Univalence Conditions of Two New Integral Operators on P-Valent Functions

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Abstract

For analytic functions in the open unit disk $\mathcal{U}$, we define two new general integral operators. The main object of the this paper is to study these two new integral operators and to determine some sufficient conditions for general p-valent integral operator to be p-th power of a univalent functions.

Keywords: analytic functions, univalent functions, integral operators, sufficient conditions, p-valent integral operators

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1. Introduction

Let $\mathcal{A}_p$ be the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, z \in \mathcal{U}, p \in \mathbb{N}^* = \{1, 2, ..., n\} \tag{1.1}$$

which are analytic and p-valent in the unit disk $\mathcal{U} = \{z : |z| < 1\}$.

A function $f \in \mathcal{A}_p$ is called p-valent starlike of order γ if $f(z)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, z \in \mathcal{U}, \tag{1.2}$$

$0 \leq \gamma < p$, $p \in \mathbb{N}^*$. We denote by $\mathcal{S}_p^*(\gamma)$ the class of all such functions.

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{S}_p^*(\gamma)$, $0 \leq \gamma < p$, $p \in \mathbb{N}^*$ if it satisfies the condition

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \gamma, z \in \mathcal{U}. \tag{1.3}$$

A function $f \in \mathcal{A}_p$ is called p-valent convex of order γ if $f(z)$ satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, z \in \mathcal{U}, \tag{1.4}$$

$0 \leq \gamma < p, p \in \mathbb{N}^*$. By $C_p(\gamma)$, we denote the class of all p-valent convex function of order γ. From (1.2) and (1.4), Çağlar showed us that:

$$f(z) \in C_p(\gamma) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\gamma). \tag{1.5}$$

A function $f \in \mathcal{A}_p$ is in the class $C_p(\gamma)$, $0 \leq \gamma < p$, $p \in \mathbb{N}^*$ if it satisfies the condition

$$\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| < p - \gamma, z \in \mathcal{U}. \tag{1.6}$$

Now we define two new general p-valent integral operators.

The first new p-valent integral operator has the following form:

$$J^p_{\alpha_i,\beta}(z) = \left\{\beta p \int_0^z t^{\beta p-1} \prod_{i=1}^{\infty} \left(\frac{f_i(t)}{t^p}\right)^{1-\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}} \tag{1.7}$$

where the functions $f_i \in \mathcal{A}_p$, $i = 1, 2, ..., n$, and the paramentes β and α_i, $\alpha_i \neq 0$, $\mathrm{Re}(\beta) > 0$ for all $i = 1, 2, ..., n$ are complex numbers such that the integral operators in (1.7).

The second new p-valent integral operator has the form

$$I^p_{\alpha_i,\beta}(z) = \left\{\beta p \int_0^z t^{\beta p-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t^p}\right)^{1-\frac{1}{\alpha_i}} \left(\frac{g_i'(t)}{pt^{p-1}}\right)^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}} \tag{1.8}$$

where the functions $f_i, g_i \in \mathcal{A}_p$, $i = 1, 2, ..., n$, and the paramentes β and α_i, $\alpha_i \neq 0$, $\mathrm{Re}(\beta) > 0$ for all $i = 1, 2, ..., n$ are complex numbers such that the integral operators in (1.8). Hallenbeck and Livingston defined p-subordination chains method and they obtained some results for $f \in \mathcal{A}_p$ to be the p-th power of a univalent functions in $\mathcal{U}$.

Theorem 1.1. *Let $f \in \mathcal{A}_p$ and α complex number such that $\mathrm{Re}\alpha > 0$ and*

$$\frac{1 - |z|^{2p\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \leq p \tag{1.9}$$

is true for all $z \in \mathcal{U}$, then the integral operator

$$\mathcal{H}_\alpha(z) = \left\{\alpha \int_0^z t^{p(\alpha-1)} f'(t) dt\right\}^{\frac{1}{\alpha}} \tag{1.10}$$

is the p-th power of a univalent function in $\mathcal{U}$ where the principal branch is considered.

2. Main Results

Firstly, we obtain sufficient conditions for p-valent integral operator defined by (1.7), to be the p-th power of a univalent function in $\mathcal{U}$.

Theorem 2.1. *Let the function $f_i \in \mathcal{S}^*_p(\gamma_i)$, $0 \leq \gamma_i < p$, $p \in \mathbb{N}^*$, $i = 1, 2, ..., n$. If $\beta, \alpha_i \in \mathbb{C}$, $\alpha_i \neq 0$, $i = 1, 2, ..., n$ and*

$$p\mathrm{Re}\beta \geq 1 - p + \sum_{i=1}^{n} \frac{|\alpha_i - 1|(p - \gamma_i)}{|\alpha_i|} \tag{2.1}$$

then the integral operator $J^p_{\alpha_i,\beta}$ defined by (1.7) is the p-th power of a univalent function in $\mathcal{U}$.

Proof. We define the function

$$h(z) = \int_o^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t^p}\right)^{1-\frac{1}{\alpha_i}} dt. \tag{2.2}$$

It is easy to see that

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z^p}\right)^{1-\frac{1}{\alpha_i}}. \tag{2.3}$$

Differentiating (2.3) logarithmically and multiplying by z, we get

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \left(1 - \frac{1}{\alpha_i}\right)\left(\frac{zf_i'(z)}{f_i(z)} - p\right). \tag{2.4}$$

Thus, we obtain

$$\begin{aligned}
&\frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|1-p+\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|1-p+\sum_{i=1}^{n}\left(1-\frac{1}{\alpha_i}\right)\left(\frac{zf_i'(z)}{f_i(z)}-p\right)\right| \\
&\leq \frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[1-p+\sum_{i=1}^{n}\left|1-\frac{1}{\alpha_i}\right|\left|\frac{zf_i'(z)}{f_i(z)}-p\right|\right] \\
&\leq \frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[\frac{|\alpha_i-1|}{|\alpha_i|}\left|\frac{zf_i'(z)}{f_i(z)}-p\right|\right] \\
&\leq \frac{1}{\mathrm{Re}\beta}\left[1-p+\sum_{i=1}^{n}\frac{|\alpha_i-1|(p-\gamma_i)}{|\alpha_i|}\right]. \qquad (2.5)
\end{aligned}$$

From the hypothesis of the Theorem 2.1, we get:

$$\begin{aligned}
&\frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|1-p+\frac{zh''(z)}{h'(z)}\right| \leq \\
&\leq \frac{1}{\mathrm{Re}\beta}\left[1-p+\sum_{i=1}^{n}\frac{|\alpha_i-1|(p-\gamma_i)}{|\alpha_i|}\right] \leq p. \qquad (2.6)
\end{aligned}$$

Applying Theorem 1.1, we get the integral operator $J^p_{\alpha_i,\beta}$ defined by (1.7) is the p-th power of a univalent function in $\mathcal{U}$. □

Letting $\beta = n = 1, \alpha_1 = \alpha, \gamma_1 = \gamma$ and $f_1 = f$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let the function $f \in \mathcal{S}^*_p(\gamma)$, $0 \leq \gamma < p$, $p \in \mathbb{N}^*$. If $\alpha \in \mathbb{C}$, $\alpha \neq -1$, $\alpha \neq \frac{1}{3}$ and*

$$p \geq \frac{|\alpha| - \gamma|\alpha - 1|}{2|\alpha| - |\alpha - 1|} \qquad (2.7)$$

then the integral operator

$$J^p_{\alpha,1}(z) = J^p_{\alpha}(z) = \int_0^z pt^{p-1}\left(\frac{f(t)}{t^p}\right)^{1-\frac{1}{\alpha}} dt \qquad (2.8)$$

is the p-th power of a univalent function in $\mathcal{U}$.

Letting $\beta = n = 1, \alpha_1 = \alpha = 1, \gamma_1 = \gamma$ and $f_1 = f$ in Theorem 2.1, we obtain following corollary.

Corollary 2.2. *Let the function $f \in \mathcal{S}^*_p(\gamma)$, $0 \leq \gamma < p$, $p \in \mathbb{N}^*$, then the integral operator*

$$J^p_{1,1}(z) = J^p(z) = p\int_0^z t^{p-1}dt \qquad (2.9)$$

is the p-th power of a univalent function in $\mathcal{U}$.

In the next theorem, we derive another sufficient condition for p-valent integral operator defined by (1.8), to be the p-th power of a univalent function in $\mathcal{U}$.

Theorem 2.2. *Let the functions $f_i \in \mathcal{S}^*_p(\gamma_i)$ and $g_i \in \mathcal{C}_p(\gamma_i)$, $0 \leq \gamma_i < p$, $p \in \mathbb{N}^*$, $i = 1, 2, ..., n$. If $\beta, \alpha_i \in \mathbb{C}$, $\alpha \neq 0$, $i = 1, 2, ..., n$ and*

$$p\mathrm{Re}\beta \geq 1 - p + \sum_{i=1}^{n}\frac{|\alpha_i - 1| + 1}{|\alpha_i|}(p - \gamma_i) \qquad (2.10)$$

then the integral operator $I^p_{\alpha_i,\beta}$ defined by (1.8) is the p-th power of a univalent function in $\mathcal{U}$.

Proof. We define

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t^p}\right)^{1-\frac{1}{\alpha_i}}\left(\frac{g_i'(t)}{pt^{p-1}}\right)^{\frac{1}{\alpha_i}} dt. \qquad (2.11)$$

It is easy to see that

$$h^{'}(z)=\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{1-\frac{1}{\alpha_i}}\left(\frac{g_i^{'}(z)}{pz^{p-1}}\right)^{\frac{1}{\alpha_i}}. \tag{2.12}$$

Differentiating (2.12) logarithmically and multiplying by z, we obtain

$$\frac{zh^{''}(z)}{h^{'}(z)}=\sum_{i=1}^{n}\left[\left(1-\frac{1}{\alpha_i}\right)\left(\frac{zf_i^{'}(z)}{f_i(z)}-p\right)+\frac{1}{\alpha_i}\left(\frac{zg^{''}(z)}{g_i^{'}(z)}-(p-1)\right)\right]. \tag{2.13}$$

Thus, we get

$$\begin{aligned}
&\frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|1-p+\frac{zh^{''}(z)}{h^{'}(z)}\right|=\\
&=\frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|1-p+\sum_{i=1}^{n}\left[\left(1-\frac{1}{\alpha_i}\right)\left(\frac{zf_i^{'}(z)}{f_i(z)}-p\right)+\frac{1}{\alpha_i}\left(\frac{zg_i^{''}(z)}{g_i^{'}(z)}-(p-1)\right)\right]\right|\\
&\le\frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[1-p+\sum_{i=1}^{n}\left(\frac{|\alpha_i-1|}{|\alpha_i|}\left|\frac{zf_i^{'}(z)}{f_i(z)}-p\right|+\frac{1}{|\alpha_i|}\left|\frac{zg_i^{''}(z)}{g^{'}(z)}-(p-1)\right|\right)\right]\\
&\le\frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[1-p+\sum_{i=1}^{n}\left(\frac{|\alpha_i-1|}{|\alpha_i|}(p-\gamma_i)+\frac{1}{|\alpha_i|}(p-\gamma_i)\right)\right]\\
&\le\frac{1}{\mathrm{Re}\beta}\left[1-p+\sum_{i=1}^{n}\frac{|\alpha_i-1|+1}{|\alpha_i|}(p-\gamma_i)\right].
\end{aligned} \tag{2.14}$$

From the hypothesis of the Theorem 2.2, we get

$$\begin{aligned}
&\frac{1-|z|^{2p\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|1-p+\frac{zh^{''}(z)}{h^{'}(z)}\right|\\
&\le\frac{1}{\mathrm{Re}\beta}\left[1-p+\sum_{i=1}^{n}\frac{|\alpha_i-1|+1}{|\alpha_i|}(p-\gamma_i)\right]\le p.
\end{aligned} \tag{2.15}$$

Applying Theorem 1.1, we get the integral operator $I^p_{\alpha_i,\beta}$ defined by (1.8) is the p-th power of a univalent function in $\mathcal{U}$. □

Letting $\beta=n=1$, $\gamma_1=\gamma$, $\alpha_1=\alpha$, $f_1=f$ and $g_1=g$ in Theorem 2.2, we get the following corollary.

Corollary 2.3. *Let the functions $f\in\mathcal{S}_p^*(\gamma)$, and $g\in\mathcal{C}_p(\gamma)$, $0\le\gamma<p$, $p\in\mathbb{N}^*$. If $\alpha\in\mathbb{C}$, $\alpha\ne-2$, $\alpha\ne\frac{2}{3}$ and*

$$p\ge\frac{|\alpha|-\gamma(|\alpha-1|+1)}{2|\alpha|-|\alpha-1|-1} \tag{2.16}$$

then the integral operator

$$I^p_{\alpha,1}(z)=I^p_{\alpha}(z)=\int_0^z pt^{p-1}\left(\frac{f(t)}{t^p}\right)^{1-\frac{1}{\alpha}}\left(\frac{g^{'}(t)}{pt^{p-1}}\right)^{\frac{1}{\alpha}}dt \tag{2.17}$$

is the p-th power of a univalent function in $\mathcal{U}$.

Letting $\beta=n=1$, $\gamma_1=\gamma$, $f_1=f$, $g_1=g$ and choosing $\alpha_1=\alpha=1$ in Theorem 2.2, we have:

Corollary 2.4. *Let the functions $f\in\mathcal{S}_p^*(\gamma)$ and $g\in\mathcal{C}_p(\gamma)$, $0\le\gamma<p$, $p\in\mathbb{N}^*$, then the integral operator*

$$I^p_{1,1}(z)=I^p(z)=\int_0^z g^{'}(t)dt \tag{2.18}$$

is the p-th power of a univalent function in $\mathcal{U}$.

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