{1}{2688}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k^2 a_{jl} c_l^2 = \frac{1}{504}$

$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k^2 a_{jl} a_{lm} c_m = \frac{1}{1008}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l a_{jm} a_{mn} c_n = \frac{1}{2016}$
$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} c_k^5 = \frac{1}{336}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k^3 a_{kl} c_l = \frac{1}{672}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k^2 a_{kl} c_l^2 = \frac{1}{1008}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k^2 a_{kl} a_{lm} c_m = \frac{1}{2016}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} a_{km} c_m c_l = \frac{1}{1344}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} c_l^3 = \frac{1}{1344}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} c_l a_{lm} c_m = \frac{1}{2688}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} a_{lm} c_m^2 = \frac{1}{4032}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} c_k a_{kl} a_{lm} a_{mn} c_n = \frac{1}{8064}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l^2 a_{km} c_m = \frac{1}{2016}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l a_{km} a_{mn} c_m = \frac{1}{4032}$
$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l^4 = \frac{1}{1680}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} c_l^2 a_{lm} c_m = \frac{1}{3360}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} c_l c_m^2 = \frac{1}{5040}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} a_{mn} c_n c_l = \frac{1}{10080}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} a_{ln} c_n c_m = \frac{1}{6720}$
$\sum_{i,j,k,l,m=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} c_m^3 = \frac{1}{6720}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} a_{mn} c_n c_m = \frac{1}{13440}$
$\sum_{i,j,k,l,m,n=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} a_{mn} c_n^2 = \frac{1}{20160}$
$\sum_{i,j,k,l,m,n,o=1}^{s} b_i a_{ij} a_{jk} a_{kl} a_{lm} a_{mn} a_{no} c_o = \frac{1}{40320}$

Appendix B

The successive derivatives of the exact solution

$\alpha_i, \beta_j \in \mathbb{N}^*$

- $\dot{y} = f$
- $\ddot{y} = f'(f)$
- $y^{(3)} = f''(f, f) + f'(f'(f))$
- $y^{(4)} = f^{(3)}(f, f, f) + 3f''(f'(f), f) + f'(f''(f, f)) + f'(f'(f'(f)))$
- $y^{(5)} = f^{(4)}(f, f, f, f) + 6f^{(3)}(f'(f), f, f) + 4f''(f''(f, f), f) + 4f''(f'(f'(f)), f) + 3f''(f'(f), f'(f)) + f'(f^{(3)}(f, f, f) + 3f'(f''(f' f, f)) + f'(f'(f''(f, f))) + f'(f'(f'(f'(f))))$
- $y^{(6)} = f^{(5)}(f, f, f, f, f) + 10f^{(4)}(f, f, f, f' f) + 10f^{(3)}(f, f, f''(f, f)) + 10f^{(3)}(f, f, f'(f'(f))) + 15f^{(3)}(f, f' f, f' f) + 5f''(f, f^{(3)}(f, f, f)) + 15f''(f, f''(f'(f), f)) + 5f''(f, f'(f''(f, f))) + 5f''(f, f'(f'(f'(f)))) + 10f''(f' f, f''(f, f)) + 10f''(f' f, f'(f'(f))) + f' f^{(4)}(f, f, f, f) + 6f' f^{(3)}(f'(f), f, f) + 4f' f''(f''(f, f), f) + 4f' f''(f'(f'(f)), f) + 3f' f''(f'(f), f'(f)) + f' f'(f^{(3)}(f, f, f) + 3f' f'(f''(f' f, f)) + f' f'(f'(f''(f, f))) + f' f'(f'(f'(f'(f))))$
- $y^{(7)} = f^{(6)}(f, f, f, f, f, f) + \alpha_1 f^{(5)}(f, f, f, f, f' f) + \alpha_2 f^{(4)}(f''(f, f), f, f, f) + \alpha_3 f^{(4)}(f'(f'(f)), f, f, f) + \alpha_4 f^{(4)}(f' f, f' f, f, f) + \alpha_5 f^{(3)}(f, f' f, f''(f, f)) + \alpha_6 f^{(3)}(f, f' f, f'(f'(f))) + \alpha_7 f^{(3)}(f, f, f^{(3)}(f, f, f)) + \alpha_8 f^{(3)}(f, f, f''(f'(f), f)) + \alpha_9 f^{(3)}(f, f, f'(f''(f, f))) + \alpha_{10} f^{(3)}(f, f, f'(f'(f'(f)))) + \alpha_{11} f^{(3)}(f' f, f' f, f' f) + \alpha_{12} f''(f, f^{(4)}(f, f, f, f)) + \alpha_{13} f''(f, f^{(3)}(f'(f), f, f)) + \alpha_{14} f''(f, f''(f''(f, f), f)) + \alpha_{15} f''(f, f''(f'(f'(f))) + \alpha_{17} f''(f, f''(f'(f), f'(f))) + \alpha_{18} f''(f, f'(f^{(3)}(f, f, f)) + \alpha_{19} f''(f, f'(f''(f' f, f))) + \alpha_{20} f''(f, f'(f'(f''(f, f)))) + \alpha_{21} f''(f, f'(f'(f'(f'(f))))) + \alpha_{22} f''(f' f, f^{(3)}(f, f, f)) + \alpha_{23} f''(f' f, f''(f'(f), f)) + \alpha_{24} f''(f' f, f'(f''(f, f))) + \alpha_{25} f''(f' f, f'(f'(f'(f)))) + \alpha_{26} f''(f''(f, f), f''(f, f)) + \alpha_{27} f''(f''(f, f), f'(f'(f))) + \alpha_{28} f''(f'(f'(f)), f'(f'(f))) + f' f^{(5)}(f, f, f, f, f) + \alpha_{29} f' f^{(4)}(f, f, f, f' f) + \alpha_{30} f' f^{(3)}(f, f, f''(f, f)) + \alpha_{31} f' f^{(3)}(f, f, f'(f'(f))) + \alpha_{32} f' f^{(3)}(f, f' f, f' f) + \alpha_{33} f' f''(f, f^{(3)}(f, f, f)) + \alpha_{34} f' f''(f, f''(f'(f), f)) + \alpha_{35} f' f''(f, f'(f''(f, f))) + \alpha_{36} f' f''(f, f'(f'(f'(f)))) + \alpha_{37} f' f''(f' f, f''(f, f)) + \alpha_{38} f' f''(f' f, f'(f'(f))) + f' f' f^{(4)}(f, f, f, f) + \alpha_{39} f' f' f^{(3)}(f'(f), f, f) + \alpha_{40} f' f' f''(f''(f, f), f) + \alpha_{41} f' f' f''(f'(f'(f)), f) + \alpha_{42} f' f' f''(f'(f), f'(f)) + f' f' f'(f^{(3)}(f, f, f) + \alpha_{43} f' f' f'(f''(f' f, f)) + f' f' f'(f'(f''(f, f))) + f' f' f'(f'(f'(f'(f))))$
- $y^{(8)} = f^{(7)}(f, f, f, f, f, f, f) + \beta_1 f^{(6)}(f, f, f, f, f, f' f) + \beta_2 f^{(5)}(f, f, f, f, f''(f, f)) + \beta_3 f^{(5)}(f, f, f, f, f'(f'(f))) + \beta_4 f^{(5)}(f' f, f' f, f, f, f) + \beta_5 f^{(4)}(f^{(3)}(f, f, f), f, f, f) + \beta_6 f^{(4)}(f''(f'(f), f), f, f, f) + \beta_7 f^{(4)}(f'(f''(f, f)), f, f, f) + \beta_8 f^{(4)}(f'(f'(f'(f))), f, f, f) + \beta_9 f^{(4)}(f, f, f' f, f''(f, f)) + \beta_{10} f^{(4)}(f, f, f' f, f'(f'(f))) + \beta_{11} f^{(4)}(f, f' f, f' f, f' f) + \beta_{12} f^{(3)}(f^{(3)}(f, f, f), f, f' f) + \beta_{13} f^{(3)}(f''(f'(f), f), f, f' f) + \beta_{14} f^{(3)}(f'(f''(f, f)), f, f' f) + \beta_{15} f^{(3)}(f'(f'(f'(f))), f, f' f) + \beta_{16} f^{(3)}(f''(f, f), f' f, f' f) + \beta_{17} f^{(3)}(f'(f'(f)), f' f, f' f) + \beta_{18} f^{(3)}(f''(f, f), f''(f, f), f) + \beta_{19} f^{(3)}(f''(f, f), f'(f'(f)), f) + \beta_{20} f^{(3)}(f'(f'(f)), f'(f'(f)), f) + \beta_{21} f^{(3)}(f, f, f^{(4)}(f, f, f, f)) + \beta_{22} f^{(3)}(f, f, f^{(3)}(f'(f), f, f)) + \beta_{23} f^{(3)}(f, f, f''(f''(f, f), f)) + \beta_{24} f^{(3)}(f, f, f''(f'(f'(f)), f)) + \beta_{25} f^{(3)}(f, f, f''(f'(f), f'(f))) + \beta_{26} f^{(3)}(f, f, f'(f^{(3)}(f, f, f)) + \beta_{27} f^{(3)}(f, f, f'(f''(f' f, f))) + 21 f^{(3)}(f, f, f'(f'(f''(f, f)))) + \beta_{28} f^{(3)}(f, f, f'(f'(f'(f'(f))))) + \beta_{29} f''(f^{(5)}(f, f, f, f, f), f) + \beta_{30} f''(10f^{(4)}(f, f, f, f' f), f) + \beta_{31} f''(10f^{(3)}(f, f, f''(f, f)), f) + \beta_{32} f''(f^{(3)}(f, f, f'(f'(f))), f) + \beta_{33} f''(f^{(3)}(f, f' f, f' f), f) + \beta_{34} f''(f''(f, f^{(3)}(f, f, f)), f) + \beta_{35} f''(f''(f, f''(f'(f), f)), f) + f''(5f''(f, f'(f''(f, f))), f) + f''(5f''(f, f'(f'(f'(f)))), f) + \beta_{36} f''(f''(f' f, f''(f, f)), f) + \beta_{37} f''(f''(f' f, f'(f'(f))), f) + \beta_{38} f''(f' f^{(4)}(f, f, f, f), f) + \beta_{39} f''(f' f^{(3)}(f'(f), f, f), f) + \beta_{40} f''(f' f''(f''(f, f), f), f) + \beta_{41} f''(f' f''(f'(f'(f)), f), f) + \beta_{42} f''(f' f''(f'(f), f'(f)), f) + \beta_{43} f''(f' f'(f^{(3)}(f, f, f), f) + \beta_{44} f''(f' f'(f''(f' f, f)), f) + \beta_{45} f''(f' f'(f'(f''(f, f))), f) + \beta_{46} f''(f' f'(f'(f'(f'(f)))), f) + \beta_{47} f''(f' f, f^{(4)}(f, f, f, f)) + \beta_{48} f''(f' f, f^{(3)}(f'(f), f, f)) + \beta_{49} f''(f' f, f''(f''(f, f), f)) + \beta_{50} f''(f' f, f''(f'(f'(f)), f)) + \beta_{51} f''(f' f, f''(f'(f), f'(f))) + \beta_{52} f''(f' f, f'(f^{(3)}(f, f, f)) + \beta_{53} f''(f' f, f'(f''(f' f, f))) + \beta_{54} f''(f' f, f'(f'(f''(f, f)))) + \beta_{55} f''(f' f, f'(f'(f'(f'(f))))) + \beta_{56} f''(f^{(3)}(f, f, f), f''(f, f)) + \beta_{57} f''(f^{(3)}(f, f, f), f'(f'(f))) + \beta_{58} f''(f''(f'(f), f), f''(f, f)) + \beta_{60} f''(f'(f''(f, f)), f'(f'(f))) + \beta_{61} f''(f'(f'(f'(f))), f'(f'(f))) + \beta_{62} f''(f''(f'(f), f), f''(f, f)) + \beta_{63} f''(f'(f''(f, f)), f'(f'(f))) + \beta_{64} f''(f'(f'(f'(f))), f''(f, f)) + f' f^{(6)}(f, f, f, f, f, f) + \beta_{65} f' f^{(5)}(f, f, f, f, f' f) + \beta_{66} f' f^{(4)}(f''(f, f), f, f, f) + \beta_{67} f' f^{(4)}(f'(f'(f)), f, f, f) + \beta_{68} f' f^{(4)}(f' f, f' f, f, f) + \beta_{69} f' f^{(3)}(f, f' f, f''(f, f)) + \beta_{70} f' f^{(3)}(f, f' f, f'(f'(f))) + \beta_{71} f' f^{(3)}(f, f, f^{(3)}(f, f, f)) + \beta_{72} f' f^{(3)}(f, f, f''(f'(f), f)) + \beta_{73} f' f^{(3)}(f, f, f'(f''(f, f))) + \beta_{74} f' f^{(3)}(f, f, f'(f'(f'(f)))) + \beta_{75} f' f^{(3)}(f' f, f' f, f' f) + \beta_{76} f' f''(f, f^{(4)}(f, f, f, f)) + \beta_{77} f' f''(f, f^{(3)}(f'(f), f, f)) + \beta_{78} f' f''(f, f''(f''(f, f), f)) + \beta_{79} f' f''(f, f''(f'(f'(f))) + \beta_{80} f' f''(f, f''(f'(f), f'(f))) + \beta_{81} f' f''(f, f'(f^{(3)}(f, f, f)) + \beta_{82} f' f''(f, f'(f''(f' f, f))) + \beta_{83} f' f''(f, f'(f'(f''(f, f)))) + \beta_{84} f' f''(f, f'(f'(f'(f'(f))))) + \beta_{85} f' f''(f' f, f^{(3)}(f, f, f)) + \beta_{86} f' f''(f' f, f''(f'(f), f)) + \beta_{87} f' f''(f' f, f'(f''(f, f))) + \beta_{88} f' f''(f' f, f'(f'(f'(f)))) + \beta_{89} f' f''(f''(f, f), f''(f, f)) + \beta_{90} f' f''(f''(f, f), f'(f'(f))) + \beta_{91} f' f''(f'(f'(f)), f'(f'(f))) + f' f' f^{(5)}(f, f, f, f, f) + \beta_{92} f' f' f^{(4)}(f, f, f, f' f) + \beta_{93} f' f' f^{(3)}(f, f, f''(f, f)) + \beta_{94} f' f' f^{(3)}(f, f, f'(f'(f))) + \beta_{95} f' f' f^{(3)}(f, f' f, f' f) + \beta_{96} f' f' f''(f, f^{(3)}(f, f, f)) + \beta_{97} f' f' f''(f, f''(f'(f), f)) + 5f' f' f''(f, f'(f''(f, f))) + \beta_{98} f' f' f''(f, f'(f'(f'(f)))) + \beta_{99} f' f' f''(f' f, f''(f, f)) + \beta_{100} f' f' f''(f' f, f'(f'(f))) + f' f' f' f^{(4)}(f, f, f, f) + \beta_{101} f' f' f' f^{(3)}(f'(f), f, f) + \beta_{102} f' f' f' f''(f''(f, f), f) + \beta_{103} f' f' f' f''(f'(f'(f)), f) + \beta_{104} f' f' f' f''(f'(f), f'(f)) + f' f' f' f'(f^{(3)}(f, f, f) + \beta_{105} f' f' f' f'(f''(f' f, f)) + f' f' f' f'(f'(f''(f, f))) + f' f' f' f'(f'(f'(f'(f))))$